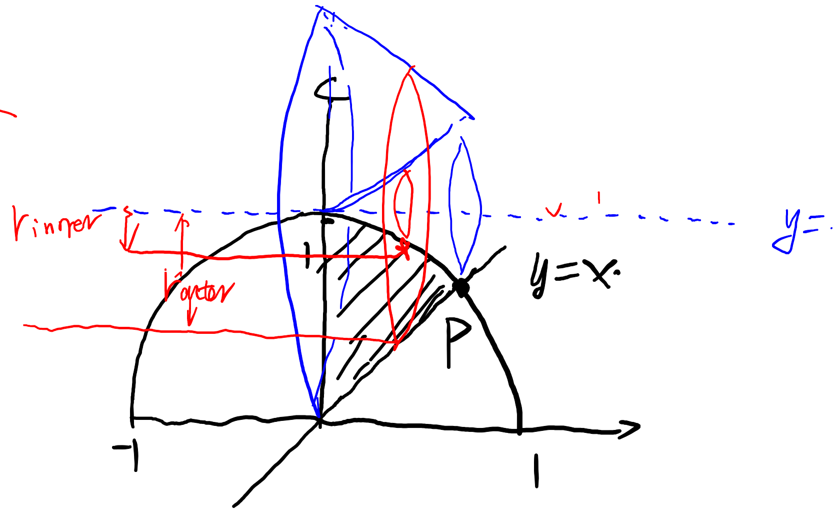
Q1[6 points] Sketch the region R bounded by $y = \sqrt{1-x^2}, x = 0,\ y = x$. Set up the volume of the solid rotating R about the axis $y = 1$. Do not evaluate.

Intersection P: $y = \sqrt{1-x^2} = x \Rightarrow 1 - x^2 = x^2 \Rightarrow 2x^2 = 1 \Rightarrow x = \frac{1}{\sqrt{2}}$ (only positive)

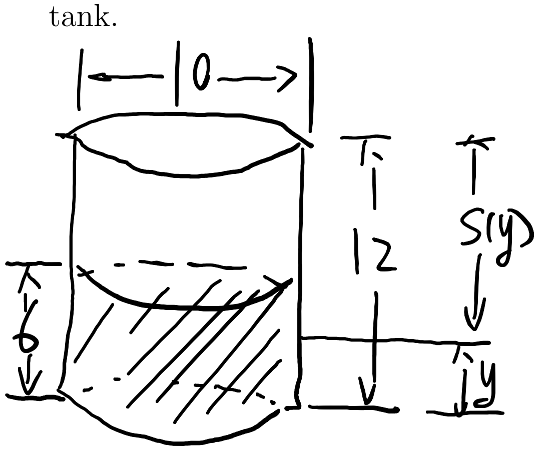
$r_{inner} = 1 - \sqrt{1-x^2}$

$r_{outer} = 1 - x$

$$\boxed{V = \int_0^{\frac{1}{\sqrt{2}}} \pi \left[(1-x)^2 - \left(1 - \sqrt{1-x^2}\right)^2 \right] dx}$$



Q2[6 points] A vertical right-circular cylindrical tank measures 12 ft high and 10 ft in diameter. It is half full of kerosene weighing 20 lb/ft^3. Find the work it would take to pump the kerosene to the top of the tank.



$\sigma = 20$, $s(y) = 12 - y$.

diameter $= 10 \Rightarrow$ radius $= 5$

$A(y) = \pi \cdot 5^2 = 25\pi$.

Work:

$$W = \int_0^6 \sigma \cdot s(y) \cdot A(y)\, dy$$

$$= \boxed{\int_0^6 20 \cdot (12-y) \cdot 25\pi \cdot dy} \quad \text{ft-lbs}$$

Q3 Evaluate the following integrals.

(a)[4 points]

$$\int_0^1 \underbrace{x}_{u}\underbrace{e^{2x}\,\mathrm{d}x}_{dv}$$

IBP: $u = x,\ dv = e^{2x}dx$

$du = dx,\ v = \frac{1}{2}e^{2x}$

$$= u\cdot v - \int v.du$$

$$= x\cdot\frac{1}{2}\cdot e^{2x}\Big|_0^1 - \int_0^1 \frac{1}{2}e^{2x}dx$$

$$= x\cdot\frac{1}{2}\cdot e^{2x}\Big|_0^1 - \frac{1}{2}\cdot\frac{1}{2}e^{2x}\Big|_0^1$$

$$= 1\cdot\frac{1}{2}\cdot e^2 - 0 - \left[\frac{1}{4}\cdot e^2 - \frac{1}{4}e^0\right] = \boxed{\frac{1}{4}e^2 + \frac{1}{4}}$$

(b)[6 points]

$$\int_0^1 \frac{1}{(x)^{4/3}}\,\mathrm{d}x$$

$$= \lim_{t\to 0^+}\int_t^1 \frac{1}{x^{\frac{4}{3}}}dx = \lim_{t\to 0^+}\int_t^1 x^{-\frac{4}{3}}dx$$

$$= \lim_{t\to 0^+}\frac{1}{-\frac{4}{3}+1}\cdot x^{-\frac{4}{3}+1}\Big|_t^1$$

$$= \lim_{t\to 0^+} -3\cdot x^{-\frac{1}{3}}\Big|_t^1$$

$$= \lim_{t\to 0^+} -3\cdot 1^{-\frac{1}{3}} + 3\cdot t^{-\frac{1}{3}}$$

$$= \lim_{t\to 0^+} -3 + \frac{3}{t^{\frac{1}{3}}} = \boxed{\infty}$$

$\boxed{\text{The improper integral is divergent.}}$

Q4 Determine whether each of the series is convergent or divergent.

(a)[5 points]

$$\sum_{n=1}^{\infty} \frac{(2n+1)(2n-1)}{n^3}$$

$a_n = \frac{(2n+1)\cdot(2n-1)}{n^3}$, $b_n = \frac{2n\cdot 2n}{n^3} = \frac{4n^2}{n^3} = \frac{4}{n}$.

$\sum b_n = \sum \frac{4}{n}$ $\boxed{\text{divergent}}$. Harmonic Series.

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{(2n+1)(2n-1)}{n^3} \cdot \frac{n}{4} = \lim_{n\to\infty} \frac{2n\cdot 2n\cdot n}{n^3\cdot 4} = \boxed{1}$.

$\boxed{\text{Limit Comparison Test implies } \sum a_n \text{ is also divergent}}$

(b)[5 points]

$$\sum_{n=1}^{\infty} \frac{5^n}{n!}$$

$a_n = \frac{5^n}{n!}$

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{5^{n+1}}{(n+1)!}}{\frac{5^n}{n!}} = \lim_{n\to\infty} \frac{5^{n+1}}{5^n} \cdot \frac{n!}{(n+1)!}$

$= \lim_{n\to\infty} 5 \cdot \frac{1}{n+1} = \boxed{0 < 1}$

$\boxed{\text{Ratio Test implies } \sum a_n \text{ is convergent}}$

Q5[10 points] Evaluate the following integral

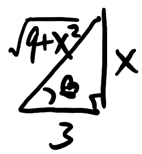
$$\int \frac{dx}{\sqrt{9+x^2}}$$

Trig-Sub: $x = 3\tan\theta \quad dx = 3\sec^2\theta \, d\theta$

$$\int \frac{dx}{\sqrt{9+x^2}} = \int \frac{3\sec^2\theta \cdot d\theta}{\sqrt{9+(3\tan\theta)^2}} = \int \frac{3\sec^2\theta}{\sqrt{9\cdot(1+\tan^2\theta)}} d\theta$$

$x = 3\tan\theta$, $\tan\theta = \frac{x}{3}$

$\sec\theta = \frac{\sqrt{9+x^2}}{3}$



$$= \int \frac{3\sec^2\theta}{\sqrt{9\sec^2\theta}} d\theta$$

$$= \int \frac{3\sec^2\theta}{3\cdot\sec\theta} d\theta = \int \sec\theta \, d\theta$$

$$= \ln|\tan\theta + \sec\theta| + C$$

$$\boxed{= \ln\left|\frac{x}{3} + \frac{\sqrt{9+x^2}}{3}\right| + C}$$

Q6[6 points] Find the partial fraction decomposition for $\frac{2x^2-x+2}{x(x^2+2)}$ in order to evaluate

$$\int \frac{2x^2-x+2}{x(x^2+2)} dx$$

(Do not need to evaluate the integral.)

$$\boxed{\frac{2x^2-x+2}{x\cdot(x^2+2)} = \frac{A}{x} + \frac{Bx+C}{x^2+2}}$$

multiply by $x\cdot(x^2+2)$ both sides.

$$2x^2-x+2 = A(x^2+2) + (Bx+C)\cdot x.$$

$$2x^2-x+2 = Ax^2+2A + Bx^2 + Cx$$

$$2x^2-x+2 = (A+B)x^2 + C\cdot x + 2A$$

$$\begin{cases} 2 = A+B. \\ -1 = C \\ 2 = 2A \end{cases} \Rightarrow \boxed{\begin{cases} A = 1 \\ B = 1 \\ C = -1 \end{cases}}$$

, i.e. $\frac{2x^2-x+2}{x\cdot(x^2+2)} = \frac{1}{x} + \frac{x-1}{x^2+2}$

$$\boxed{= \frac{1}{x} + \frac{x}{x^2+2} - \frac{1}{x^2+2}}$$

Q7[6 points]Find the equation of the line tangent to the parametric curve

$$x(t) = \arctan(2t), \quad y(t) = 3^t \qquad \text{at} \qquad (x, y) = (0, 1)$$

$x = \arctan(2t) = 0, \quad y = 3^t = 1 \Rightarrow \boxed{t = 0}$

$\frac{dx}{dt} = \frac{1}{1+(2t)^2} \cdot 2, \quad \frac{dy}{dt} = \ln 3 \cdot 3^t$

slope: $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\ln 3 \cdot 3^t}{\frac{1}{1+4t^2} \cdot 2} \overset{t=0}{=\!=} \frac{\ln 3 \cdot 3^0}{\frac{1}{1+0} \cdot 2} = \boxed{\frac{\ln 3}{2}}$

$y - 1 = \frac{\ln 3}{2} \cdot (x - 0)$

$\Rightarrow \boxed{y = \frac{\ln 3}{2} \cdot x + 1}$

Q8[6 points] Solve $y(x)$ if

$$y'(x) = e^{-2y}x, \; y(0) = 0$$

$y'(x) = \frac{dy}{dx} = e^{-2y} x$

$e^{2y} dy = x \, dx$

$\int e^{2y} dy = \int x \, dx$

$\frac{1}{2} \cdot e^{2y} = \frac{1}{2} \cdot x^2 + C$

$y(0) = 0 \Rightarrow x = 0, y = 0$

$\frac{1}{2} \cdot e^0 = \frac{1}{2} \cdot 0^2 + C \Rightarrow C = \frac{1}{2}$

$\frac{1}{2} \cdot e^{2y} = \frac{1}{2} \cdot x^2 + \frac{1}{2}$

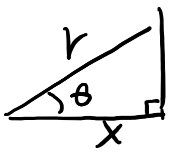
$e^{2y} = x^2 + 1$

$2y = \ln(x^2+1)$

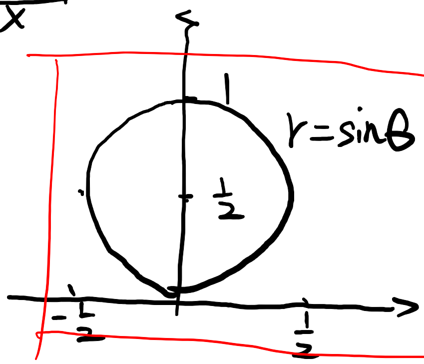
$\boxed{y = \frac{1}{2} \cdot \ln(x^2+1)}$

Q9

(a)[**6 points**] Find the Cartesian equation of the polar curve given by $r = \sin\theta$. What curve is it? Sketch the curve.



$\sin\theta = \frac{y}{r} \Rightarrow r = \frac{y}{r} \Rightarrow r^2 = y \Rightarrow \boxed{x^2+y^2=y}$

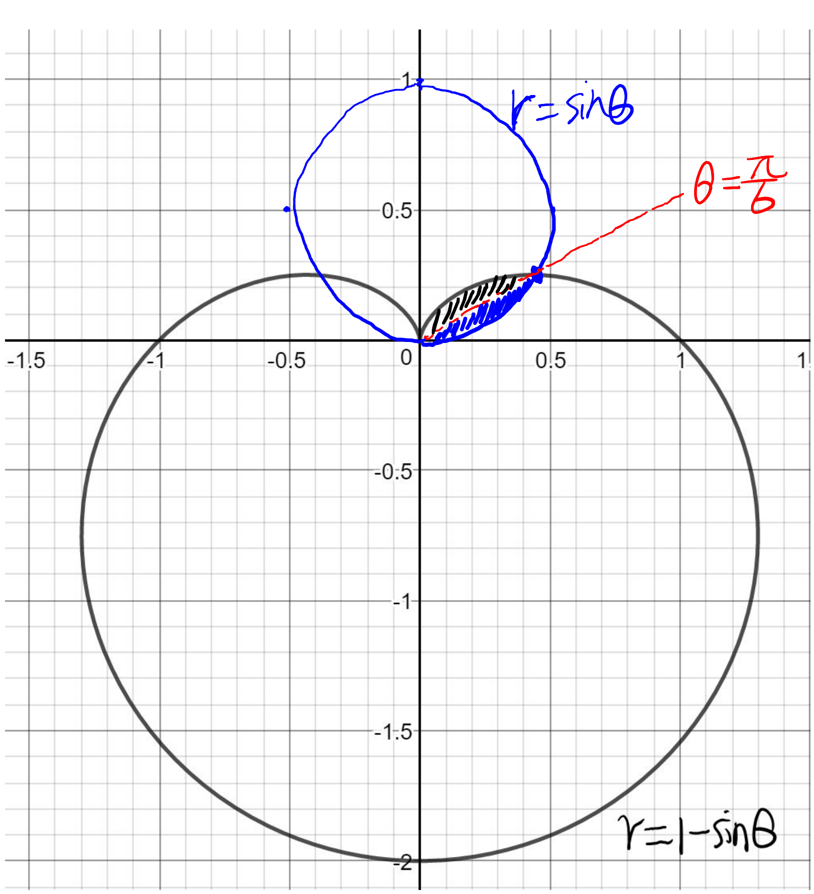


$x^2+y^2-y+\frac{1}{4}=\frac{1}{4}$

$x^2+(y-\frac{1}{2})^2=(\frac{1}{2})^2$

Circle centered at $(0,\frac{1}{2})$ with radius $\frac{1}{2}$

(b)[**6 points**] Give the graph of Cardioid $r = 1 - \sin\theta$ as below. Find the (r, θ) coordinates of the intersection of $r = \sin\theta$ and $r = 1 - \sin\theta$ in the first quadrant. Set up the integral for the area shared by these two polar curves **in the first quadrant**. Do not evaluate.



Intersection:

$r = 1-\sin\theta = \sin\theta \Rightarrow \sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}$

$r = \frac{1}{2}$, $(r,\theta) = (\frac{1}{2}, \frac{\pi}{6})$

$\text{Area1} = \int_0^{\frac{\pi}{6}} \frac{1}{2}\sin^2\theta\, d\theta$

$\text{Area2} = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{1}{2}(1-\sin\theta)^2\, d\theta$

$\text{Area} = \text{Area1} + \text{Area2}$

$$= \boxed{\int_0^{\frac{\pi}{6}} \frac{1}{2}\sin^2\theta\, d\theta + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{1}{2}(1-\sin\theta)^2\, d\theta}$$

Multiple Choice. Circle the best answer. No work needed.

Q10[3 points] Evaluate the integral

$$\int \sin\theta \cdot \sin(\cos\theta)\, d\theta$$

$u = \cos\theta$, $du = -\sin\theta\, d\theta$

$= \int \sin(\cos\theta) \cdot \underline{\sin\theta\, d\theta}$

$= \int \sin(u) \cdot (-du)$

$= \cos u + C$

$= \cos(\cos\theta) + C$.

A $\cos(\cos\theta) + C$

B $\cos(\sin\theta) + C$

C $\sin(\cos\theta) + C$

D $-\cos(\cos\theta) + C$

E $-\sin(\sin\theta) + C$

Q11[3 points] Find the open interval of convergence of the following power series.

$$\sum_{n=1}^{\infty} n(2x+1)^n$$

$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)(2x+1)^{n+1}}{n \cdot (2x+1)^n} \right|$

$= \lim_{n\to\infty} \left| \frac{(n+1)}{n} \cdot (2x+1) \right|$

$= 1 \cdot |2x+1| < 1$

$-1 < 2x+1 < 1 \Rightarrow -2 < 2x < 0$

$\Rightarrow -1 < x < 0$

A $(0, 1)$

B $(0, \frac{1}{2})$

C $(-\frac{1}{2}, \frac{1}{2})$

D $(-1, 0)$

E $(-1, 1)$

Q12[3 points] Evaluate the following limit

$$\lim_{x\to 0^+} (\cos x)^{\frac{1}{x}} = \lim_{x\to 0^+} e^{\ln(\cos x)^{\frac{1}{x}}} = e^{\lim_{x\to 0^+} \frac{\ln(\cos x)}{x}}$$

$= e^0 = 1$

$\lim_{x\to 0^+} \frac{\ln(\cos x)}{x}$ $\frac{\ln 1}{0} = \frac{0}{0}$

$\stackrel{\text{H}}{=} \lim_{x\to 0^+} \frac{\frac{1}{\cos x} \cdot (-\sin x)}{1} = \lim_{x\to 0^+} \frac{-\sin x}{\cos x} = -\frac{\sin 0}{\cos 0} = -\frac{0}{1} = 0$

A e

B 1

C e^2

D $-\infty$

E 0

Q13[3 points] Find the second degree Maclaurin polynomial of the function

$$f(x) = \frac{\ln(1+x)}{1-x}$$

$$= \ln(1+x) \cdot \frac{1}{1-x}$$

$$= \left(x - \frac{x^2}{2} + \cdots\right)\left(1 + x + x^2 + \cdots\right)$$

$$\approx \left(x - \frac{x^2}{2}\right)\left(1 + x + x^2\right)$$

$$= x + x^2 + x^3 - \frac{x^2}{2} - \frac{x^3}{2} - \frac{x^4}{2}$$

$$= \boxed{x + \frac{1}{2}x^2} + \cancel{\frac{1}{2}x^3 - \frac{x^4}{2}}$$

A $x - \frac{1}{2}x^2$

B $x + \frac{1}{2}x^2$

C $1 + x + \frac{3}{2}x^2$

D $x + \frac{3}{2}x^2$

E $1 + x + x^2$

Q14[3 points] Which point given by the polar coordinates (r, θ) is in the second quadrant on the XY plane?

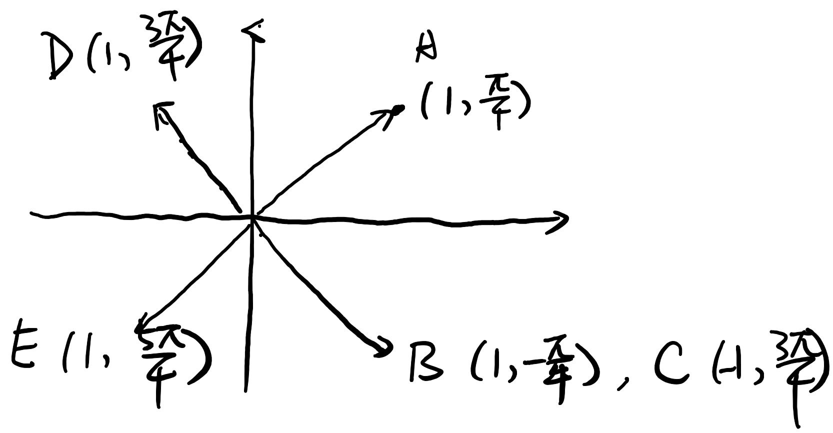
A $(r, \theta) = (1, \pi/4)$

B $(r, \theta) = (1, -\pi/4)$

C $(r, \theta) = (-1, 3\pi/4)$

D $(r, \theta) = (1, 3\pi/4)$

E $(r, \theta) = (1, 5\pi/4)$



$$\lim_{k\to\infty} \sec\left(\frac{2}{k}\right) = \sec 0$$

$$= \frac{1}{\cos 0} = 1.$$

$$\neq 0$$

Q15[3 points] Consider the sequence $a_k = \sec(\frac{2}{k})$ and the series $\sum \sec(\frac{2}{k})$

A Both the sequence and the series diverge.

B The sequence a_k converges to 1 and nth term test is inconclusive for the series.

C The sequence a_k converges to 0 and nth term test tells the series is divergent.

a_k conv

D The sequence a_k converges to 1 and nth term test tells the series is divergent.

$\sum a_k$ DIV

E Both the sequence and the series converge.

Q16[3 points] A variable force of $\frac{6}{x^2}$ pounds moves an object along a straight line when it is x feet from the origin. Calculate the work W done in moving the object from $x = 2$ ft to $x = 3$ ft.

A 1 ft-lb

B -1 ft-lb

C 6 ft-lb

D $\frac{6}{3^2} - \frac{6}{2^2}$ ft-lb

E $\frac{6}{2^2} - \frac{6}{3^2}$ ft-lb

$$W = \int_2^3 \frac{6}{x^2}\,dx$$

$$= \frac{-6}{x}\Big|_2^3$$

$$= \frac{-6}{3} + \frac{6}{2} = -2+3 = 1.$$

Q17[3 points] Which integral represents the arc-length of the parametric curve given by $x = 2\sin t, y = 3\cos t$ from $t = 0$ to $t = \pi$

A

$$\int_0^\pi \sqrt{2\sin t + 3\cos t}\; dt$$

B

$$\int_0^\pi \sqrt{2\cos t - 3\sin t}\; dt$$

C

$$\int_0^\pi \sqrt{2(\cos t)^2 + 3(\sin t)^2}\; dt$$

D

$$\int_0^\pi \sqrt{(2\sin t)^2 + (3\cos t)^2}\; dt$$

E

$$\int_0^\pi \sqrt{4(\cos t)^2 + 9(\sin t)^2}\; dt$$

$\frac{dx}{dt} = 2\cos t$, $\frac{dy}{dt} = -3\sin t$.

$$\text{Arc-length} = \int_0^\pi \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\cdot dt$$

$$= \int_0^\pi \sqrt{(2\cos t)^2 + (-3\sin t)^2}\, dt$$

$$= \int_0^\pi \sqrt{4\cos^2 t + 9\sin^2 t}\, dt.$$

Q18[3 points] Which trig-substitution can be used to evaluate

$$\int \frac{dx}{x^2\sqrt{4x^2 - 9}}$$

A $x = \tan\theta$

B $x = 4\sec^2\theta - 9$

C $x = \frac{3}{2}\sin\theta$

D $x = \frac{3}{2}\sec\theta$

E $x = \frac{2}{3}\sec\theta$

$\sec^2\theta - 1 = \tan^2\theta$

$4x^2 - 9$

$(2x)^2 - 3^2$

$2x = 3\sec\theta$

$x = \frac{3}{2}\sec\theta.$

Q19[3 points] Let

$$\tan x = \sum_{n=0}^{\infty} c_n (x - \frac{\pi}{4})^n$$

be the Taylor expansion for $\tan x$ centered at $x = \frac{\pi}{4}$. Then c_1 is

A 0

B 1

C $\sqrt{2}$

D 2

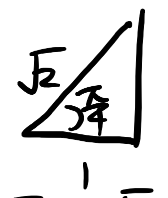
E $1/\sqrt{2}$

$$f(x) = \underbrace{f(a)}_{C_0} + \underbrace{f'(a)}_{C_1}\cdot(x-a) + \underbrace{\frac{f''(a)}{2}}_{C_2}\cdot(x-a)^2 + \cdots$$

$C_1 = (\tan x)'$ evaluated at $a = \frac{\pi}{4}$.

$= \sec^2 x.$

$C_1 = \sec^2(\frac{\pi}{4}) = (\sqrt{2})^2 = 2$



$\sec\frac{\pi}{4} = \frac{\sqrt{2}}{1}$

$\tan\frac{\pi}{4} = 1$

Q20[3 points] Evaluate

$$\int_0^{\pi/4} \tan t \ \sec^2 t \ dt$$

$u = \tan t \quad du = \sec^2 t$

$\int u \cdot du$

$= \frac{1}{2} u^2 = \frac{1}{2}\tan^2 t \Big|_0^{\frac{\pi}{4}} = \frac{1}{2}\cdot\tan^2\frac{\pi}{4} - \frac{1}{2}\tan^2 0$

$= \underline{\frac{1}{2}\cdot 1^2} - \frac{1}{2}\cdot 0$

A $\frac{1}{2}$

B 1

C $\pi/4$

D $\tan^2(\pi/4)$

E $\sec(\pi/4)$

Q21[3 points] Find the derivative of

$$\sin^{-1}(5^x)$$

$(\sin^{-1}(5^x))' = \frac{1}{\sqrt{1-(5^x)^2}} \cdot (5^x)'$

$= \frac{1}{\sqrt{1-5^{2x}}} \cdot 5^x \cdot \ln 5.$

A $\cos 5^x$

B $\frac{1}{\sqrt{1-5^{2x}}}$

C $\frac{5^x \ln 5}{\sqrt{1-5^{2x}}}$

D $\frac{5^x \ln 5}{\sqrt{1-5^x}}$

E $\frac{5^x}{\sqrt{1-5^{2x}}}$