one ☆ : IMPORTANT . X : not required.

two ☆☆ : concepts/methods/formulas are required, but the problems will be easier.

Standard Response Questions. Show all work to receive credit. Please $\boxed{\textbf{BOX}}$ your final answer.

1. Evaluate the following integrals

(a) (6 points) $\int e^x(1+e^x)^3 dx$

u-sub: $u = 1+e^x$, $du = e^x dx$.

$= \int u^3 \cdot du = \frac{1}{4}u^4 + C = \frac{1}{4}(1+e^x)^4 + C.$

(b) (6 points) $\int \sin x \cos^2 x dx$

u-sub for trig-integral (odd rule)

$u = \cos x$, $du = -\sin x \cdot dx$

$= \int u^2 (-du)$

$= -\frac{1}{3}u^3 + C$

$= \boxed{-\frac{1}{3}\cos^3 x + C}$

☆ (c) (6 points) $\int x \sin 2x dx$

IBP: poly × sin. $u = x$. $dv = \sin 2x \, dx$ $v = \int \sin 2x \, dx$.

$du = dx$, $v = -\frac{1}{2}\cos 2x$.

$= x \cdot (-\frac{1}{2}\cos 2x) - \int -\frac{1}{2}\cos 2x \, dx$

$= -\frac{x}{2}\cos 2x + \frac{1}{2} \cdot \int \cos 2x \, dx$

$= \boxed{-\frac{x}{2}\cos 2x + \frac{1}{2} \cdot \frac{1}{2}\sin 2x + C}$

2. Evaluate the following integrals

(a) (9 points) $\int \frac{x^3}{\sqrt{x^2+4}}dx$ $\to x^2 = u-4$

u-sub: $u = x^2+4$, $du = 2\cdot x\cdot dx$.

$$= \int \frac{x^2 \cdot x \cdot dx}{\sqrt{x^2+4}}$$

$$= \int \frac{(u-4)}{\sqrt{u}} \cdot \frac{1}{2}du$$

$$= \int \left(\frac{u}{\sqrt{u}} - \frac{4}{\sqrt{u}}\right)\cdot\frac{1}{2}du$$

$$= \int \left(\sqrt{u} - \frac{4}{\sqrt{u}}\right)\frac{1}{2}du$$

$$= \left(\frac{2}{3}u^{\frac{3}{2}} - 4\cdot 2\cdot u^{\frac{1}{2}}\right)\cdot\frac{1}{2} + C = \frac{1}{3}u^{\frac{3}{2}} - 4u^{\frac{1}{2}} + C$$

$$= \boxed{\frac{1}{3}(x^2+4)^{\frac{3}{2}} - 4\cdot(x^2+4)^{\frac{1}{2}} + C}$$

(b) (9 points) $\int \frac{3\,dx}{x^2+x-2}$

P.F.D. $\frac{3}{x^2+x-2} = \frac{3}{(x+2)(x-1)} = \frac{A}{x+2} + \frac{B}{x-1}$. times $(x+2)\cdot(x-1)$ both sides,

$3 = A\cdot(x-1) + B\cdot(x+2)$

$x=-2$ $3 = A\cdot(-3) + B\cdot 0$. $\Rightarrow A = -1$

$x=1$ $3 = A\cdot 0 + B\cdot 3$ $\Rightarrow B = 1$

$$\int \frac{3\,dx}{x^2+x-2} = \int \frac{-1}{x+2} + \frac{1}{x-1}dx = \boxed{-\ln|x+2| + \ln|x-1| + C}$$

3. Evaluate the following limits

(a) (9 points) $\lim_{x\to+\infty} \frac{(\ln x)^2}{\sqrt[3]{x}}$ $\quad \frac{\infty}{\infty}$ L'Hospital

$$\overset{\text{l'Hosp}}{=} \lim_{x\to+\infty} \frac{[(\ln x)^2]'}{[x^{\frac{1}{3}}]'}$$

$$= \lim_{x\to+\infty} \frac{2\ln x \cdot \frac{1}{x}}{\frac{1}{3}\cdot x^{-\frac{2}{3}}}$$

(simplify) $= \lim_{x\to+\infty} 6 \cdot \frac{\ln x}{x^{-\frac{2}{3}}\cdot x}$

$\frac{\infty}{\infty}$. L'Hosp $= \lim_{x\to+\infty} 6 \cdot \frac{\ln x}{x^{\frac{1}{3}}}$

$$= \lim_{x\to+\infty} 6 \frac{\frac{1}{x}}{\frac{1}{3}\cdot x^{-\frac{2}{3}}} = \lim_{x\to+\infty} 18 \cdot \frac{1}{x^{\frac{1}{3}}} = \boxed{0}$$

(b) (9 points) $\lim_{x\to+\infty} 5\left(1-\frac{1}{x}\right)^{5x}$ exp-log, l'Hospital. 1^{∞}

$$= \lim_{x\to+\infty} 5 \cdot e^{\boxed{\ln(1-\frac{1}{x})^{5x}}}$$

$$\lim_{x\to+\infty} \ln(1-\frac{1}{x})^{5x} = \lim_{x\to+\infty} 5x \cdot \ln(1-\frac{1}{x}) \quad , \infty\cdot 0\text{-type}$$

$$= \lim_{x\to+\infty} \frac{\ln(1-\frac{1}{x})}{\frac{1}{5x}} \quad \frac{0}{0}\text{-type}.$$

$$\overset{\text{l'Hospital}}{=} \lim_{x\to+\infty} \frac{\frac{1}{1-\frac{1}{x}}\cdot(-\,-\cancel{\frac{1}{x^2}})}{\frac{-1}{5\cancel{x^2}}} \overset{\text{simplify}}{=} \lim_{x\to+\infty} \frac{\frac{1}{1-\frac{1}{x}}}{\frac{-1}{5}} = \frac{\frac{1}{1-0}}{\frac{-1}{5}} = -5$$

$$\Rightarrow \lim_{x\to+\infty} 5(1-\frac{1}{x})^{5x} = \lim_{x\to+\infty} 5\cdot e^{\boxed{\ln(1-\frac{1}{x})^{5x}}} = \boxed{5\cdot e^{-5}}$$

4. (18 points) Find the Maclaurin series of the function $f(x) = e^{-x^4}$

Apply the formula $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ directly.

$$e^{\square} = \sum_{n=0}^{\infty} \frac{\square^n}{n!} \quad \text{with } \square = -x^4$$

$$e^{-x^4} = \sum_{n=0}^{\infty} \frac{(-x^4)^n}{n!}$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \cdot x^{4n}}$$

★★. (18 points) Show that the series $\sum_{n=1}^{\infty} \frac{(-1)^n (n!)^2 3^n}{(2n+1)!}$ is convergent by using the Ratio Test.

$$a_n = \frac{(-1)^n \cdot (n!)^2 \cdot 3^n}{(2n+1)!}, \quad a_{n+1} = \frac{(-1)^{n+1} \cdot \left((n+1)!\right)^2 \cdot 3^{n+1}}{\left(2(n+1)+1\right)!}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{[(n+1)!]^2 \cdot 3^{n+1}}{(2n+3)!} \cdot \frac{(2n+1)!}{[n!]^2 \cdot 3^n}$$

$$= \frac{[(n+1)!]^2}{[n!]^2} \cdot \frac{3^{n+1}}{3^n} \cdot \frac{(2n+1)!}{(2n+3)!}$$

$$= \left[\frac{(n+1)!}{n!}\right]^2 \cdot 3 \cdot \frac{1 \times 2 \times 3 \times \cdots \times (2n) \times (2n+1)}{1 \times 2 \times 3 \times \cdots \times (2n) \times (2n+1) \times (2n+2) \times (2n+3)}$$

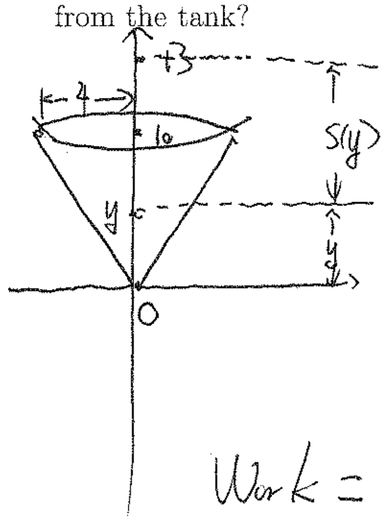
$$= (n+1)^2 \cdot 3 \cdot \frac{1}{(2n+2)\cdot(2n+3)} = \frac{3(n+1)^2}{(2n+2)\cdot(2n+3)}.$$

$$= \frac{3(n+1)^2}{2(n+1)(2n+3)} = \frac{3(n+1)}{2(2n+3)}$$

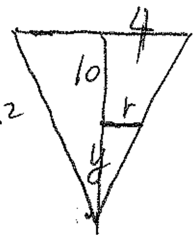
$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{3(n+1)}{2(2n+3)} = \boxed{\frac{3}{4} < 1.}$$

According to Ratio Test, $\sum a_n$ is convergent.

☆ (18 points) A conical water tank with a top radius of 4 feet and height of 10 feet is standing at ground level. Water weighing 60 pounds per cubic foot is pumped from the tank to an outlet 3 feet above the top of the tank. If the tank is full, how many foot-pounds of work are required to pump all of the water from the tank?



$\Longrightarrow A(y)=\pi\cdot r^2$



$\frac{r}{y}=\frac{4}{10} \Rightarrow r=\frac{2}{5}y$

$A(y)=\pi\left(\frac{2}{5}y\right)^2$

$$\text{Work} = \int_0^{10} 6\cdot s(y)\cdot A(y)\,dy$$

$$= \int_0^{10} 60\cdot(13-y)\cdot\pi\cdot\left(\frac{2}{5}y\right)^2 dy$$

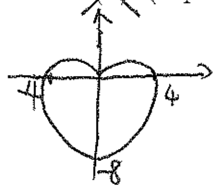
$$= 60\cdot\pi\cdot\frac{4}{25}\cdot\int_0^{10}(13-y)\cdot y^2\,dy$$

$$= \frac{48}{5}\pi\cdot\int_0^{10} 13y^2-y^3\,dy$$

$$= \frac{48}{5}\pi\cdot\left(13\cdot\frac{1}{3}y^3-\frac{1}{4}y^4\right)\Big|_0^{10}$$

$$= \boxed{\frac{48}{5}\pi\cdot\left(\frac{13}{3}\cdot 10^3-\frac{1}{4}\cdot 10^4\right)}$$

7. ~~(a)~~ (9 points) Graph the curve $r = 4(1-\sin\theta)$ and ~~determine its arc length~~.



$$\text{Arclength} = \int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{16(1-\sin\theta)^2 + (-4\cos\theta)^2}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{16 + 16\sin^2\theta + 16\cos^2\theta - 32\sin\theta}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{32(1-\sin\theta)}\, d\theta$$

$$= \sqrt{32}\int_0^{2\pi} \sqrt{\left(\sin\frac{\theta}{2} - \cos\frac{\theta}{2}\right)^2}\, d\theta$$

$$= \sqrt{32}\int_0^{2\pi} \left|\sin\frac{\theta}{2} - \cos\frac{\theta}{2}\right| d\theta$$

$$= \sqrt{32}\left[\int_0^{\frac{\pi}{2}} \left(\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\right) d\theta + \int_{\frac{\pi}{2}}^{2\pi} \left(\sin\frac{\theta}{2} - \cos\frac{\theta}{2}\right) d\theta\right]$$

$$1-\sin\theta$$
$$= \sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2} - 2\sin\frac{\theta}{2}\cdot\cos\frac{\theta}{2}$$
$$= \left(\sin\frac{\theta}{2} - \cos\frac{\theta}{2}\right)^2$$

$$= \sqrt{32}\left[\left(+2\sin\frac{\theta}{2} + 2\cos\frac{\theta}{2}\right)\Bigg|_0^{\frac{\pi}{2}} + \left(-2\cos\frac{\theta}{2} - 2\sin\frac{\theta}{2}\right)\Bigg|_{\frac{\pi}{2}}^{2\pi}\right]$$

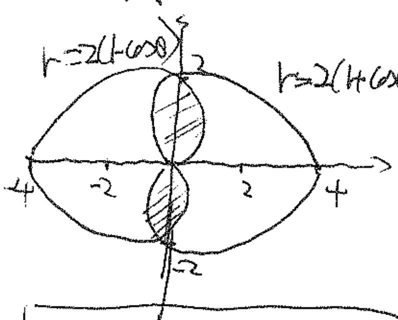
$$= \sqrt{32}\left((\sqrt{2}+\sqrt{2}) - (0+2) + (+2+0) - (-\sqrt{2}-\sqrt{2})\right)$$

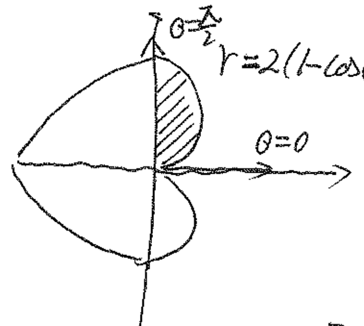
$$= \sqrt{32}\left(2\sqrt{2} - 2 + 2 + 2\sqrt{2}\right)$$

$$= \sqrt{32}\cdot 4\sqrt{2}$$

$$= \boxed{32}$$

(9 points) Sketch and find the area shared by the cardioids $r = 2(1+\cos\theta)$ and $r = 2(1-\cos\theta)$





$r = 2(1-\cos\theta)$ from $\theta = 0$ to $\theta = \frac{\pi}{2}$

$$\text{Area} = \int_0^{\frac{\pi}{2}} \frac{1}{2} r^2\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{2}\left[2(1-\cos\theta)\right]^2 d\theta$$

$$= \int_0^{\frac{\pi}{2}} 2(1-\cos\theta)^2\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} 2(1 - 2\cos\theta + \cos^2\theta)\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} 2 - 4\cos\theta + 2\cdot\frac{1+\cos 2\theta}{2}\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} 3 - 4\cos\theta + \cos 2\theta\, d\theta$$

$$= 3\theta - 4\sin\theta + \frac{1}{2}\sin 2\theta\Big|_0^{\frac{\pi}{2}}$$

$$= 3\cdot\frac{\pi}{2} - 4\cdot 1 + 0 - 0$$

$$= \frac{3}{2}\pi - 4$$

$$\boxed{\text{Area} = 4\times\left[\frac{3}{2}\pi - 4\right]}$$

Multiple Choice. Circle the best answer. No work needed. No partial credit available.

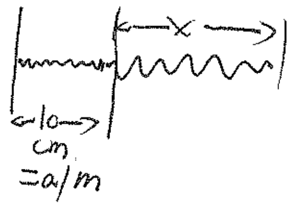
8. (7 points) A spring has a natural length of 10 cm. If a 25-N force is required to keep it stretched to a length of 20 cm, how much work is required to stretch it from 10 cm to 15 cm

A. 0.3125 J
B. 0.4135 J
C. 25 J
D. 0 J
E. 41 J

$F(x) = k \cdot x$.

$25 = k \cdot \left(\frac{20-10}{100}\right) = k \cdot 0.1$

$k = 250$

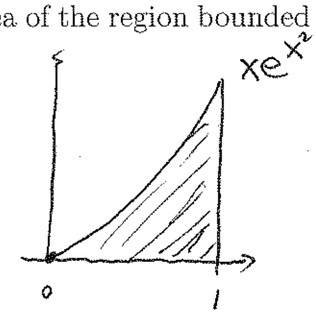


$$W = \int_0^{0.05} F(x)\,dx = \int_0^{0.05} 250 \cdot x\,dx = 250 \tfrac{1}{2} x^2 \Big|_0^{0.05}$$

$$= 250 \cdot \tfrac{1}{2} \cdot (0.05)^2 = 0.3125 \text{ J}$$

9. (7 points) Find the area of the region bounded above by the graph of $y = xe^{x^2}$ and below by the x-axis, with $0 \leq x \leq 1$

A. $\frac{1}{2}e$
B. $\frac{1}{2}(e-1)$
C. 1
D. $e-1$
E. $2(e-1)$



$$\text{Area} = \int_0^1 x \cdot e^{x^2}\,dx$$

u-sub

$u = x^2$, $du = 2x\,dx$

$$= \int e^u \cdot \tfrac{1}{2}\,du$$

$$= \tfrac{1}{2} e^u = \tfrac{1}{2} e^{x^2} \Big|_0^1$$

$$= \tfrac{1}{2} e^1 - \tfrac{1}{2} \cdot e^0 = \tfrac{1}{2}(e-1)$$

10. (7 points) Consider the series $\sum_{n=1}^{\infty} \frac{n+1}{n^2}$. Which of the following tests gives correct determination

A. The alternating series test
B. Comparison Test with the term of the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$
C. Comparison Test with the term of the series $\sum_{n=1}^{\infty} \frac{1}{n}$
D. n^{th} Term Test to show it diverges
E. None of the above.

Direct Comparison.

$$\frac{n+1}{n^2} > \frac{n}{n^2} = \frac{1}{n}.$$

$$\sum \frac{n+1}{n^2} \geq \sum \frac{1}{n}$$

DIV

☆14. (7 points) Find the solution of the initial value problem $y' = -y^2, \quad y(0) = 1/2$

A. $y = \frac{1}{2}e^{-2t}$

B. $y = \frac{1}{2}\ln(1+t)$

C. $y = \frac{1}{2} \cdot \frac{1}{t+1}$

D. $y = \frac{1}{t+2}$ ✓

E. None of the above.

$\frac{dy}{dx} = -y^2$

$\frac{1}{y^2}dy = -dx$.

$\int \frac{1}{y^2}dy = \int -dx$.

$-\frac{1}{y} = -x + C$. $\quad -\frac{1}{y} = -x - 2$

$y(0) = \frac{1}{2} \Rightarrow \begin{cases} x = 0 \\ y = \frac{1}{2} \end{cases} \quad -2 = 0 + C \Rightarrow C = -2 \quad \frac{1}{y} = x + 2$

$y = \frac{1}{x+2}$.

☆☆15. (7 points) Evaluate the improper integral: $\int_{-2}^{1} \frac{dx}{(x)^{4/3}}$

A. inf ✓

B. $-3(1 + \frac{1}{2^{1/3}})$

C. $-3(1 + 2^{1/3})$

D. $-$ inf

E. None of the above.

$= \int_{-2}^{0} \frac{dx}{x^{4/3}} + \int_{0}^{1} \frac{dx}{x^{4/3}}$

If one of them is divergent, then the sum is divergent.

$\int_0^1 \frac{dx}{x^{4/3}} = \lim_{t \to 0^+} \int_t^1 x^{-\frac{4}{3}} dx$

$= \lim_{t \to 0^+} \frac{1}{-\frac{1}{3}} \cdot x^{-\frac{1}{3}} \Big|_t^1 = \lim_{t \to 0^+} -3 + 3t^{-\frac{1}{3}} = +\infty$ (+inf)

$\int_{-2} \frac{dx}{x^{4/3}} = \lim_{t \to 0^-} \int_{-2}^{t} x^{-\frac{4}{3}} dx = \lim_{t \to 0^-} -3 \cdot x^{-\frac{1}{3}} \Big|_{-2}^{t} = \lim_{t \to 0^-} -3 \cdot t^{-\frac{1}{3}} + 3(-2)^{-\frac{1}{3}} = (-3)(-\infty) = +\infty$

☆☆16. (7 points) Consider the series $\sum_{n=1}^{\infty} \frac{\cos(n)}{n^3}$. Which of the following tests gives correct determination

A. By the Alternating Series Test, the series converges.

B. By the Alternating Series Test, the series diverges.

C. The series converges and also absolutely converges. ✓

D. The series converges but does not converge absolutely.

E. None of the above.

$\left| \frac{\cos n}{n^3} \right| \leq \frac{1}{n^3}$

$\sum \frac{1}{n^3}$ is convergent. ($p = 3 > 1$)

$\Rightarrow \sum \left| \frac{\cos n}{n^3} \right|$ is convergent (comparison test)

$\Rightarrow \sum \frac{\cos n}{n^3}$ ABS convergent.

11. (7 points) What is the Cartesian equation for the curve given as

$$x = -3\cancel{\cosh}(4t), \qquad y = 3\cancel{\sinh}(4t), \qquad t \in (-\infty, \infty)$$

A. $x^2 + y^2 = 9$
✓B. $x^2 - y^2 = 9$
C. $y^2 - x^2 = 9$
D. $y^2 - x^2 = 3$
E. None of the above.

$\cosh^2\square - \sinh^2\square = 1 \Rightarrow \cosh^2(4t) - \sinh^2(4t) = 1.$

$\Rightarrow \left(\frac{x}{-3}\right)^2 - \left(\frac{y}{3}\right)^2 = 1.$

$\Rightarrow \frac{x^2}{9} - \frac{y^2}{9} = 1$

$\Rightarrow x^2 - y^2 = 9$

12. (7 points) What is the coefficient next to the x term of the ~~binomial~~ series for the function:
(Maclaurin)

$$f(x) = \left(1 - \frac{x}{3}\right)^{-3}$$

✓A. 1
B. -1
C. 3
D. -3
E. None of the above.

$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \cdot x^n$

$= f(0) + \boxed{f'(0) \cdot x} + \cdots$

$f'(x) = (-3)\cdot\left(1-\frac{x}{3}\right)^{-4}\cdot\left(-\frac{1}{3}\right)$

$= \left(1-\frac{x}{3}\right)^{-4}$

$f'(0) = 1^{-4} = 1.$

13. (7 points) Given the function $f(x) = x^3 + x^2 + x$ on the domain $x \geq 0$, find the value of $\frac{d(f^{-1})}{dx}$ at the point $x = 3 = f(1)$.

A. $\frac{1}{16}$
✓B. $\frac{1}{6}$
C. $\frac{-6}{9}$
D. $\frac{-16}{39^2}$
E. None of the above.

$\frac{d(f^{-1})}{dx} = (f^{-1})'(3) = \frac{1}{f'[f^{-1}(3)]}$

$= \frac{1}{6}.$

$3 = f(1) \Rightarrow f^{-1}(3) = 1.$

$f'(x) = 3x^2 + 2x + 1.$

$f'[f^{-1}(3)] = f'(1) = 3 + 2 + 1 = 6$

17. (7 points) Find the interval of convergence of the series $\sum_{n=1}^{\infty}(-2)^{n+1}(1+x)^n$.

A. 1

B. $\frac{1}{2}$

C. $(-\frac{3}{2}, -\frac{1}{2})$

D. $(\frac{1}{2}, \frac{3}{2})$

E. None of the above.

Center : $1+x=0 \Rightarrow \boxed{x=-1}$

radius of Conv: $a_n=(-2)^{n+1}\cdot(1+x)^n$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-2)^{n+2}\cdot(1+x)^{n+1}}{(-2)^{n+1}\cdot(1+x)^n}\right|$$

$$=\lim_{n\to\infty} 2\cdot|1+x| = 2|x+1| < 1.$$

$|x+1| < \boxed{\frac{1}{2}}$ ← R: radius

←$\frac{1}{2}$→ ←$\frac{1}{2}$→

$-\frac{3}{2}$ -1 $-\frac{1}{2}$

center

18. (7 points) If $g(x) = (1+x^2)^x$, find the derivative $g'(x)$.

A. $g'(x) = 2x(1+x^2)^x$

B. $g'(x) = 2x(x) + (1+x^2)$

C. $g'(x) = x(1+x^2)^{x-1}(2x)$

D. $g'(x) = (1+x^2)^x(\ln(1+x^2) + \frac{2x^2}{1+x^2})$

E. None of the above.

$$\ln g(x) = \ln(1+x^2)^x = x\cdot\ln(1+x^2)$$

$$\frac{g'(x)}{g(x)} = 1\cdot\ln(1+x^2) + x\cdot\frac{1}{1+x^2}\cdot 2x.$$

$$g'(x) = (1+x^2)^x\cdot\left[\ln(1+x^2) + \frac{2x^2}{1+x^2}\right]$$

19. (7 points) Find the Cartesian equation for the line tangent to the parametric curve $x(t) = 6\sin(\pi t)$, $y(t) = t^2 - \frac{1}{2}t + 1$ at the point $(0,1)$. $=(x,y)$. $\Rightarrow t=0$.

A. $y = -\frac{1}{12\pi}x + 1$

B. $y = -\frac{1}{6\pi}x + 1$

C. $y = -\frac{1}{12\pi}x - 1$

D. $y = \frac{3}{12\pi}x - 1$

E. None of the above.

$$\frac{dx(t)}{dt} = (6\sin\pi t)' = 6\cdot\cos\pi t\cdot\pi = 6\pi\cos 0 = 6\pi$$

$$\frac{dy(t)}{dt} = (t^2 - \tfrac{1}{2}t + 1)' = 2t - \frac{1}{2} = -\frac{1}{2}.$$

$$\frac{dy}{dx} = \frac{-\frac{1}{2}}{6\pi} = -\frac{1}{12\pi}$$ → slope, point $(x,y)=(0,1)$

$$y = 1 - \frac{1}{12\pi}(x-0) = -\frac{1}{12\pi}x + 1$$

FORMULA SHEET PAGE 1

Integrals

- **Volume**: Suppose $A(x)$ is the cross-sectional area of the solid S perpendicular to the x-axis, then the volume of S is given by
$$V = \int_a^b A(x)\,dx$$
- **Work**: Suppose $f(x)$ is a force function. The work in moving an object from a to b is given by:
$$W = \int_a^b f(x)\,dx$$
- $\displaystyle\int \frac{1}{x}\,dx = \ln|x| + C$
- $\displaystyle\int \tan x\,dx = \ln|\sec x| + C$
- $\displaystyle\int \sec x\,dx = \ln|\sec x + \tan x| + C$
- $\displaystyle\int a^x\,dx = \frac{a^x}{\ln a} + C \qquad$ for $a \neq 1$
- **Integration by Parts**:
$$\int u\,dv = uv - \int v\,du$$
- Arc Length Formula:
$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx$$

Derivatives

- $\displaystyle\frac{d}{dx}(\sinh x) = \cosh x \qquad \frac{d}{dx}(\cosh x) = \sinh x$
- Inverse Trigonometric Functions:
$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\csc^{-1} x) = \frac{-1}{x\sqrt{x^2-1}}$$
$$\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$$
$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$$
- If f is a one-to-one differentiable function with inverse function f^{-1} and $f'(f^{-1}(a)) \neq 0$, then the inverse function is differentiable at a and
$$(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}$$

Hyperbolic and Trig Identities

- Hyperbolic Functions
$$\sinh(x) = \frac{e^x - e^{-x}}{2} \qquad \operatorname{csch}(x) = \frac{1}{\sinh x}$$
$$\cosh(x) = \frac{e^x + e^{-x}}{2} \qquad \operatorname{sech}(x) = \frac{1}{\cosh x}$$
$$\tanh(x) = \frac{\sinh x}{\cosh x} \qquad \coth(x) = \frac{\cosh x}{\sinh x}$$
- $\cosh^2 x - \sinh^2 x = 1$
- $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$
- $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$
- $\sin(2x) = 2\sin x \cos x$
- $\sin A \cos B = \frac{1}{2}[\sin(A-B) + \sin(A+B)]$
- $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$
- $\cos A \cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$

Parametric

- $\displaystyle\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \qquad$ if $\displaystyle\frac{dx}{dt} \neq 0$
- Arc Length: $\displaystyle L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$

Polar

- $x = r\cos\theta \qquad y = r\sin\theta$
- $r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x}$
- Area: $\displaystyle A = \int_a^b \frac{1}{2} r(\theta)^2\,d\theta$
- $\displaystyle L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta$

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Series

- **nth term test for divergence:** If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

- **The p-series:** $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$.

- **Geometric:** If $|r| < 1$ then $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$

- **The Integral Test:** Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then
  - (i) If $\int_1^{\infty} f(x)\, dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.
  - (ii) If $\int_1^{\infty} f(x)\, dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

- **The Comparison Test:** Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.
  - (i) If $\sum b_n$ is convergent and $a_n \leq b_n$ for all n, then $\sum a_n$ is also convergent.
  - (ii) If $\sum b_n$ is divergent and $a_n \geq b_n$ for all n, then $\sum a_n$ is also divergent.

- **The Limit Comparison Test:** Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If
$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$
where c is a finite number and $c > 0$, then either both series converge or both diverge.

- **Alternating Series Test:** If the alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$ satisfies
  - (i) $0 < b_{n+1} \leq b_n$ for all n
  - (ii) $\lim_{n\to\infty} b_n = 0$

  then the series is convergent.

- **The Ratio Test**
  - (i) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.
  - (ii) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.
  - (iii) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, the Ratio Test is inconclusive.

- **Maclaurin Series:** $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$

- **Taylor's Inequality** If $|f^{(n+1)}(x)| \leq M$ for $|x - a| \leq d$, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality
$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1} \quad \text{for } |x-a| \leq d$$

- Some Power Series
  - $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ $\quad R = \infty$
  - $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ $\quad R = \infty$
  - $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$ $\quad R = \infty$
  - $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$ $\quad R = 1$
  - $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ $\quad R = 1$