Name: ______________________________

Section: ______________ **Recitation Instructor**: ______________________________

INSTRUCTIONS

- Fill in your name, etc. on this first page.
- Without fully opening the exam, check that you have pages 1 through 12.
- **Show all your work** on the standard response questions. Write your answers clearly! Include enough steps for the grader to be able to follow your work. Don't skip limits or equal signs, etc. Include words to clarify your reasoning.
- Do first all of the problems you know how to do immediately. Do not spend too much time on any particular problem. Return to difficult problems later.
- If you have any questions please raise your hand and a proctor will come to you.
- You will be given exactly 90 minutes for this exam.
- Remove and utilize the formula sheet provided to you at the end of this exam.
- This is a practice exam. The actual exam may differ significantly from this practice exam because there are many varieties of problems that can test each concept.

ACADEMIC HONESTY

- Do not open the exam booklet until you are instructed to do so.
- Do not seek or obtain any kind of help from anyone to answer questions on this exam. If you have questions, consult only the proctor(s).
- Books, notes, calculators, phones, or any other electronic devices are not allowed on the exam. Students should store them in their backpacks.
- No scratch paper is permitted. If you need more room use the back of a page.
- Anyone who violates these instructions will have committed an act of academic dishonesty. Penalties for academic dishonesty can be very severe. All cases of academic dishonesty will be reported immediately to the Dean of Undergraduate Studies and added to the student's academic record.

I have read and understand the above instructions and statements regarding academic honesty: ______________________________

SIGNATURE

Standard Response Questions. Show all work to receive credit. Please $\boxed{\textbf{BOX}}$ your final answer.

1. Determine whether the series is convergent, or divergent. State which test you used and all necessary conditions to use the test.

(a) (3 points) $\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n}$

$\cos\pi = -1$, $\cos 2\pi = 1$, $\cos 3\pi = -1$, $\cdots$, $\cos n\pi = (-1)^n$.

$= \sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{n}$, $b_n = \frac{1}{n}$ positive and decreasing, $\lim_{n\to\infty} \frac{1}{n} = 0$

Alternating Series Test implies $\sum \frac{\cos n\pi}{n}$ conv.

(b) (3 points) $\sum_{n=1}^{\infty} \frac{n^2+n+5}{\left(\frac{n^2}{2}\right)}$.

$\lim_{n\to\infty} \frac{n^2 \cancel{+n+5}}{\frac{n^2}{2}} = \lim_{n\to\infty} \frac{n^2}{\frac{n^2}{2}} = 2 \neq 0.$

nth term test for DIV

$\Rightarrow$ The series is divergent

(c) (4 points) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n+5^n}$

s/n 1: $b_n = \frac{1}{n+5^n}$ positive and decreasing, $\lim_{n\to\infty} \frac{1}{n+5^n} = 0$

A.S.T. $\Rightarrow$ conv.

s/n 2: $\left|\frac{(-1)^n}{n+5^n}\right| = \frac{1}{n+5^n} \leq \frac{1}{5^n}$, $\sum \frac{1}{5^n}$ conv $\Rightarrow \sum \left|\frac{(-1)^n}{n+5^n}\right|$ conv $\Rightarrow \sum \frac{(-1)^n}{n+5^n}$ conv.

comparison test.

(d) (4 points) $\sum_{n=1}^{\infty} \frac{9^n}{1+15^n}$

$\frac{9^n}{1+15^n} \leq \frac{9^n}{15^n} = \left(\frac{3}{5}\right)^n$, $\frac{3}{5} < 1$, $\sum \left(\frac{3}{5}\right)^n$ conv $\Rightarrow$

$\sum \frac{9^n}{1+15^n}$ conv (Comparism Test)

(e) (4 points) $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ $\sim \int_1^{\infty} \frac{\ln x}{x} dx$, $f(x) = \frac{\ln x}{x}$ positive and decreasing

$\int_1^{\infty} \frac{\ln x}{x} dx = \lim_{t\to\infty} \int_1^t \frac{\ln x}{x} dx$

$f'(x) = \frac{\frac{1}{x}\cdot x - \ln x \cdot 1}{x^2} = \frac{1-\ln x}{x^2} < 0$

for $x \geq 3$

$u = \ln x$
$du = \frac{1}{x}dx$
$= \lim_{t\to\infty} \int_1^t u \cdot du = \lim_{t\to\infty} \frac{1}{2}u^2 = \lim_{t\to\infty} \frac{1}{2}(\ln x)^2 \Big|_1^t = \lim_{t\to\infty} \frac{1}{2}(\ln t)^2 - 0 = \infty$

Improper Integral DIV $\Rightarrow$ Series DIV (integral test)

2. Find the series' radius and interval of convergence.

(a) (6 points) $\sum_{n=1}^{\infty} \frac{nx^n}{n+5}$

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(n+1)\cdot x^{n+1}}{n+1+5}}{\frac{n\cdot x^n}{n+5}}\right| = \lim_{n\to\infty}\left|\frac{(n+1)(n+5)}{(n+6)\,n}\cdot x\right|$

Radius of conv $R=1$. $= 1\cdot|x| < 1$

Open interval: $|x|<1 \Leftrightarrow -1<x<1 \Leftrightarrow (-1,1)$.

left endpoint $x=-1$. $\sum_{n=1}^{\infty}\frac{n\cdot(-1)^n}{n+5}$, $\lim_{n\to\infty}\frac{n(-1)^n}{n+5} = \pm 1$ (D.I.V.E). DIV.

Right endpoint $x=1$ $\sum_{n=1}^{\infty}\frac{n}{n+5}$ $\lim_{n\to\infty}\frac{n}{n+5} = 1 \neq 0$, DIV

Both $x=\pm 1$ are not included.

$\boxed{\text{Interval}: (-1,1)}$.

(b) (6 points) $\sum_{n=1}^{\infty} (2x)^n$

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(2x)^{n+1}}{(2x)^n}\right| = \lim_{n\to\infty}|2x| = |2x| < 1 \Leftrightarrow |x| < \frac{1}{2}$.

Radius of conV: $R=\frac{1}{2}$. Open interval: $-\frac{1}{2}<x<\frac{1}{2} \Leftrightarrow (-\frac{1}{2},\frac{1}{2})$

left endpoint $x=-\frac{1}{2}$, $\sum(-1)^n$ DIV. Not included.

Right endpoint $x=\frac{1}{2}$, $\sum 1^n$ DIV Not included.

Interval of conV: $(-\frac{1}{2},\frac{1}{2})$.

(c) (6 points) $\sum_{n=1}^{\infty} \frac{(3x-2)^n}{n}$

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(3x-2)^{n+1}}{n+1}}{\frac{(3x-2)^n}{n}}\right| = \lim_{n\to\infty}\left|\frac{n}{n+1}\cdot(3x-2)\right|$

Radius of conV: $R=\frac{1}{3}$. $= |3x-2| < 1$

$|3x-2|<1 \Leftrightarrow -1<3x-2<1 \Leftrightarrow 1<3x<3$ $\Leftrightarrow |x-\frac{2}{3}|<\frac{1}{3}$.

$\Leftrightarrow \frac{1}{3}<x<1$. Open interval $(\frac{1}{3},1)$.

left endpoint: $x=\frac{1}{3}$, $\sum_{n=1}^{\infty}\frac{(1-2)^n}{n} = \sum\frac{(-1)^n}{n}$, Alternating Series Test $\Rightarrow$ ConV.

$x=\frac{1}{3}$ included.

Right endpoint: $x=1$ $\sum\frac{(3-2)^n}{n} = \sum\frac{1}{n}$. DIV. $x=1$ NOT included.

Interval of ConV: $[\frac{1}{3},1)$.

3. (18 points) Find the Taylor polynomial of degree 3 generated by $f(x) = \sqrt{x}$ centered at $a = 4$.

$f(x) = x^{\frac{1}{2}}$ $\qquad x = a = 4$. $\quad f(4) = \sqrt{4} = 2$

$f'(x) = \frac{1}{2} \cdot x^{-\frac{1}{2}}$ $\qquad f'(4) = \frac{1}{2} \cdot 4^{-\frac{1}{2}} = \frac{1}{2} \cdot \frac{1}{\sqrt{4}} = \frac{1}{4}$

$f''(x) = \frac{1}{2} \cdot \left(-\frac{1}{2}\right) \cdot x^{-\frac{3}{2}}$ $\qquad f''(4) = -\frac{1}{4} \cdot 4^{-\frac{3}{2}} = -\frac{1}{4} \cdot \frac{1}{(\sqrt{4})^3} = -\frac{1}{4} \cdot \frac{1}{8} = -\frac{1}{32}$

$f'''(x) = \frac{1}{2} \cdot \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right) \cdot x^{-\frac{5}{2}}$ $\qquad f'''(4) = \frac{3}{8} \cdot 4^{-\frac{5}{2}} = \frac{3}{8} \cdot \frac{1}{(\sqrt{4})^5} = \frac{3}{8} \cdot \frac{1}{32}$

$$T_3(x) = f(4) + f'(4)(x-4) + \frac{f''(4)}{2!} \cdot (x-4)^2 + \frac{f'''(4)}{3!} \cdot (x-4)^3$$

$$= \boxed{2 + \frac{1}{4} \cdot (x-4) - \frac{\frac{1}{32}}{2!} (x-4)^2 + \frac{\frac{3}{8} \cdot \frac{1}{32}}{3!} (x-4)^3}$$

4. (18 points) Find the first four nonzero terms of the power series representation for the function

$$f(x) = \int_0^x t\sin(t)\ dt$$

$$\sin t = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \cdot t^{2n+1} = 1 - t + \frac{t^3}{3!} - \frac{t^5}{5!} + \cdots$$

$$f(x) = \int_0^x t \cdot \sin t\, dt = \int_0^x t\left(1 - t + \frac{t^3}{3!} - \frac{t^5}{5!} + \cdots\right) dt$$

$$= \int_0^x t - t^2 + \frac{t^4}{3!} - \frac{t^6}{5!} + \cdots\ dt$$

$$= \frac{1}{2}t^2 - \frac{1}{3}t^3 + \frac{1}{3!} \cdot \frac{1}{5}t^5 - \frac{1}{5!} \cdot \frac{1}{7}t^7 + \cdots \Big|_0^x$$

$$= \boxed{\frac{1}{2}x^2 - \frac{1}{3} \cdot x^3 + \frac{1}{3!} \cdot \frac{1}{5} \cdot x^5 - \frac{1}{5!} \cdot \frac{1}{7} \cdot x^7} + \cdots$$

5. (18 points) Consider the curve given by $y = \dfrac{x^3}{6} + \dfrac{1}{2x}$

(a) Write an integral that expresses the length of the arc of the curve on $[1, 3]$.

$$y' = \frac{3x^2}{6} + \frac{1}{2}\cdot\left(-\frac{1}{x^2}\right) = \frac{1}{2}x^2 - \frac{1}{2x^2}$$

$$\text{Arc length} = \int_1^3 \sqrt{1 + [y']^2}\, dx.$$

$$= \int_1^3 \sqrt{1 + \left(\frac{1}{2}x^2 - \frac{1}{2x^2}\right)^2}\, dx.$$

(b) Evaluate the integral from (a) to find the length of the arc.

$$= \int_1^3 \sqrt{1 + \left(\frac{1}{2}x^2\right)^2 - 2\cdot\frac{1}{2}x^2\cdot\frac{1}{2x^2} + \left(\frac{1}{2x^2}\right)^2}\, dx.$$

$$= \int_1^3 \sqrt{1 + \left(\frac{1}{2}x^2\right)^2 - \frac{1}{2} + \left(\frac{1}{2x^2}\right)^2}\, dx$$

$$= \int_1^3 \sqrt{\left(\frac{1}{2}x^2\right)^2 + \frac{1}{2} + \left(\frac{1}{2x^2}\right)^2}\, dx.$$

$$= \int_1^3 \sqrt{\left(\frac{1}{2}x^2 + \frac{1}{2x^2}\right)^2}\, dx.$$

$$= \int_1^3 \frac{1}{2}x^2 + \frac{1}{2x^2}\, dx$$

$$= \frac{1}{2}\cdot\frac{1}{3}x^3 + \frac{1}{2}\cdot\left(-\frac{1}{x}\right)\Big|_1^3$$

$$= \boxed{\frac{1}{6}\cdot 3^3 - \frac{1}{2}\cdot\frac{1}{3} - \left(\frac{1}{6}\cdot 1 - \frac{1}{2}\right)}$$

Multiple Choice. Circle the best answer. No work needed.
No partial credit available. No credit will be given for choices not clearly marked.

6. (7 points) Evaluate the series: $\sum_{n=1}^{\infty}(-1)^{n+1}(0.1)^n$ Geometric Series

A. 0

B. ∞

C. 1/11

D. $-1/11$

E. None of the above

First term $n=1$, $(-1)^2 \cdot 0.1 = 0.1$

Second term $n=2$, $(-1)^3 \cdot 0.1^2 = -0.01$

$a = 0.1$, $r = \frac{-0.01}{0.1} = -0.1$.

$\frac{a}{1-r} = \frac{0.1}{1-(-0.1)} = \frac{0.1}{1.1} = \frac{1}{11}$

7. (7 points) What can be said about the series $\sum_{n=1}^{\infty}\sin(1/n)$?

A. The Comparison Test does not apply here because the hypothesis is not met.

B. The Comparison Test shows that it converges.

C. The Limit Comparison Test shows that it diverges.

D. The n^{th} Term Test shows that it diverges .

E. None of the above

$\lim \frac{\sin(\frac{1}{n})}{\frac{1}{n}} \overset{L'H}{=} \lim_{n\to\infty} \frac{\cos(\frac{1}{n})\cdot(-\frac{1}{n^2})}{-\frac{1}{n^2}}$

$= \lim_{n\to\infty} \cos(\frac{1}{n})$

$= \cos 0 = 1$.

Limit Comparison: $\sum \frac{1}{n}$ DIV $\Rightarrow \sum \sin(\frac{1}{n})$ DIV.

8. (7 points) Which of the following is true about the series $\sum_{n=1}^{\infty}\left(\frac{-3n}{n+1}\right)^{5n}$?

A. The series doesn't converge.

B. The series converges absolutely, but doesn't converge.

C. The series converges, but doesn't converge absolutely.

D. The series converges and converges absolutely.

E. None of the above

$\lim_{n\to\infty}\left(\frac{-3n}{n+1}\right)^{5n} = (-3)^{\infty} = \pm\infty \neq 0$. nth term test for DIV

$\Rightarrow$ DIV.

9. (7 points) Find a power series representation for the function $f(x) = \dfrac{5}{1-4x^2}$

A. $5 + 4x^2 + 16x^4 + 64x^6 + \cdots$
B. $5 - 4x^2 + 16x^4 - 64x^6 + \cdots$
C. $5 + 20x^2 + 80x^4 + 320x^5 + \cdots$
D. $5 + 20x^2 + 80x^4 + 320x^6 + \cdots$ ✓
E. None of the above

$$5 \cdot \sum_{n=0}^{\infty} (4x^2)^n$$
$$= 5 \cdot \left[1 + 4x^2 + (4x^2)^2 + (4x^2)^3 + \cdots\right]$$
$$= 5 + 5 \cdot 4x^2 + 5 \cdot 16x^4 + 5 \cdot 4^3 \cdot x^6 + \cdots$$

10. (7 points) Determine the interval of convergence of the power series from the previous question

$$f(x) = \frac{5}{1-4x^2}$$

A. $(-\frac{1}{4}, \frac{1}{4})$
B. $(-\frac{1}{2}, \frac{1}{2})$ ✓
C. $(-\frac{1}{16}, \frac{1}{16})$
D. $(-\infty, \infty)$
E. None of the above

$$\boxed{} = 4x^2$$
$$|4x^2| < 1 \iff x^2 < \frac{1}{4}.$$
$$\iff |x| < \sqrt{\frac{1}{4}} = \frac{1}{2}$$
$$\iff -\frac{1}{2} < x < \frac{1}{2}$$

11. (7 points) The series $\sum_{n=1}^{\infty} \dfrac{(-1)^n (n+6)!}{7! \cdot n! \cdot 7^n}$ is

A. divergent by the Divergence Test
B. divergent by the Comparison Test
C. convergent by the Limit Comparison Test
D. convergent by the Ratio Test ✓
E. None of the above

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{\dfrac{(-1)^{n+1} \cdot (n+7)!}{\cancel{7!} \cdot (n+1)! \cdot 7^{n+1}}}{\dfrac{(-1)^n \cdot (n+6)!}{\cancel{7!} \cdot n! \cdot 7^n}} \right|$$
$$= \lim_{n\to\infty} \frac{(n+7)!}{(n+1)! \cdot 7^{n+1}} \cdot \frac{n! \cdot 7^n}{(n+6)!}$$
$$= \lim_{n\to\infty} \frac{(n+7)!}{(n+6)!} \cdot \frac{n!}{(n+1)!} \cdot \frac{7^n}{7^{n+1}}$$
$$= \lim_{n\to\infty} (n+7) \cdot \frac{1}{n+1} \cdot \frac{1}{7} = \frac{1}{7} < 1$$

12. (7 points) Find the Maclaurin series for $f(x) = \dfrac{x^2}{2+x}$ $= \dfrac{x^2}{2} \cdot \dfrac{1}{1+\frac{x}{2}}$

$= \dfrac{x^2}{2} \cdot \sum_{n=0}^{\infty} \left(-\dfrac{x}{2}\right)^n$

$= \dfrac{x^2}{2}\left(1 - \dfrac{x}{2} + \dfrac{x^2}{4} - \dfrac{x^3}{8} + \cdots\right)$

$= \dfrac{x^2}{2} - \dfrac{x^3}{4} + \dfrac{x^4}{8} - \dfrac{x^5}{16} + \cdots$

A. $x^2/2 + x^3/2 + x^4/2 + x^5/2 + \cdots$

B. $x^2/2 - x^3/2 + x^4/2 - x^5/2 + \cdots$

C. $x^2/2 + x^3/4 + x^4/8 + x^5/16 + \cdots$

D. $x^2/2 - x^3/4 + x^4/8 - x^5/16 + \cdots$ ✓

E. None of the above

13. (7 points) Find the Maclaurin series for $f(x) = x\cos\sqrt{x}$

$x \cdot \sum_{n=0}^{\infty} \dfrac{(-1)^n}{(2n)!}(\sqrt{x})^{2n}$ $\quad (\sqrt{x})^{2n} = (x^{\frac{1}{2}})^{2n} = x^n$

$= \sum_{n=0}^{\infty} x \cdot \dfrac{(-1)^n}{(2n)!} \cdot x^n$

$= \sum_{n=0}^{\infty} \dfrac{(-1)^n}{(2n)!} \cdot x^{n+1}$

A. $\sum_{n=0}^{\infty} (-1)^{n+1} \dfrac{x^{2n+1}}{(2n)!}$

B. $\sum_{n=0}^{\infty} (-1)^{n} \dfrac{x^{n}}{(2n)!}$

C. $\sum_{n=0}^{\infty} (-1)^{n+1} \dfrac{x^{n+1}}{(2n)!}$

D. $\sum_{n=0}^{\infty} (-1)^{n} \dfrac{x^{n+1}}{(2n)!}$ ✓

E. None of the above

14. (7 points) What can be said about the limit

$$\lim_{x\to\infty} \frac{n^2}{e^{2n}} + \frac{n^2+2}{3-2n^2}$$

$\lim_{n\to\infty} \dfrac{n^2}{e^{2n}} \overset{\text{l'H}}{=} \lim_{n\to\infty} \dfrac{2n}{2 \cdot e^{2n}} \overset{\text{l'H}}{=} \lim_{n\to\infty} \dfrac{2}{2 \cdot 2 e^{2n}} = 0$

$\lim_{n\to\infty} \dfrac{n^2+2}{3-2n^2} = \lim_{n} \dfrac{n^2}{-2n^2} = -\dfrac{1}{2}$

A. The limit does not exist because $\frac{n^2}{e^{2n}}$ doesn't have a limit.

B. The limit does not exist because $\frac{n^2+2}{3-2n^2}$ doesn't have a limit.

C. The limit L exists and $L = \frac{1}{3}$.

D. The limit L exists and $L = -\frac{1}{2}$. ✓

E. None of the above.

$\lim = 0 - \dfrac{1}{2} = -\dfrac{1}{2}$

Congratulations you are now done with the exam!
Go back and check your solutions for accuracy and clarity. Make sure your final answers are $\boxed{\text{BOXED}}$.

When you are completely happy with your work please bring your exam to the front to be handed in. **Please have your MSU student ID ready** so that is can be checked.

DO NOT WRITE BELOW THIS LINE.

Page	Points	Score
2	18	
3	18	
4	18	
5	18	
6	18	
7	21	
8	21	
9	21	
Total:	153	

No more than 150 points may be earned on the exam.

FORMULA SHEET PAGE 1

Integrals

- **Volume**: Suppose $A(x)$ is the cross-sectional area of the solid S perpendicular to the x-axis, then the volume of S is given by

$$V = \int_a^b A(x)\ dx$$

- **Work**: Suppose $f(x)$ is a force function. The work in moving an object from a to b is given by:

$$W = \int_a^b f(x)\ dx$$

- $\int \frac{1}{x}\ dx = \ln|x| + C$
- $\int \tan x\ dx = \ln|\sec x| + C$
- $\int \sec x\ dx = \ln|\sec x + \tan x| + C$
- $\int a^x\ dx = \frac{a^x}{\ln a} + C \qquad$ for $a \neq 1$
- **Integration by Parts**:

$$\int u\ dv = uv - \int v\ du$$

- Arc Length Formula:

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\ dx$$

Derivatives

- $\frac{d}{dx}(\sinh x) = \cosh x \qquad \frac{d}{dx}(\cosh x) = \sinh x$
- Inverse Trigonometric Functions:

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\csc^{-1} x) = \frac{-1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$$

- If f is a one-to-one differentiable function with inverse function f^{-1} and $f'(f^{-1}(a)) \neq 0$, then the inverse function is differentiable at a and

$$(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}$$

Hyperbolic and Trig Identities

- Hyperbolic Functions

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \qquad \operatorname{csch}(x) = \frac{1}{\sinh x}$$

$$\cosh(x) = \frac{e^x + e^{-x}}{2} \qquad \operatorname{sech}(x) = \frac{1}{\cosh x}$$

$$\tanh(x) = \frac{\sinh x}{\cosh x} \qquad \coth(x) = \frac{\cosh x}{\sinh x}$$

- $\cosh^2 x - \sinh^2 x = 1$
- $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$
- $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$
- $\sin(2x) = 2\sin x \cos x$
- $\sin A \cos B = \frac{1}{2}[\sin(A-B) + \sin(A+B)]$
- $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$
- $\cos A \cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$

FORMULA SHEET PAGE 2

Series

- **nth term test for divergence:** If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.
- **The p-series:** $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$.
- **Geometric:** If $|r| < 1$ then $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$
- **The Integral Test:** Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then
  - (i) If $\int_1^{\infty} f(x)\, dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.
  - (ii) If $\int_1^{\infty} f(x)\, dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.
- **The Comparison Test:** Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.
  - (i) If $\sum b_n$ is convergent and $a_n \leq b_n$ for all n, then $\sum a_n$ is also convergent.
  - (ii) If $\sum b_n$ is divergent and $a_n \geq b_n$ for all n, then $\sum a_n$ is also divergent.
- **The Limit Comparison Test:** Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

  where c is a finite number and $c > 0$, then either both series converge or both diverge.

- **Alternating Series Test:** If the alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$ satisfies
  - (i) $0 < b_{n+1} \leq b_n$ for all n
  - (ii) $\lim_{n\to\infty} b_n = 0$

  then the series is convergent.
- **The Ratio Test**
  - (i) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.
  - (ii) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.
  - (iii) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, the Ratio Test is inconclusive.
- **Maclaurin Series**: $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}\, x^n$
- **Taylor's Inequality** If $|f^{(n+1)}(x)| \leq M$ for $|x-a| \leq d$, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1} \quad \text{for } |x-a| \leq d$$

- Some Power Series
  - $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ $\qquad R = \infty$
  - $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ $\qquad R = \infty$
  - $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$ $\qquad R = \infty$
  - $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$ $\qquad R = 1$
  - $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ $\qquad R = 1$