Integrals

- **Volume**: Suppose $A(x)$ is the cross-sectional area of the solid S perpendicular to the x-axis, then the volume of S is given by
$$V = \int_a^b A(x)\ dx$$
- **Work**: Suppose $f(x)$ is a force function. The work in moving an object from a to b is given by:
$$W = \int_a^b f(x)\ dx$$
- $\displaystyle\int \frac{1}{x}\ dx = \ln|x| + C$
- $\displaystyle\int \tan x\ dx = \ln|\sec x| + C$
- $\displaystyle\int \sec x\ dx = \ln|\sec x + \tan x| + C$
- $\displaystyle\int a^x\ dx = \frac{a^x}{\ln a} + C \qquad \text{for } a \neq 1$
- **Integration by Parts**:
$$\int u\ dv = uv - \int v\ du$$
- Arc Length Formula:
$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\ dx$$

Derivatives

- $\displaystyle\frac{d}{dx}(\sinh x) = \cosh x \qquad \frac{d}{dx}(\cosh x) = \sinh x$
- Inverse Trigonometric Functions:
$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\csc^{-1} x) = \frac{-1}{x\sqrt{x^2-1}}$$
$$\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$$
$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$$
- If f is a one-to-one differentiable function with inverse function f^{-1} and $f'(f^{-1}(a)) \neq 0$, then the inverse function is differentiable at a and
$$(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}$$

Hyperbolic and Trig Identities

- Hyperbolic Functions
$$\sinh(x) = \frac{e^x - e^{-x}}{2} \qquad \operatorname{csch}(x) = \frac{1}{\sinh x}$$
$$\cosh(x) = \frac{e^x + e^{-x}}{2} \qquad \operatorname{sech}(x) = \frac{1}{\cosh x}$$
$$\tanh(x) = \frac{\sinh x}{\cosh x} \qquad \coth(x) = \frac{\cosh x}{\sinh x}$$
- $\cosh^2 x - \sinh^2 x = 1$
- $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$
- $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$
- $\sin(2x) = 2\sin x \cos x$
- $\sin A \cos B = \frac{1}{2}[\sin(A-B) + \sin(A+B)]$
- $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$
- $\cos A \cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$

Parametric

- $\displaystyle\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \qquad \text{if} \quad \frac{dx}{dt} \neq 0$
- Arc Length: $\displaystyle L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$

Polar

- $x = r\cos\theta \qquad y = r\sin\theta$
- $r^2 = x^2 + y^2 \qquad \tan\theta = \dfrac{y}{x}$
- Area: $\displaystyle A = \int_a^b \frac{1}{2} r(\theta)^2\ d\theta$
- $\displaystyle L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\ d\theta$

Series

- **nth term test for divergence:** If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

- **The p-series:** $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$.

- **Geometric:** If $|r| < 1$ then $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$

- **The Integral Test:** Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then

  (i) If $\int_1^{\infty} f(x)\, dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.

  (ii) If $\int_1^{\infty} f(x)\, dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

- **The Comparison Test:** Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

  (i) If $\sum b_n$ is convergent and $a_n \leq b_n$ for all n, then $\sum a_n$ is also convergent.

  (ii) If $\sum b_n$ is divergent and $a_n \geq b_n$ for all n, then $\sum a_n$ is also divergent.

- **The Limit Comparison Test:** Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

  where c is a finite number and $c > 0$, then either both series converge or both diverge.

- **Alternating Series Test:** If the alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$ satisfies

  (i) $0 < b_{n+1} \leq b_n$ for all n

  (ii) $\lim_{n\to\infty} b_n = 0$

  then the series is convergent.

- **The Ratio Test**

  (i) If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

  (ii) If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L > 1$ or $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

  (iii) If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$, the Ratio Test is inconclusive.

- **Maclaurin Series**: $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}\, x^n$

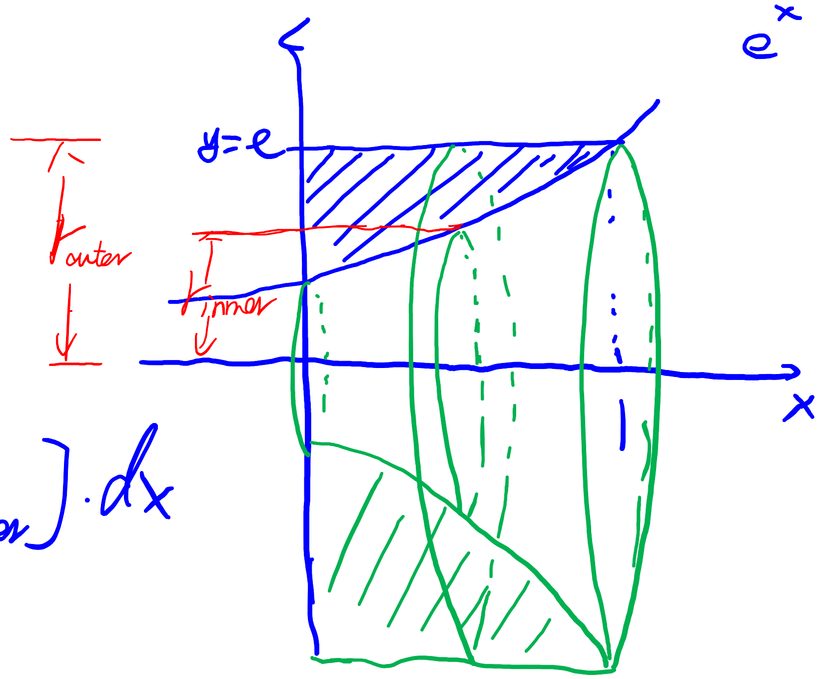
- **Taylor's Inequality** If $|f^{(n+1)}(x)| \leq M$ for $|x - a| \leq d$, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality

$$|R_n(x)| \leq \frac{M}{(n+1)!} |x-a|^{n+1} \quad \text{for } |x-a| \leq d$$

- Some Power Series
  - $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ $\qquad R = \infty$
  - $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ $\qquad R = \infty$
  - $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$ $\qquad R = \infty$
  - $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$ $\qquad R = 1$
  - $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ $\qquad R = 1$

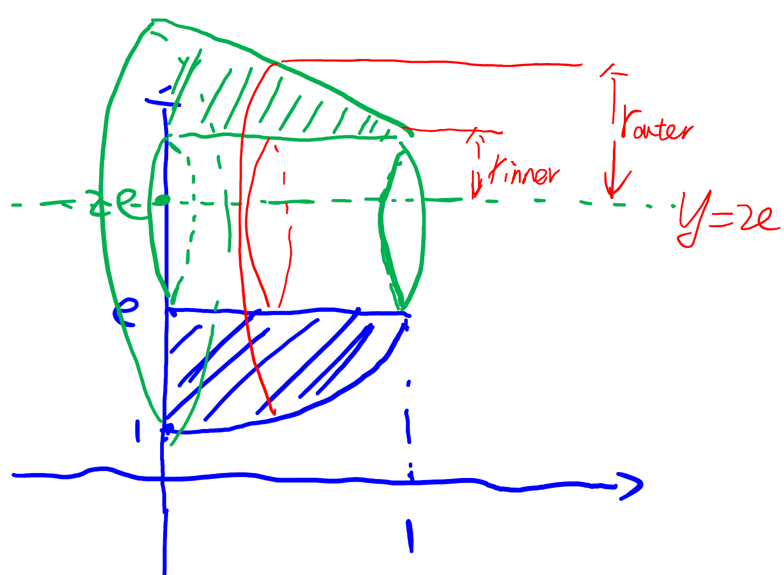
Q1[*Sec5.2*, rotating solid, Washer] Sketch the region R bounded by $y = e^x, y = e,\ x = 0$.

(a) Set up an integral for the volume of the solid rotating R about the $x-$axis. Do not evaluate the integral.

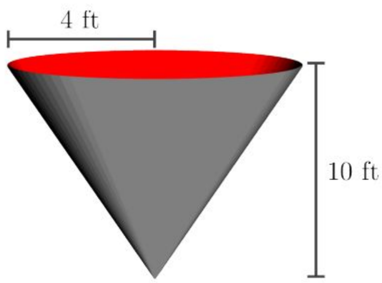


$$V = \int_0^1 \pi \cdot \left[r_{outer}^2 - r_{inner}^2 \right] \cdot dx$$

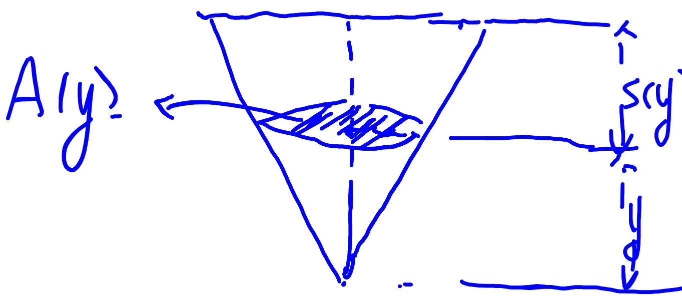
(b) Set up an integral for the volume of the solid rotating R about the axis $y = 2e$. Do not evaluate the integral.



Q2[*Sec5.4, Water-Pumping*] A conical water tank with a top diameter of 8 feet and height of 10 feet is standing at ground level as shown in the sketch below. Water weighing 60 pounds per cubic foot is pumped from the tank to an outlet 3 feet above the top of the tank. If the tank is full, how many foot-pounds of work are required to pump all of the water from the tank?



$$V = \int_a^b \delta \cdot s(y) \cdot A(y)\, dy$$



Q3[*Sec6.1, derivative formula for inverse functions*] Let $f(x) = 3^x - 3x, x > 0$. Find $(f^{-1})'(0)$. Notice that $f(1) = 0$.

$$(f^{-1})'(a) = \frac{1}{f'[f^{-1}(a)]}$$

$f(1) = 0 \Rightarrow f^{-1}(0) = 1$

$(a^x)' = \ln a \cdot a^x$

Q4[*Sec6.2-6.4, exp/log functions*] Find the derivative of the following functions.

(a)

$$f(x) = \ln(\sinh x)$$

$$(\ln x)' = \frac{1}{x}$$

$$(\sinh x)' = \cosh x.$$

(b)

$$f(x) = \sec(\arctan x)$$

$$(\sec x)' = \tan x \cdot \sec x$$

$$(\arctan x)' = \frac{1}{1+x^2}$$

Q5[*Sec6.5/9.3, initial value problem*] Find $y = y(x)$ if

$$\sec x \, \frac{\mathrm{d}y}{\mathrm{d}x} - \sqrt{y} = 0, \;\; y(0) = 4$$

$$\sec x \cdot \frac{dy}{dx} = \sqrt{y}$$

$$\sec x = \frac{1}{\cos x} \Rightarrow \frac{1}{\sec x} = \cos x$$

$$\frac{1}{\sqrt{y}} \cdot dy = \frac{1}{\sec x} dx$$

$$\int \frac{1}{\sqrt{y}} dy = \int \cos x \cdot dx$$

Q6[*Sec6.8/10.1, L'Hospital's Rule*] Evaluate the following limits.

(a)[∞^0**-type**]

$$\lim_{n\to\infty} \sqrt[n]{n^2+1}$$

$$(n^2+1)^{\frac{1}{n}} = e^{\boxed{\ln (n^2+1)^{\frac{1}{n}}.}}$$

Compute the limit $\frac{\ln(n^2+1)}{n}$ by l'H.

(b)[$\infty \cdot 0$**-type**]

$$\lim_{n\to\infty} n^2 \left(\cos\frac{1}{n} - 1\right) = \frac{\cos\left(\frac{1}{n}\right) - 1}{\frac{1}{n^2}}$$

$$\frac{\cos 0 - 1}{0} = \frac{0}{0}$$

Q7[*Sec7.1-7.2, integration by parts and trig-integral*] Evaluate the following integrals.

(a) ***[Sec7.1,IBP for polynomial***$\times \sin / \cos / exp$***-type]***

$$\int (x+1)\sin x \; \mathrm{d}x$$

$u = x+1$, $dv = \sin x \, dx$.

$du = dx$, $v = -\cos x$

I.B.P. $\int u \cdot dv = u \cdot v - \int v \cdot du$.

(b) ***[Sec7.2, Odd/Even rule for*** sin-cos-***type]***

$$\int_0^{\pi/6} (2+\cos\theta)^2 \; \mathrm{d}\theta$$

D.A.F.

$\cos^2\theta = \frac{1+\cos 2\theta}{2}$.

$= \int_0^{\frac{\pi}{6}} 4 + 4 \cdot \cos\theta + \cos^2\theta \, . \, d\theta$

$\int \cos 2\theta \, d\theta = \frac{1}{2} \sin 2\theta$.

(c) ***[Sec7.2,*** tan-sec ***type]***

$$\int \sec^2(2x)\tan(2x) \; \mathrm{d}x$$

$u = \tan(2x) \Rightarrow du = 2 \cdot \sec^2(2x) dx$.

u-sub.

Q8[*Sec7.3-7.4, trig-sub and partial fraction decomposition*] Evaluate the following integrals.

(a) ***[Trig-sub]***

$$\int \sqrt{4 - x^2}\, \mathrm{d}x$$

$$\sqrt{a^2 - b^2x^2} \xlongequal{bx = a\sin\theta} a\cdot\sqrt{1-\sin^2\theta} = a\cdot\cos\theta$$

$$= \int \sqrt{4 - 4\sin^2\theta} \cdot 2\cdot\cos\theta\, d\theta$$

$$= \int \sqrt{4\cdot\cos^2\theta} \cdot 2\cos\theta \cdot d\theta$$

(c) ***[Sec7.4, P.F.D. linear product type]***

$$\int \frac{2}{t^2 - 1}\, \mathrm{d}t$$

$$\frac{2}{t^2-1} = \frac{2}{(t+1)(t-1)} = \frac{A}{t+1} + \frac{B}{t-1}$$

Q9[*Sec7.8, improper integral*] Determine whether each of the improper integral is convergent or divergent. Evaluate the improper integral if it is convergent.

(a)

$$\int_0^{\infty} \frac{2}{1+9x^2}\,\mathrm{d}x$$

$$\int \frac{1}{1+\square^2}\,d\square = \arctan\square$$

$$1+9x^2 = 1+(3x)^2 \qquad u=3x.$$

(b)

$$\int_0^{1/2} \frac{1}{\sqrt{1-2x}}\,\mathrm{d}x$$

$$1-2x=0 \quad x=\frac{1}{2} \quad , \quad u=1-2x$$

$$=\lim_{t\to\frac{1}{2}^-} \int_0^t \frac{1}{\sqrt{1-2x}}\,dx.$$

Q10[*Sec11.2-11.6*] Determine whether each of the series is convergent or divergent. Please show your work and name any test(s) that are used.

(a)***[Sec11.2,n-th term test for DIV]***

$$\sum_{n=1}^{\infty} \cos\left(\frac{1}{3n^2}\right)$$

$$\lim \cos\left(\frac{1}{3n^2}\right) = \cos 0 = 1 \neq 0$$

(b)***[Sec11.4,(limit) Comparison Test]***

$$\sum_{n=1}^{\infty} \frac{\sqrt{n}+1}{n^2+1} = a_n, \qquad b_n = \frac{\sqrt{n}}{n^2} = \frac{1}{n^{1.5}}.$$

(c)***[Sec11.6, Ration Test]***

$$\sum_{n=1}^{\infty} \frac{n}{2^n}$$

$$\frac{a_{n+1}}{a_n} = \frac{\frac{n+1}{2^{n+1}}}{\frac{n}{2^n}} = \frac{n+1}{n} \cdot \frac{1}{2}.$$

Q11[*Sec11.6,11.8, ratio test for the radius of convergence of power series*] Consider the following power series. Find its center and radius of convergence.

$x-(-5)$ center

$$\sum_{n=1}^{\infty} \frac{3^n(x+5)^n}{n!}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{3^{n+1}\cdot(x+5)^{n+1}}{(n+1)!}}{\frac{3^n\cdot(x+5)^n}{n!}}\right|$$

Q12[*Sec11.9/11.10, power series representation*] Let

$$f(x) = \frac{2+x}{1+x^2}.$$

Find the **third degree** Maclaurin polynomial of $f(x)$.

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)}$$

$$(2+x)\cdot\sum_{n=0}^{\infty}(-x^2)^n$$

$$(2+x)(1-x^2+x^4-\cdots)$$

$$= 2-x^2+x^4+\cdots$$

$$+x-x^3+x^5+\cdots$$

$$= \boxed{2+x-x^2-x^3}+\cdots$$

Q12, Taylor series Let $f(x) = (1+x^2)\sin x$.

(a) Find the 2nd degree Taylor polynomial of $f(x)$ at $x = \frac{\pi}{2}$.

$$f\left(\tfrac{\pi}{2}\right) = \left[1+\left(\tfrac{\pi}{2}\right)^2\right]\cdot \sin\tfrac{\pi}{2} = 1+\frac{\pi^2}{4}$$

$$f'(x) = 2x\cdot\sin x + (1+x^2)\cdot\cos x. \qquad f'\left(\tfrac{\pi}{2}\right) = 2\cdot\tfrac{\pi}{2}\cdot\sin\tfrac{\pi}{2} + 0$$

$$= \pi.$$

$$f''(x) = 2\sin x + 2x\cdot\cos x + 2x\cdot\cos x + (1+x^2)(-\sin x)$$

$$f''\left(\tfrac{\pi}{2}\right) = 2\cdot 1 + 2\cdot\tfrac{\pi}{2}\cdot 0 + 2\cdot\tfrac{\pi}{2}\cdot 0 + \left(1+\tfrac{\pi^2}{4}\right)\cdot(-1) = 1-\frac{\pi^2}{4}.$$

$$T_2(x) = f\left(\tfrac{\pi}{2}\right) + f'\left(\tfrac{\pi}{2}\right)\cdot\left(x-\tfrac{\pi}{2}\right) + \frac{f''\left(\frac{\pi}{2}\right)}{2}\left(x-\tfrac{\pi}{2}\right)^2$$

(b) Find the first four nonzero terms in the Maclaurin series of $f(x)$.

$$(1+x^2)\cdot\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right]$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + x^3 - \frac{x^5}{3!} + \frac{x^7}{5!} - \frac{x^9}{7!} + \cdots$$

$$= \boxed{x + \left(1-\frac{1}{3!}\right)\cdot x^3 + \left(\frac{1}{5!}-\frac{1}{3!}\right)x^5 + \left(\frac{1}{5!}-\frac{1}{7!}\right)x^7} + \cdots$$

Q13[***Sec10.1,10.2, Parametric curves/equations***] Consider the parametric equations given by

$$x(t) = \ln(t+1), \quad y(t) = \sqrt{t+3} \qquad \text{at}$$

(a) Find the equation of the tangent line to the parametric curve at $(x, y) = (0, \sqrt{3})$

$(x, y) = (0, \sqrt{3}) \quad 0 = \ln(t+1), \; \sqrt{3} = \sqrt{t+3} \Rightarrow t = 0.$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \longleftarrow \text{Plug in } t=0$$

Point-slope formula

(b) Set up an integral to find the arc length of the curve from $t = 0$ to $t = 1$. DO NOT EVALUATE THE INTEGRAL.

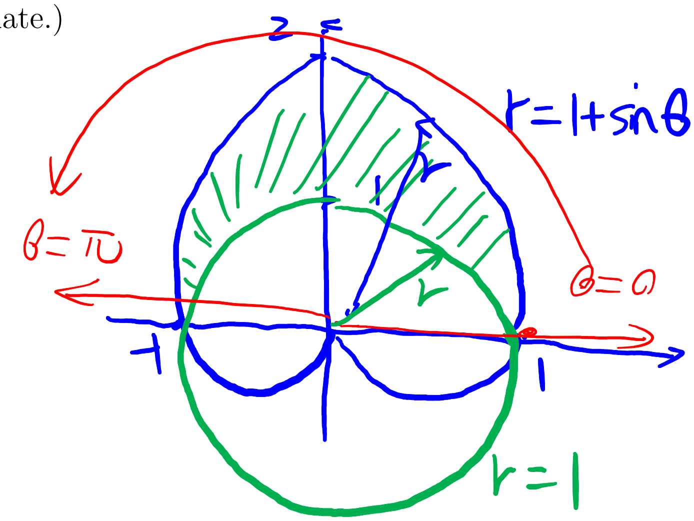
$$\text{Arc length} = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \cdot dt$$

Q14[***Sec10.3,10.4, polar curve***]

(a) Sketch the curves $r = 1 + \sin\theta$, and $r = 1$. Set up the integral for the area of the region inside $r = 1(1 + \sin\theta)$ and outside $r = 1$.(Do not evaluate.)

$r = 1 + \sin\theta = 1 \Rightarrow \sin\theta = 0$

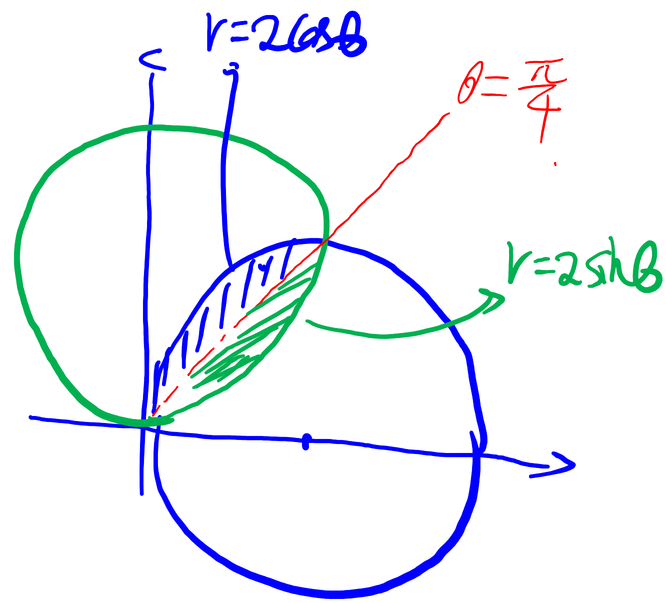
$\Rightarrow \theta = 0, \theta = \pi$



$$A = \int_0^{\frac{\pi}{2}} \frac{1}{2} \cdot r(\theta)^2 \cdot d\theta$$

$$A = \text{Area1} - \text{Area2}.$$

(b) Sketch the curves $r = 2\sin\theta$, and $r = 2\cos\theta$. Set up the integral for the area of the region shared by $r = 2\sin\theta$, and $r = 2\cos\theta$. (Do not evaluate.)



$$\text{Area} = \text{Area1} + \text{Area2}$$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \qquad \int_0^{\frac{\pi}{4}}$$