Decomposition of Schramm-Loewner evolution along its curve

Dapeng Zhan*

Michigan State University

March 30, 2016

Abstract

We show that, for $\kappa \in (0, 8)$, the integral of the laws of two-sided radial SLE_κ curves through different interior points against a measure with SLE_κ Green function density is the law of a chordal SLE_κ curve, biased by the path's natural length. We also show that, for $\kappa > 0$, the integral of the laws of extended $\mathrm{SLE}_\kappa(-8)$ curves through different interior points against a measure with a closed formula density restricted in a bounded set is the law of a chordal SLE_κ curve, biased by the path's capacity length restricted in that set. Another result is that, for $\kappa \in (4, 8)$, if one integrates the laws of two-sided chordal SLE_κ curves through different force points on $\mathbb{R}$ against a measure with density on $\mathbb{R}$, then one also gets a law that is absolutely continuous w.r.t. that of a chordal SLE_κ curve. To obtain these results, we develop a framework to study stochastic processes with random lifetime, and improve the traditional Girsanov's Theorem.

Keywords: SLE, Girsanov's Theorem, Doob-Meyer decomposition

1 Introduction

The Schramm-Loewner evolution (SLE), first introduced by Oded Schramm in 1999 ([22]), is a one-parameter ($\kappa \in (0, \infty)$) family of measures on non-self-crossing curves, which has received a lot of attention over the past fifteen years. It has been shown that, modulo time parametrization, several discrete random paths on grids (e.g., loop-erased random walk [15], critical percolation explorer [24, 6]) have SLE as a scaling limit.

SLE is defined using Loewner's differential equation, and is originally parameterized by capacity. For the discrete random paths which converge to SLE, in order to show the convergence, people have to first reparameterize them by capacity and then prove that the reparametrized curve converge to SLE with capacity parametrization. The convergence does not take into consideration of the discrete length of the path.

In order to upgrade the convergence results, Lawler and Sheffield introduced natural parametrization of SLE in [14], and conjectured that those discrete random paths with

*Research partially supported by NSF grants DMS-1056840 and Sloan fellowship

their original length suitably rescaled, converge to the SLE with natural parametrization. Their construction used Doob-Meyer decomposition, and they proved the existence of the natural parametrization of SLE_κ for $\kappa < 5.021....$ This result was later improved by [16], where it was shown that the natural parametrization of SLE_κ exists for all $\kappa \in (0, 8)$.

It was proved later in [13] that the natural parametrization agrees with the d-dimensional Minkowski content of the SLE_κ curve, where $d = 1 + \frac{\kappa}{8}$ is the Hausdorff dimension of the curve (c.f. [4]). While the convergence of the discrete random paths with natural length to SLE with natural parametrization are still conjectures, some work have been done on loop-erased random walk (c.f. [5]) towards this direction.

There are two major versions of SLE: chordal SLE and radial SLE. Most of the study focus on chordal SLE, which describes a curve in a simply connected domain from one prime end (c.f. [1]) to another prime end. Two-sided radial SLE_κ and SLE_κ Green function for $\kappa \in (0, 8)$ were introduced in [14]. A two-sided radial SLE_κ curve has two arms: the first arm from a prime end to an interior point is a chordal $\mathrm{SLE}_\kappa(\kappa - 8)$ process, and the second arm from the interior point to another prime end is a chordal SLE_κ curve conditioned on the first arm. It can be understood as a chordal SLE_κ curve conditioned to pass through a fixed interior point. While that event has probability zero, some limiting procedure was used to make this idea rigorous. SLE_κ Green function was defined by a closed formula, and turned out to be the density of the SLE_κ curve with natural parametrization. These two objects have discrete analogues. The two-sided radial SLE_κ curve corresponds to the discrete random path conditioned to pass through a fixed vertex, and the SLE_κ Green function corresponds to the density of the path.

Field recently proved in [7] that, for $\kappa \in (0, 4]$, if one integrates the laws of two-sided radial SLE_κ in a bounded analytic domain D passing through different interior points (with the two ends fixed) against the measure with density (w.r.t. the Legesgue measure on $\mathbb{C}$) being the SLE_κ Green function in D, then one gets the law of a chordal SLE_κ curve biased by the curve's length in the *natural parametrization*. This is analogous to a simple fact of discrete random paths: if one integrates the laws of the path conditioned to pass through a fixed vertex against the density of the path, one should get a measure on paths, which is absolutely continuous w.r.t. the law of the original discrete random path, and the Radon-Nikodym derivative is the total number of vertices on the path, which is due to the repetition of counting. Field's proof used the reversibility of two-sided radial SLE, and some escape estimate of SLE derived in [8].

This paper is motivated by Field's work. We extend his result from $\kappa \in (0, 4]$ to $\kappa \in (0, 8)$ (see Corollary 4.3) using a simpler approach. We do not need to assume that the domain is bounded or analytic. The main tools used here are from Probability Theory. In fact, Fields' theorem essentially follows from the Doob-Meyer decomposition used to define the natural parametrization.

We then find other applications of the new technique. We define the extended chordal $\mathrm{SLE}_\kappa(\rho)$ curve in the upper half plane $\mathbb{H}$ for $\rho \leq \frac{\kappa}{2} - 4$, which is composed of two arms: the first arm growing from 0 to $z_0 \in \mathbb{H}$ is a chordal $\mathrm{SLE}_\kappa(\rho)$ curve, and the second arm growing from z_0 to ∞ is a chordal SLE_κ curve conditioned on the first arm. In particular, a two-sided radial SLE_κ curve is an extended chordal $\mathrm{SLE}_\kappa(\kappa - 8)$ curve. We prove that, for any $\kappa > 0$, there is a positive function $G^{\kappa,-8}(z)$ with closed formula such that for any measurable $U \subset \mathbb{H}$ on which $G^{\kappa,-8}$ is integrable, if one integrates the laws of extended chordal $\mathrm{SLE}_\kappa(-8)$ curve through different z against a measure with density $\mathbf{1}_U G^{\kappa,-8}(z)$, then one gets the law of a chordal SLE_κ curve biased by $C_{\kappa,1}$ times the total time that

the curve spends in U in the *capacity parametrization* (see Corollary 5.1), where $C_{\kappa,1}$ is a positive constant depending only on κ.

The above two main results of this paper immediately imply that, if we sample a point on a chordal SLE_κ curve in $\mathbb{H}$ according to a law, which is absolutely continuous w.r.t. the natural parametrization (resp. capacity parametrization), and stop the curve at that point, then we get a random curve, whose law is absolutely continuous w.r.t. that of a chordal $\mathrm{SLE}_\kappa(\kappa-8)$ (resp. $\mathrm{SLE}_\kappa(-8)$) curve in $\mathbb{H}$.

The application of our new technique on capacity parametrization is motivated by the author's work [28]. One of its results ([28, Remark 2 after Theorem 6.6]) is that, for $\kappa \in (0,4)$, if a chordal SLE_κ curve is stopped at a random time, whose law is absolutely continuous w.r.t. the capacity parametrization, then the tip of the curve is "similar" to the initial segment of a whole-plane $\mathrm{SLE}_\kappa(\kappa+2)$ curve. From the reversibility of whole-plane $\mathrm{SLE}_\kappa(\rho)$ curve (c.f. [18]) and the SLE coordinate changes (c,f. [23]), we see that the initial segment of a whole-plane $\mathrm{SLE}_\kappa(\kappa+2)$ curve is "similar" to the end segment of a chordal $\mathrm{SLE}_\kappa(-8)$ curve. From this observation, we see that this paper extends that result in [28] from $\kappa \in (0,4)$ to $\kappa \in (0,\infty)$.

Another application of the new technique is to study the intersection of SLE curve with the boundary. Using a Doob-Meyer decomposition, Albert and Sheffield constructed in [2] a measure supported by the intersection of an SLE_κ curve γ in $\mathbb{H}$ for $\kappa \in (4,8)$ with $\mathbb{R}$, and conjectured that the measure agrees with the $(2-\frac{8}{\kappa})$-dimensional Minkowski content of $\gamma \cap \mathbb{R}$. Their work used two-sided chordal SLE_κ, which can be understood as a chordal SLE_κ curve conditioned to pass through a fixed point on the boundary. Using their result, we prove in this paper that, if one integrates the laws of two-sided chordal SLE_κ curve in $\mathbb{H}$ from 0 to ∞ through different $x \in \mathbb{R}$ against a suitable measure, then one gets a measure on curves, which is absolutely continuous w.r.t. the law of a chordal SLE_κ curve.

The new technique has applications beyond the SLE area. For example, we use it to decompose a planar Brownian motion restricted in a simply connected domain.

The paper is organized as follows. In Section 2, in order to study the driving functions of $\mathrm{SLE}_\kappa(\rho)$ curves, we develop a framework on stochastic processes with random lifetime. We introduce the "locally absolutely continuity" between these processes, and extend the traditional Girsanov's theorem. In Section 3, we review the definitions and basic properties of SLE_κ and its variance $\mathrm{SLE}_\kappa(\rho)$ processes. In Section 4, we prove our main result about natural parametrization and two-sided radial SLE curves. In Section 5, we prove the result about capacity parametrization and extended chordal $\mathrm{SLE}_\kappa(-8)$ curves. In Section 6, we show the application on boundary measure and two-sided chordal SLE curves. In Section 7, we use the technique to decompose planar Brownian motions. In the appendix, we prove the transience property of chordal $\mathrm{SLE}_\kappa(\rho)$ curves.

Acknowledgement

The author acknowledges the support from the National Science Foundation under the grant DMS-1056840 and the support from the Alfred P. Sloan Foundation.

2 Stochastic Processes with Random Lifetime

For $0 < T \leq \infty$, let $C([0,T))$ denote the space of real valued continuous functions on $[0,T)$. Let

$$\Sigma = \bigcup_{0<T\leq\infty} C([0,T)).$$

For each $f \in \Sigma$, let T_f be such that $[0,T_f)$ is the domain of f.

We define two basic operations on Σ: killing and continuing. For $0 < \tau \leq \infty$, we define the killing map $\mathcal{K}_\tau : \Sigma \to \Sigma$ such that if $g = \mathcal{K}_\tau(f)$, then $T_g = \tau \wedge T_f$ and $g = f|_{[0,T_g)}$. Let $\Sigma^\oplus = \{f \in \Sigma : T_f < \infty, f(T_f^-) := \lim_{t\to T_f^-} f(t) \in \mathbb{R}\}$ and $\Sigma_\oplus = \{f \in \Sigma : f(0) = 0\}$. For example, if $T_f > \tau > 0$, then $\mathcal{K}_\tau(f) \in \Sigma^\oplus$. For $f \in \Sigma^\oplus$ and $g \in \Sigma_\oplus$, we may continue f using g and get the function $f \oplus g \in \Sigma$, which is defined by $T_{f\oplus g} = T_f + T_g$ and

$$f \oplus g(t) = \begin{cases} f(t), & 0 \leq t < T_f; \\ f(T_f^-) + g(t - T_f), & T_f \leq t < T_f + T_g. \end{cases}$$

Sometimes we want to record the time that the two functions are joined together. For this purpose, we define $f\widehat{\oplus}g = (f \oplus g, T_f)$. Then we can use $f\widehat{\oplus}g$ to recover f and g.

For $0 \leq t < \infty$, let $\mathcal{F}_t^0$ be the σ-algebra generated by

$$\{f \in \Sigma : s < T_f, f(s) \in U\}, \quad 0 \leq s \leq t, U \in \mathcal{B}(\mathbb{R}).$$

Then $(\mathcal{F}_t^0)$ is a filtration on Σ. Let $\mathcal{F}^0 = \vee_{0\leq t<\infty}\mathcal{F}_t^0$. We will mainly work on the measurable space $(\Omega, \mathcal{F}^0)$ or its completion w.r.t. certain measure. Every probability measure on $(\Sigma, \mathcal{F}^0)$ is the law of a continuous stochastic process with random lifetime. For any measure μ on $(\Sigma, \mathcal{F}^0)$, we use $\mathcal{F}^\mu$ and $\mathcal{F}_t^\mu$ to denote the μ-completion of $\mathcal{F}^0$ and $\mathcal{F}_t^0$, respectively.

The continuing maps $(f,g) \mapsto f \oplus g$ and $(f,g) \mapsto f\widehat{\oplus}g$ are measurable. If μ and ν are σ-finite measures supported by $\Sigma^\oplus$ and $\Sigma_\oplus$, respectively, we use $\mu \oplus \nu$ and $\mu\widehat{\oplus}\nu$ to denote the pushfoward measures of the product measure $\mu \otimes \nu$ under the maps $(f,g) \mapsto f \oplus g$ and $(f,g) \mapsto f\widehat{\oplus}g$, respectively.

Let's recall an important notation in Probability: kernel. Suppose $(U,\mathcal{U})$ and $(V,\mathcal{V})$ are two measurable spaces. A kernel from $(U,\mathcal{U})$ to $(V,\mathcal{V})$ is a map $\nu : U \times \mathcal{V} \to [0,\infty]$ such that (i) for every $u \in U$, $\nu(u,\cdot)$ is a measure on $(V,\mathcal{V})$, and (ii) for every $F \in \mathcal{V}$, $\nu(\cdot,F)$ is $\mathcal{U}$-measurable. Let μ be a σ-finite measure on $(U,\mathcal{U})$. Let $\mathcal{U}^\mu$ be the μ-completion of $\mathcal{U}$. A μ-kernel from $(U,\mathcal{U})$ to $(V,\mathcal{V})$ is a kernel from $(U^\mu,\mathcal{U}^\mu \cap U^\mu)$ to $(V,\mathcal{V})$, where $U^\mu \subset U$ is such that $U \setminus U^\mu$ is a μ-null. The μ-kernel is said to be finite if for μ-a.s. every $u \in U$, $\nu(u,V) < \infty$; and is said to be σ-finite if there is a sequence $F_n \in \mathcal{V}$, $n \in \mathbb{N}$, with $V = \bigcup F_n$ such that for any $n \in \mathbb{N}$, and μ-a.s. every $u \in U$, $\nu(u,F_n) < \infty$. If ν is a σ-finite μ-kernel from $(U,\mathcal{U})$ to $(V,\mathcal{V})$, then we may define a measure $\mu \otimes \nu$ on $\mathcal{U} \times \mathcal{V}$ such that

$$\mu \otimes \nu(E \times F) = \int_E \nu(u,F)d\mu(u), \quad E \in \mathcal{U}, \quad \mathcal{F} \in \mathcal{V}.$$

This new measure is first defined on the semi-ring $\{E \times F : E \in \mathcal{U}, \mathcal{F} \in \mathcal{V}\}$, and then extended to a measure on $\mathcal{U} \times \mathcal{V}$. Carathéodory's extension theorem guarantee the existence of the extension. The σ-finiteness of μ and ν ensures that the extension is unique, and $\mu \otimes \nu$ is also σ-finite. We use $\mu \cdot \nu$ to denote the marginal of $\mu \otimes \nu$ on $(V,\mathcal{V})$, i.e., $\mu \cdot \nu(F) = \int_U \nu(u,F)d\mu(u)$, $F \in \mathcal{V}$. Similarly, if ν is a σ-finite measure on $(V,\mathcal{V})$, and μ is a σ-finite

ν-kernel from $(V, \mathcal{V})$ to $(U, \mathcal{U})$, then we also use $\mu \otimes \nu$ to denote the measure on $\mathcal{U} \times \mathcal{V}$ such that $\mu \otimes \nu(E \times F) = \int_F \mu(v, E) d\nu(v)$ for $E \in \mathcal{U}$ and $F \in \mathcal{V}$, when there is no confusion.

The killing map $\mathcal{K} : (f, r) \mapsto \mathcal{K}_r(f)$ is also measurable. If ν is a σ-finite μ-kernel from Σ to $(0, \infty)$, we use $\mathcal{K}_\nu(\mu)$ to denote the pushforward measure of $\mu \otimes \nu$ under $\mathcal{K}$.

Let $\Sigma_t = \{f \in \Sigma : T_f > t\}$, $t \geq 0$. Then (Σ_t) is a decreasing family of subspaces of Σ with $\Sigma_0 = \Sigma$ and $\Sigma_\infty := \cap_{t=0}^\infty \Sigma_t = C([0, \infty))$. We will be interested in the restriction of $\mathcal{F}_t^0$ to Σ_t, i.e., $\mathcal{F}_t^0 \cap \Sigma_t$. Note that $\bigcup_{0 \leq t < \infty} \mathcal{F}_t^0 \cap \Sigma_t$ is a π-system, which generates the σ-algebra $\mathcal{F}^0$, and $\Sigma = \Sigma_0 \in \mathcal{F}_0^0 \cap \Sigma_0$. This enables us to apply Dynkin's $\pi - \lambda$ theorem. For example, if two finite measures on $(\Sigma, \mathcal{F}^0)$ agree on each $\mathcal{F}_t^0 \cap \Sigma_t$, then they are equal.

Let μ and ν be two measures on $(\Sigma, \mathcal{F}^0)$, which are σ-finite on $\mathcal{F}_0^0$ (and so are σ-finite on each $\mathcal{F}_t^0$). We say that ν is locally absolutely continuous w.r.t. μ, and write $\nu \lhd \mu$, if for any $0 \leq t < \infty$, $\nu|_{\mathcal{F}_t^0 \cap \Sigma_t}$ is absolutely continuous w.r.t. $\mu|_{\mathcal{F}_t^0 \cap \Sigma_t}$. This is certainly the case if $\nu \ll \mu$, i.e., ν is (globally) absolutely continuous w.r.t. μ. The process $M_t := \frac{d\nu|_{\mathcal{F}_t^0 \cap \Sigma_t}}{d\mu|_{\mathcal{F}_t^0 \cap \Sigma_t}}$, $0 \leq t < \infty$, are called the local Radon-Nikodym derivatives of ν w.r.t. μ. From Dynkin's $\pi - \lambda$ theorem, we see that ν is determined by μ and (M_t). Thus, we say that ν can be obtained by locally weighting μ by (M_t).

Proposition 2.1. *Let μ be a measure on $(\Sigma, \mathcal{F}^0)$, which is σ-finite on $\mathcal{F}_0^0$. Let $(\Upsilon, \mathcal{G})$ be a measurable space. Let $\nu : \Upsilon \times \mathcal{F}^0 \to [0, \infty]$ be such that for every $v \in \Upsilon$, $\nu(v, \cdot)$ is a finite measure on $\mathcal{F}^0$ that is locally absolutely continuous w.r.t. μ. Moreover, suppose that the local Radon-Nikodym derivatives are equal to $(M_t(v, \cdot))$, where $M_t : (\Upsilon, \mathcal{G}) \times (\Sigma, \mathcal{F}_t) \to [0, \infty)$ is measurable for every $t \geq 0$. Then ν is a kernel from $(\Upsilon, \mathcal{G})$ to $(\Sigma, \mathcal{F}^0)$. Moreover, if ξ is a σ-finite measure on $(\Upsilon, \mathcal{G})$ such that μ-a.s., $\int_\Upsilon M_t(v, \cdot) d\xi(v) < \infty$ for all $t \geq 0$, then $\xi \cdot \nu \lhd \mu$, and the local Radon-Nikodym derivatives are $\int_\Upsilon M_t(v, \cdot) d\xi(v)$, $0 \leq t < \infty$.*

Proof. From Dynkin's $\pi - \lambda$ theorem, to prove that $v \mapsto \nu(v, \cdot)$ is measurable, it suffices to show that, for any $t \in [0, \infty)$ and any $A \in \mathcal{F}_t^0 \cap \Sigma_t$, $v \mapsto \nu(v, A)$ is measurable, which easily follows from Tonelli's theorem because $\nu(v, A) = \int_A M_t(v, f) d\mu(f)$. To prove that $\xi \cdot \nu \lhd \mu$ and find the local Radon-Nikodym derivatives, we apply Tonelli's theorem again and get

$$\xi \cdot \nu(A) = \int_\Upsilon \nu(v, A) d\xi(v) = \int_\Upsilon \int_A M_t(v, f) d\mu(f) d\xi(v) = \int_A \int_\Upsilon M_t(v, f) d\xi(v) d\mu(f)$$

for $A \in \mathcal{F}_t^0 \cap \Sigma_t$. Then we conclude that $\xi \cdot \nu \lhd \mu$, and conclude that the local Radon-Nikodym derivatives are $\int_\Upsilon M_t(v, \cdot) d\xi(v)$, $0 \leq t < \infty$. □

Proposition 2.2. *Let μ be a probability measure on $(\Sigma, \mathcal{F}^0)$. Let ξ be a μ-kernel from $(\Sigma, \mathcal{F}^0)$ to $(0, \infty)$ that satisfies $\mathbb{E}_\mu[|\xi|] < \infty$. Then $\mathcal{K}_\xi(\mu) \lhd \mu$, and the local Radon-Nikodym derivatives are $\mathbb{E}_\mu[\xi((t, \infty)) | \mathcal{F}_t^0]$, $0 \leq t < \infty$.*

Proof. Fix $t \in [0, \infty)$, and $E \in \mathcal{F}_t^0 \cap \Sigma_t$. We claim that $\mathcal{K}_r(f) \in E$ iff $f \in E$ and $r > t$. On the one hand, if $f \in E$ and $r > t$, then from $\mathcal{K}_r(f)(s) = f(s)$, $0 \leq s \leq t$, and $f \in E \in \mathcal{F}_t^0$, we get $\mathcal{K}_r(f) \in E$. On the other hand, if $g = \mathcal{K}_r(f) \in E$, then $r = T_g > t$. From $g(s) = f(s)$, $0 \leq s \leq t$, and $g \in E \in \mathcal{F}_t^0$, we get $f \in E$. This proves the claim, which implies that

$$\mathcal{K}_\xi(\mu)(E) = \mu \otimes \xi(E \times (t, \infty)) = \int_E \xi(f, (t, \infty)) d\mu(f) = \int_E \mathbb{E}_\mu[\xi((t, \infty)) | \mathcal{F}_t^0] d\mu.$$

Thus, we conclude that $\mathcal{K}_\xi(\mu) \lhd \mu$, and $\mathbb{E}_\mu[\xi((t, \infty)) | \mathcal{F}_t^0]$, $0 \leq t < \infty$, are the local Radon-Nikodym derivatives. □

Remark. Proposition 2.2 will be mainly applied to the case that $\xi = d\theta$, where (θ_t) is an $(\mathcal{F}_t^\mu)$-adapted right-continuous increasing process with $\theta_0 = \theta_{0^-} = 0$ and $\mathbb{E}_\mu[\theta_\infty] < \infty$. Applying the proposition, we find that $\mathcal{K}_{d\theta}(\mu) \lhd \mu$, and

$$\mu - \text{a.s.}, \quad \frac{d\mathcal{K}_{d\theta}(\mu)|_{\mathcal{F}_t^0 \cap \Sigma_t}}{d\mu|_{\mathcal{F}_t^0 \cap \Sigma_t}} = \mathbb{E}_\mu[\theta_\infty | \mathcal{F}_t^\mu] - \theta_t. \tag{2.1}$$

Fix $\kappa > 0$. Let $\mathbb{P}_\kappa$ be the law of $\sqrt{\kappa}$ times a standard Brownian motion. This means that $\frac{1}{\sqrt{\kappa}}$ times the coordinate process on Σ under $\mathbb{P}_\kappa$ is a standard Brownian motion. For this reason, we use (B_t) to denote the above standard Brownian motion on Σ, i.e., $\frac{1}{\sqrt{\kappa}}$ times the coordinate process. We observe that $\mathbb{P}_\kappa$ is supported by $\Sigma_\infty \cap \Sigma_\oplus$. Let $\mathcal{F}_t^B$ and $\mathcal{F}^B$ be the $\mathbb{P}_\kappa$-completion of $\mathcal{F}_t^0$ and $\mathcal{F}^0$, respectively; and $\mathbb{E}_\kappa$ denote the expectation w.r.t. $\mathbb{P}_\kappa$.

We now use Girsanov's theorem to derive local Radon-Nikodym derivatives. Recall that when we used Girsanov's theorem to weight a probability measure by a positive local martingale, we had to stop the process at some stopping time to get a bounded martingale. The following proposition says that we do not need to do the stopping, and the local martingale valued at different times are just the local Radon-Nikodym derivatives.

Proposition 2.3. *Suppose that $(X_t)_{0 \le t < T_0}$ satisfies $X_0 = 0$ and the $(\mathcal{F}_t^B)$-adapted SDE:*

$$dX_t = \sqrt{\kappa} dB_t + \sigma_t dt, \quad 0 \le t < T_0,$$

where T_0 is a positive $(\mathcal{F}_t^B)$-stopping time. Let $\mathbb{P}^{\kappa,\sigma}$ denote the law of X. Then $\mathbb{P}^{\kappa,\sigma} \lhd \mathbb{P}_\kappa$. Moreover, if M_t, $0 \le t < T_0$, is an $(\mathcal{F}_t^B)$-adapted continuous local martingale that satisfies $M_0 = 1$ and the SDE:

$$dM_t = M_t \frac{\sigma_t}{\sqrt{\kappa}} dB_t, \quad 0 \le t < T_0. \tag{2.2}$$

then

$$\frac{d\mathbb{P}^{\kappa,\sigma}|_{\mathcal{F}_t^0 \cap \Sigma_t}}{d\mathbb{P}_\kappa|_{\mathcal{F}_t^0 \cap \Sigma_t}} = \mathbf{1}_{T_0 > t} M_t, \quad 0 \le t < \infty. \tag{2.3}$$

Proof. Let $M_t = \exp(\int_0^t \frac{\sigma_s}{\sqrt{\kappa}} dB_s - \int_0^t \frac{\sigma_s^2}{2\kappa} ds)$, $0 \le t < T_0$. From Itô's formula (c.f. [19]), we see that (M_t) satisfies $M_0 = 1$ and (2.2). Thus, it suffices to prove (2.3).

Fix $N \in \mathbb{N}$ and let $\tau_N = \inf\{0 \le t < T_0 : |M_t| \ge N\}$. Here we set $\inf \emptyset = T_0$. Then τ_N is a stopping time with $\tau_N \le T_0$, and M_t, $0 \le t < \tau_N$, is uniformly bounded by N. If $\tau_N = T_0$, then $\mathbb{P}_\kappa$-a.s. $\lim_{t \to T_0^-} M_t$ exists. Let

$$M_t^{\tau_N} := \begin{cases} M_{t \wedge \tau_N}, & t \wedge \tau_N < T_0; \\ \liminf_{s \to T_0^-} M_s, & t \ge \tau_N = T_0. \end{cases}$$

Then $(M_t^{\tau_N}, 0 \le t < \infty)$ is a uniformly bounded ($\mathbb{P}_\kappa$-a.s.) continuous local martingale, and satisfies the SDE:

$$dM_t^{\tau_N} = M_t^{\tau_N} \mathbf{1}_{t < \tau_N} \frac{\sigma_t}{\sqrt{\kappa}} dB_t, \quad 0 \le t < \infty. \tag{2.4}$$

Let $M_\infty^{\tau_N} = \liminf_{t \to \infty} M_t^{\tau_N}$. Then for any $0 \le t < \infty$, a.s. $\mathbb{E}_\kappa[M_\infty^{\tau_N} | \mathcal{F}_t^B] = M_t^{\tau_N}$. In particular, since $M_0^{\tau_N} = 1$, we have $\mathbb{E}_\kappa[M_\infty^{\tau_N}] = 1$. Define $\mathbb{P}_N^{\kappa,\sigma}$ such that $\frac{d\mathbb{P}_N^{\kappa,\sigma}}{d\mathbb{P}_\kappa} = M_\infty^{\tau_N}$. Then $\mathbb{P}_\kappa^N$ is also a probability measure on $(\Sigma, \mathcal{F}^0)$. Let

$$B_t^N = B_t - \int_0^{\tau_N \wedge t} \frac{\sigma_s}{\sqrt{\kappa}} ds, \quad 0 \le t < \infty.$$

From Girsanov's theorem (c.f. [19]) and (2.4), we know that the law of B^N under $\mathbb{P}_N^{\kappa,\sigma}$ is also that of a standard Brownian motion. From

$$\sqrt{\kappa}B_t = \sqrt{\kappa}B_t^N + \int_0^t \sigma_s ds, \quad 0 \le t < \tau_N.$$

we see that the law of $(\sqrt{\kappa}B_t, 0 \le t < \tau_N)$ under $\mathbb{P}_N^{\kappa,\sigma}$ is the same as the law of $(X_t, 0 \le t < \tau_N)$ under $\mathbb{P}_\kappa$.

Fix $t \in [0,\infty)$ and $E \in \mathcal{F}_t^0 \cap \Sigma_t$. Since T_0 is the lifetime of X, and $T_0 = \sup_{N\in\mathbb{N}} \tau_N$, we have

$$X^{-1}(E) \subset X^{-1}(\Sigma_t) \subset \{T_0 > t\} = \bigcup_{N\in\mathbb{N}} \{\tau_N > t\}. \tag{2.5}$$

Since the law of $(\sqrt{\kappa}B_t, 0 \le t < \tau_N)$ under $\mathbb{P}_N^{\kappa,\sigma}$ is the same as the law of $(X_t, 0 \le t < \tau_N)$ under $\mathbb{P}_\kappa$, and $(\sqrt{\kappa}B_t)$ is the coordinate process, we get

$$\begin{aligned}\mathbb{P}_\kappa[X^{-1}(E) \cap \{\tau_N > t\}] &= \mathbb{P}_N^{\kappa,\sigma}[E \cap \{\tau_N > t\}] = \mathbb{E}_\kappa[\mathbf{1}_{E\cap\{\tau_N>t\}} M_\infty^{\tau_N}] \\ &= \mathbb{E}_\kappa[\mathbf{1}_{E\cap\{\tau_N>t\}} M_t^{\tau_N}] = \mathbb{E}_\kappa[\mathbf{1}_E \mathbf{1}_{\{\tau_N>t\}} M_t],\end{aligned}$$

where the third equality follows from the optional stopping theorem, and the last equality holds because $M_t^{\tau_N} = M_t$ on $\{\tau_N > t\}$. This together with (2.5) implies that

$$\mathbb{P}^{\kappa,\sigma}(E) = \mathbb{P}_\kappa[X^{-1}(E)] = \lim_{N\to\infty} \mathbb{P}_\kappa[X^{-1}(E) \cap \{\tau_N > t\}] = \mathbb{E}_\kappa[\mathbf{1}_E \mathbf{1}_{\{T_0>t\}} M_t].$$

So we get (2.3) and finish the proof. □

At the end of this section, we state and prove the following proposition, which extends the strong Markov property of Brownian motions.

Proposition 2.4. *Let $(\theta_t)_{0\le t<\infty}$ be a right-continuous increasing $(\mathcal{F}_t^B)$-adapted process that satisfies $\theta_0 = \theta_{0^+} = 0$ and $\mathbb{E}_\kappa[\theta_\infty] < \infty$. Then*

$$\mathcal{K}_{d\theta}(\mathbb{P}_\kappa) \widehat{\oplus} \mathbb{P}_\kappa = \mathbb{P}_\kappa \otimes d\theta. \tag{2.6}$$

Thus, $\mathcal{K}_{d\theta}(\mathbb{P}_\kappa) \oplus \mathbb{P}_\kappa \ll \mathbb{P}_\kappa$, and θ_∞ is the Radon-Nikodym derivative.

Remark. If $\theta_t = \mathbf{1}_{\tau\le t}$, where τ is a positive finite $(\mathcal{F}_t^B)$-stopping time, then the proposition reduces to the strong Markov property of $(\sqrt{\kappa}B_t)$, i.e., $\mathcal{K}_{\delta_\tau}(\mathbb{P}_\kappa) \oplus \mathbb{P}_\kappa = \mathbb{P}_\kappa$.

Proof. First, assume that there is $t_0 \in (0,\infty)$ and $E \in \mathcal{F}_{t_0}^B$ such that $\theta_t(f) = \mathbf{1}_E(f) \cdot \mathbf{1}_{[t_0,\infty)}(t)$. Fix $t_0' \in (0,t_0)$, $A \in \mathcal{F}_{t_0'}^B$ and $B \in \mathcal{F}^B$. For every $r \in [0,\infty)$, define $\mathcal{S}_r : \Sigma_r \to \Sigma_\oplus$ such that if $g = \mathcal{S}_r(f)$, then $T_g = T_f - r$, and $g(t) = f(r+t) - f(r)$, $0 \le t < T_g$. Let $A \oplus_{t_0} B = \{f \in A \cap \Sigma_{t_0} : \mathcal{S}_{t_0}(f) \in B\}$. Since $\mathcal{S}_{t_0}(\mathcal{K}_{t_0}(f) \oplus g) = g$, and $\mathcal{K}_{t_0}(f) \in A$ iff $f \in A$ and $r > t_0'$, we get

$$\mathcal{K}_{d\theta}(\mathbb{P}_\kappa) \oplus \mathbb{P}_\kappa(A \oplus_{t_0} B) = \mathcal{K}_{d\theta}(\mathbb{P}_\kappa)(A)\mathbb{P}_\kappa(B) = \mathbb{P}_\kappa \otimes d\theta(A \times (t_0',\infty))\mathbb{P}_\kappa(B) = \mathbb{P}_\kappa(A \cap E)\mathbb{P}_\kappa(B).$$

From the Markov property of $(\sqrt{\kappa}B_t)$, we get

$$\int_{A\oplus_{t_0}B} \theta_\infty d\mathbb{P}_\kappa = \int_{A\oplus_{t_0}B} \mathbf{1}_E d\mathbb{P}_\kappa = \mathbb{P}_\kappa((A \cap E) \oplus_{t_0} B) = \mathbb{P}_\kappa(A \cap E) \cdot \mathbb{P}_\kappa(B).$$

Define $\mathbb{P}_\kappa^\theta$ such that $d\mathbb{P}_\kappa^\theta/d\mathbb{P}_\kappa = \theta_\infty$. From the above two displayed formulas, we see that $\mathcal{K}_{d\theta}(\mathbb{P}_\kappa) \oplus \mathbb{P}_\kappa$ agrees with $\mathbb{P}_\kappa^\theta$ on the sets $A \oplus_{t_0} B$, where $A \in \mathcal{F}_{t_0'}^B$, $t_0' \in (0, t_0)$, and $B \in \mathcal{F}^B$. Since these sets form a π-system, Dynkin's $\pi - \lambda$ theorem implies that the two measures agree on the σ-algebra generated by these sets, which agrees with $\mathcal{F}^B$ restricted to Σ_{t_0}. Since both measures are supported by $\Sigma_\infty \subset \Sigma_{t_0}$, we get $\mathcal{K}_{d\theta}(\mathbb{P}_\kappa) \oplus \mathbb{P}_\kappa = \mathbb{P}_\kappa^\theta$. Since these two measures are the projections of $\mathcal{K}_{d\theta}(\mathbb{P}_\kappa)\widehat{\oplus}\mathbb{P}_\kappa$ and $\mathbb{P}_\kappa \otimes d\theta$, respectively, to Σ, and the projections of them to $(0, \infty)$ are both concentrated at t_0, we get (2.6) in this special case.

Second, assume that (θ_t) has the form of $\sum_{n=1}^\infty C_n \theta_t^{(n)}$, where each C_n is a nonnegative real number and each $\theta_t^{(n)}$ satisfies the condition in the previous paragraph. In this case, we get (2.6) using the result in the above paragraph and the fact that both sides of (2.6) satisfy the countable linearity in (θ_t).

Finally, we consider the general case. Since $\mathbb{E}_\kappa[\theta_\infty] < \infty$, from the linearity of both sides of (2.6) in $d\theta$, we may assume that $\mathbb{E}_\kappa[\theta_\infty] = 1$. In this case both sides of (2.6) are probability measures. For $n \in \mathbb{N}$, define $U^{(n)}$ and $L^{(n)}$ from $[0, \infty)$ to $[0, \infty)$ such that

$$U^{(n)}(t) = \frac{\lfloor 2^n \cdot t \rfloor + 1}{2^n}, \quad L^{(n)}(t) = 0 \vee \frac{\lceil 2^n \cdot t \rceil - 1}{2^n}.$$

Then $U^{(n)}(t) \downarrow t$ and $L^{(n)}(t) \uparrow t$ for any $t \in [0, \infty)$. Moreover, we have

$$L^{(n)}(t) \le s \quad \text{if and only if} \quad t \le U^{(n)}(s), \quad \forall t, s \in [0, \infty). \tag{2.7}$$

This equivalence holds because for any $t, s \ge 0$, both side are equivalent to that there is an integer n such that $2^n t \le n \le 2^n s + 1$. Define $(\theta_t^{(n)})$ such that $\theta_t^{(n)} = \theta_{U^{(n)}(t)}$. Then $(\theta_t^{(n)})$ has the form of that in the above paragraph since it takes values only at $k/2^n$, $k \in \mathbb{Z}$. Thus, $\mathcal{K}_{d\theta^{(n)}}(\mathbb{P}_\kappa)\widehat{\oplus}\mathbb{P}_\kappa = \mathbb{P}_\kappa \otimes d\theta^{(n)}$ for each $n \in \mathbb{N}$. From (2.7), we get

$$L_*^{(n)}(d\theta(f, \cdot)) = d\theta^{(n)}(f, \cdot), \quad \forall f \in \Sigma, n \in \mathbb{N}. \tag{2.8}$$

We assign Σ the topology of locally uniform convergence. It suffices to show that $\mathcal{K}_{d\theta^{(n)}}(\mathbb{P}_\kappa)\widehat{\oplus}\mathbb{P}_\kappa$ and $\mathbb{P}_\kappa \otimes d\theta^{(n)}$ converge weakly to $\mathcal{K}_{d\theta}(\mathbb{P}_\kappa)\widehat{\oplus}\mathbb{P}_\kappa$ and $\mathbb{P}_\kappa \otimes d\theta$, respectively. To prove that $\mathcal{K}_{d\theta^{(n)}}(\mathbb{P}_\kappa)\widehat{\oplus}\mathbb{P}_\kappa \to \mathcal{K}_{d\theta}(\mathbb{P}_\kappa)\widehat{\oplus}\mathbb{P}_\kappa$, we define $\Sigma \times (0, \infty)$-valued random variables $h^{(n)}$ and h on a probability space $(\Omega, \mathbb{P})$ such that their laws are the above measures, and a.s. $h^{(n)} \to h$. For this purpose, we choose $\Omega = (\Sigma_\infty \times (0, \infty)) \times \Sigma_\oplus$, $\mathbb{P} = (\mathbb{P}_\kappa \otimes d\theta) \times \mathbb{P}_\kappa$, $h((f, t), g) = (\mathcal{K}_t(f) \oplus g, t)$, and $h^{(n)}(f, t, g) = h((f, L^{(n)}(t)), g)$. Using (2.8) and the convergence $L^{(n)}(t) \uparrow t$ it is easy to check that $h^{(n)}$ and h satisfy the desired properties. To prove that $\mathbb{P}_\kappa \otimes d\theta^{(n)} \to \mathbb{P}_\kappa \otimes d\theta$, we choose $\Omega = \Sigma \times (0, \infty)$, $\mathbb{P} = \mathbb{P}_\kappa \otimes d\theta$, $h = \mathrm{id}_\Omega$, and $h^{(n)}(f, t) = (f, L^{(n)}(t))$. Then a.s. $h^{(n)} \to h$, and from (2.8) we see that the laws of $h^{(n)}$ and h are $\mathbb{P}_\kappa \otimes d\theta^{(n)}$ and $\mathbb{P}_\kappa \otimes d\theta$, respectively. So we get (2.6) in the general case.

The statement after (2.6) follows from projecting both sides of (2.6) to Σ. □

3 Schramm-Loewner Evolution

In this section, we review the Loewner equations and the Schramm-Loewner Evolution (SLE). See [10, 20] for more details. We focus on chordal SLE, and will often omit the word "chordal" before "Loewner equation" or "SLE" when there is no confusion.

The definition of SLE uses the Loewner equations. Let's first review the (chordal) Loewner equation. Let $\lambda \in C([0,T))$, where $T \in (0,\infty]$. The Loewner equation driven by λ is the following differential equation in the complex plane:

$$\partial g_t(z) = \frac{2}{g_t(z) - \lambda(t)}, \quad 0 \le t < T; \quad g_0(z) = z.$$

Let $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$. For $0 \le t < T$, let K_t denote the set of $z \in \mathbb{H}$ such that the solution $s \mapsto g_s(z)$ blows up before or at t. It turns out that g_t maps $\mathbb{H} \setminus K_t$ conformally onto $\mathbb{H}$, and satisfies $g_t(z) = z + \frac{2t}{z} + O(|z|^{-2})$ as $z \to \infty$. We call g_t and K_t the Loewner maps and hulls, respectively, driven by λ.

Suppose for every $t \in [0,T)$, g_t^{-1} extends continuously from $\mathbb{H}$ to $\overline{\mathbb{H}}$. Throughout, we use f_t to denote the continuation of g_t^{-1} from $\overline{\mathbb{H}}$ into $\overline{\mathbb{H}}$. Also suppose that $\gamma(t) := f_t(\lambda(t))$, $0 \le t < T$, is a continuous curve in $\overline{\mathbb{H}}$. Then we say that γ is the Loewner curve driven by λ. In this case, for $0 \le t < T$, $\mathbb{H} \setminus K_t$ is the unbounded connected component of $\mathbb{H} \setminus \gamma([0,t])$. The Loewner curve driven by λ may not exist in general.

The Loewner equations satisfy the following scaling and translation properties. Suppose $\lambda(t)$, $0 \le t < T$, generates Loewner maps g_t and hulls K_t, $0 \le t < T$. Let $a > 0$ and $b \in \mathbb{R}$, and $\lambda^{a,b}(t) = b + a \cdot \lambda(t/a^2)$, $0 \le t < a^2 T$. Then $\lambda^{a,b}$ generates the Loewner maps $z \mapsto b + a \cdot g_{t/a^2}((z-b)/a)$ and hulls $b + a \cdot K_{t/a^2}$, $0 \le t < a^2 T$. If λ generates a Loewner curve γ, then $\lambda^{a,b}$ also generates a Loewner curve, which is $b + a \cdot \gamma(\cdot/a^2)$.

Another simple and useful property of the Loewner equations is the renewal property. Suppose $\lambda(t)$, $0 \le t < T$, generates Loewner maps g_t and hulls K_t, $0 \le t < T$. Let $\tau \in [0,T)$. Then $\lambda(\tau + t)$, $0 \le t < T - \tau$, generates Loewner maps $g_{\tau+t} \circ g_\tau^{-1}$ and hulls $g_\tau(K_{\tau+t} \setminus K_\tau)$, $0 \le t < T - \tau$. If λ and $\lambda(\tau + \cdot)$ generate Loewner curves γ and γ_τ, respectively, then $\gamma(\tau + t) = f_\tau(\gamma_\tau(t))$, $0 \le t < T - \tau$.

Let $\Sigma^{\mathbb{C}}$ denote the counterpart of Σ with real valued continuous functions replaced by complex valued continuous functions. Let $\Sigma^{\mathcal{L}}$ denote the set of $\lambda \in \Sigma$, which generates a Loewner curve γ. Then $\Sigma^{\mathcal{L}} \in \mathcal{F}^0$, and the Loewner map $\mathcal{L} : \lambda \mapsto \gamma$ from $\Sigma^{\mathcal{L}}$ to $\Sigma^{\mathbb{C}}$ is measurable. We also define here the extended Loewner map $\widehat{\mathcal{L}}$ from $\{(\lambda, t) : \lambda \in \Sigma^{\mathcal{L}}, 0 \le t < T_\lambda\}$ to $\Sigma^{\mathbb{C}} \times \overline{\mathbb{H}}$ such that $\widehat{\mathcal{L}}(\lambda, t) = (\mathcal{L}(\lambda), \mathcal{L}(\lambda)(t))$.

Fix $\kappa > 0$. Let $B(t)$, $0 \le t < \infty$, be a standard Brownian motion. The SLE_κ process is defined by taking $\lambda(t) = \sqrt{\kappa} B(t)$, $0 \le t < \infty$, in the Loewner equation. In this case, the Loewner curve γ driven by λ a.s. exists, and satisfies $\gamma(0) = 0$ and $\lim_{t\to\infty} \gamma(t) = \infty$. Such γ is called a standard SLE_κ curve (in $\mathbb{H}$ from 0 to ∞). In terms of measures, this means that $\mathbb{P}_\kappa$ (the law of $(\sqrt{\kappa} B_t)$) is supported by $\Sigma^{\mathcal{L}}$. The pushforward measure $\mathcal{L}_*(\mathbb{P}_\kappa)$ is then the law of a standard SLE_κ curve.

The scaling property of Loewner equations and the scaling property of Brownian motions together imply the scaling property of the SLE curve: if γ is a standard SLE_κ curve, then $t \mapsto a\gamma(t/a^2)$ is also a standard SLE_κ curve. The renewal and translation properties of Loewner equations and the strong Markov property of Brownian motions together imply the domain Markov property of SLE: if γ is a standard SLE_κ curve, and τ is a finite stopping time, then conditioned on $\gamma(t)$, $t \le \tau$, there is a standard SLE_κ curve $\widehat{\gamma}$ such that $\gamma(\tau + t) = f_\tau(\widehat{\gamma}(t) + \lambda_\tau)$, $t \ge 0$.

The definition of SLE_κ extends to other simply connected domains by conformal maps. Let D be a simply connected domain with locally connected boundary. Let a and b be two distinct prime ends ([1]) of D. Let f be a conformal map from $\mathbb{H}$ onto D such that $f(0) = a$ and $f(\infty) = b$. Let γ be a standard SLE_κ curve. Then $f \circ \gamma$ is called an SLE_κ

curve in D from a to b. The locally connectedness of ∂D is used to guarantee that f extends continuously to $\overline{\mathbb{H}}$ so that $f \circ \gamma$ is a continuous curve in $\overline{D}$. This condition may be weakened in some cases. Although the f is not unique, the law of $f \circ \gamma$ is unique up to a linear time-change, thanks to the scaling property of a standard SLE_κ curve. From now on, an SLE_κ curve without the domain and two prime ends specified is always a standard SLE_κ curve, and the word "standard" will often be omitted

The behavior of an SLE_κ curve depends on the value of κ. Let γ be an SLE_κ curve. If $\kappa \in (0, 4]$, then γ is a simple curve, and does not intersect $\mathbb{R}$ after the time 0; if $\kappa > 4$, then γ will intersect itself and $\mathbb{R}$ after the time 0. If $\kappa \geq 8$, γ is space-filling: it visits every point in $\overline{\mathbb{H}}$; if $\kappa < 8$, then for every $z_0 \in \overline{\mathbb{H}} \setminus \{0\}$, the probability that γ visits z_0 is 0. Besides, the Hausdorff dimension of γ is $\min\{1 + \frac{\kappa}{8}, 2\}$ (c.f. [4]).

$\mathrm{SLE}_\kappa(\rho)$ is a variant of SLE_κ. Its definition involves one or more force points, which lie on the boundary or in the interior of the domain. For the purpose here, we consider the case that there is only one force point, which is an interior point. Let $\rho \in \mathbb{R}$, $a_0 \in \mathbb{R}$ and $z_0 \in \mathbb{H}$. An $\mathrm{SLE}_\kappa(\rho)$ process started from $a_0 \in \mathbb{R}$ with force point at $z_0 \in \mathbb{H}$ is the solution of the Loewner equation driven by λ_t, $0 \leq t < T_{z_0}$, which is the solution of the SDE:

$$d\lambda_t = \sqrt{\kappa} dB_t + \mathrm{Re}\, \frac{\rho}{\lambda_t - g_t^\lambda(z_0)}\, dt, \quad \lambda_0 = a_0.$$

Here g_t^λ are the Loewner maps driven by λ, and $[0, T_{z_0})$ is the maximal solution interval. Let $\mathbb{P}_{z_0}^{\kappa,\rho}$ denote the law of λ_t, $0 \leq t < T_{z_0}$. From Proposition 2.3, we know that $\mathbb{P}_{z_0}^{\kappa,\rho} \lhd \mathbb{P}_\kappa$. Thus, $\mathbb{P}_{z_0}^{\kappa,\rho}$ is supported by $\Sigma^{\mathcal{L}}$, i.e., the Loewner curve γ driven by λ a.s. exists, which is called a (standard) $\mathrm{SLE}_\kappa(\rho)$ curve started from a_0 with force point at z_0. The following proposition is about the end point of $\mathrm{SLE}_\kappa(\rho)$ curves. To reduce the amount of arguments in this section, we move its proof to Appendix.

Proposition 3.1. *For any $\kappa > 0$ and $\rho \leq \frac{\kappa}{2} - 4$,*

(i) a.s. $T_{z_0} < \infty$ and $\lim_{t \to T_{z_0}^-} \lambda_t \in \mathbb{R}$;

(ii) a.s. $\lim_{t \to T_{z_0}^-} \gamma(t) = z_0$.

From the above proposition we see that, if $\rho \leq \frac{\kappa}{2} - 4$, then $\mathbb{P}_{z_0}^{\kappa,\rho}$ is supported by $\Sigma^{\oplus}$, and we may define $\mathbb{P}_{z_0}^{\kappa,\rho} \oplus \mathbb{P}_\kappa$, which is supported by $\Sigma_{\mathcal{L}}$. The pushforward measure $\mathcal{L}_*(\mathbb{P}_{z_0}^{\kappa,\rho} \oplus \mathbb{P}_\kappa)$ is called the law of a standard extended $\mathrm{SLE}_\kappa(\rho)$ curve through z_0, which is supported by the continuous curves in $\overline{\mathbb{H}}$ from 0 to ∞ that pass through z_0. In other words, a standard extended $\mathrm{SLE}_\kappa(\rho)$ curve through z_0 is defined by continuing an $\mathrm{SLE}_\kappa(\rho)$ curve started from 0 with force point at z_0 by an SLE_κ curve in the remaining domain from z_0 to ∞.

Using conformal maps, we may define an extended $\mathrm{SLE}_\kappa(\rho)$ curve (for $\rho \leq \frac{\kappa}{2} - 4$) in a simply connected domain from one prime end to another prime end through an interior point. We will mainly work on extended $\mathrm{SLE}_\kappa(\rho)$ curves in $\mathbb{H}$ from 0 to ∞ through some $z_0 \in \mathbb{H}$, and so will omit the word "standard".

We now derive the local Radon-Nikodym derivative of $\mathbb{P}_{z_0}^{\kappa,\rho}$ w.r.t. $\mathbb{P}_\kappa$. Let $\lambda_t = \sqrt{\kappa} B_t$, and g_t be the Loewner maps driven by λ. Fix $z_0 = x_0 + iy_0 \in \mathbb{H}$. For $0 \leq t < T_{z_0}$, let

$$Z_t = g_t(z_0) - \lambda_t, \quad X_t = \mathrm{Re}\, Z_t, \quad Y_t = \mathrm{Im}\, Z_t, \quad D_t = |g_t'(z_0)|. \tag{3.1}$$

From Loewner's equation, we see that Y_t and D_t satisfy the ODEs:

$$\frac{dY_t}{Y_t} = \frac{-2}{X_t^2 + Y_t^2} dt, \quad \frac{dD_t}{D_t} = \frac{-2(X_t^2 - Y_t^2)}{(X_t^2 + Y_t^2)^2} dt; \tag{3.2}$$

and X_t satisfies the SDE

$$dX_t = -\sqrt{\kappa}dB_t + \frac{2X_t}{X_t^2 + Y_t^2}dt. \tag{3.3}$$

Define

$$M_t^{\kappa,\rho}(z_0) = |Z_t|^{\frac{\rho}{\kappa}} \cdot Y_t^{\frac{\rho^2}{8\kappa}} \cdot D_t^{\frac{\rho}{\kappa}(1-\frac{\kappa}{4}+\frac{\rho}{8})}, \quad 0 \le t < T_{z_0}, \tag{3.4}$$

and

$$G^{\kappa,\rho}(z_0) = M_0^{\kappa,\rho}(z_0) = |z_0|^{\frac{\rho}{\kappa}} \cdot \mathrm{Im}(z_0)^{\frac{\rho^2}{8\kappa}}. \tag{3.5}$$

Using Itô's formula and (3.2)-(3.3), it is straightforward to check that $(M_t^{\kappa,\rho}(z_0))$ is an $(\mathcal{F}_t^0)$-adapted continuous local martingale, and satisfies the SDE:

$$dM_t^{\kappa,\rho}(z_0) = M_t^{\kappa,\rho}(z_0) \cdot \mathrm{Re}\,\frac{\rho/\sqrt{\kappa}}{\lambda_t - g_t(z_0)}\,dB_t, \quad 0 \le t < T_{z_0}.$$

We further define

$$M_t^{\kappa,\rho}(z_0) = 0, \quad T_{z_0} \le t < \infty. \tag{3.6}$$

For any measurable subset U of $\overline{\mathbb{H}}$, define

$$\Psi_t^{\kappa,\rho}(U) = \int_{U\cap\mathbb{H}} M_t^{\kappa,\rho}(z)dA(z). \tag{3.7}$$

Throughout this paper, we use dA to denote the Lebesgue measure on $\mathbb{C}$. Since $(M_t^{\kappa,\rho}(z))$ is a positive local martingale, it is also a supermartingale. Thus, if $\int_U G^{\kappa,\rho}(z)dA(z) < \infty$, then $(\Psi_t^{\kappa,\rho}(U))$ is also a supermartingale, which has to be $\mathbb{P}_\kappa$-a.s. finite.

From Proposition 2.3 we know that

$$\frac{d\mathbb{P}_{z_0}^{\kappa,\rho}|_{\mathcal{F}_t^0\cap\Sigma_t}}{d\mathbb{P}_\kappa|_{\mathcal{F}_t^0\cap\Sigma_t}} = \frac{M_t^{\kappa,\rho}(z_0)}{G^{\kappa,\rho}(z_0)}, \quad 0 \le t < \infty. \tag{3.8}$$

From Proposition 2.1, $(z, E) \mapsto \mathbb{P}_z^{\kappa,\rho}(E)$ is a probability kernel from $\mathbb{H}$ to $(\Sigma, \mathcal{F}^0)$. Thus, for any measurable subset U of $\overline{\mathbb{H}}$, we may define the measure

$$\mathbb{P}_U^{\kappa,\rho} = \int_{U\cap\mathbb{H}} \mathbb{P}_z^{\kappa,\rho} G^{\kappa,\rho}(z)dA(z). \tag{3.9}$$

Moreover, if $\int_U G^{\kappa,\rho}(z)dA(z) < \infty$, then $\mathbb{P}_U^{\kappa,\rho} \lhd \mathbb{P}_\kappa$, and

$$\frac{d\mathbb{P}_U^{\kappa,\rho}|_{\mathcal{F}_t^0\cap\Sigma_t}}{d\mathbb{P}_\kappa|_{\mathcal{F}_t^0\cap\Sigma_t}} = \Psi_t^{\kappa,\rho}(U), \quad 0 \le t < \infty. \tag{3.10}$$

4 Natural Parametrization

Fix $\kappa \in (0, 8)$. Let $d = 1 + \frac{\kappa}{8}$ be the Hausdorff dimension of SLE_κ curves. We will review the natural parametrization of SLE_κ in this section.

First, we review the definition of SLE Green function. The Green function for an SLE_κ curve γ in a simply connected domain D from one prime end a to another prime end b is

$$G_{(D;a,b)}(z) := \lim_{r\to 0^+} r^{d-2}\mathbb{P}[\mathrm{dist}(z,\gamma) < r], \quad z \in D,$$

provided that the limit exists. It is clear that, if the SLE_κ Green function exists for one configuration $(D; a, b)$, then it exists for all configurations, and it satisfies the conformal covariance, i.e., if f maps $(D; a, b)$ conformally onto $(D'; a', b')$, then

$$G_{(D;a,b)}(z) = |f'(z)|^{2-d} G_{(D';a',b')}(f(z)).$$

It is proved in [13] that there is an unknown positive constant $C_\kappa > 0$ depending only on κ such that

$$G_{(\mathbb{H};0,\infty)}(z) = C_\kappa |z|^{d-2} \sin^{\frac{\kappa}{8}+\frac{8}{\kappa}-2}(\arg z), \quad z \in \mathbb{H}, \tag{4.1}$$

We will write $G(z)$ for $G_{(\mathbb{H};0,\infty)}(z)$. Define

$$M_t(z) = |g_t'(z)|^{2-d} G(g_t(z) - \lambda(t)), \quad 0 \le t < T_z.$$

and $M_t(z) = 0$ for $t \ge T_z$. For any measurable set $U \subset \overline{\mathbb{H}}$, define $\Psi_t(U) = \int_{U\cap\mathbb{H}} M_t(z) dA(z)$. It is easy to check that $G(z)$, $M_t(z)$, $\Psi_t(U)$ agree with the C_κ times $G^{\kappa,\kappa-8}(z)$, $M_t^{\kappa,\kappa-8}(z)$, $\Psi_t^{\kappa,\kappa-8}(U)$, respectively, defined by (3.4)-(3.7). From (3.8) we see that, locally weighting $\sqrt{\kappa} B_t$ using $(M_t(z_0)/G(z_0))$ generates a driving $\mathrm{SLE}_\kappa(\kappa-8)$ process with force point at z_0.

A two-sided radial SLE_κ curve is just an extended $\mathrm{SLE}_\kappa(\kappa-8)$ curve. This means that the law of a (standard) two-sided radial SLE_κ curve through $z_0 \in \mathbb{H}$ can be expressed by $\mathcal{L}_*(\mathbb{P}_{z_0}^{\kappa,\kappa-8} \oplus \mathbb{P}_\kappa)$. Two-sided radial SLE is important because it can be understood as an SLE_κ curve conditioned to pass though an interior point. To make this rigorous, one may condition an SLE_κ curve γ on the event that $\mathrm{dist}(z_0, \gamma) < r$ for a fixed interior point z_0, and then pass the limit $r \to 0$.

From the reversibility of chordal SLE_κ (c.f. [25, 17]) it is easy to see that two-sided radial SLE_κ also satisfies reversibility, i.e., the time-reversal of a two-sided radial SLE_κ curve in a simply connected domain D from a to b through z_0 agrees with a two-sided radial SLE_κ curve in D from b to a, up to a reparametrization. This property is not satisfied by extended $\mathrm{SLE}_\kappa(\rho)$ processes for $\rho \ne \kappa - 8$.

Recall that $\mathbb{P}_\cdot^{\kappa,\kappa-8}$ is a kernel from $\mathbb{H}$ to $(\Sigma, \mathcal{F}^0)$. So we can defined the measure $\mathbb{P}_U = \int_{U\cap\mathbb{H}} \mathbb{P}_t^{\kappa,\kappa-8}(z) G(z) dA(z)$ for any measurable set $U \subset \overline{\mathbb{H}}$. Then $\mathbb{P}_U$ equals C_κ times the $\mathbb{P}_U^{\kappa,\kappa-8}$ defined by (3.9). From (3.10), if $\int_U G(z) dA(z) < \infty$, then $\mathbb{P}_U \lhd \mathbb{P}_\kappa$ and

$$\frac{d\mathbb{P}_U|_{\mathcal{F}_t^0\cap\Sigma_t}}{d\mathbb{P}_\kappa|_{\mathcal{F}_t^0\cap\Sigma_t}} = \Psi_t(U), \quad 0 \le t < \infty. \tag{4.2}$$

It is proved by Lawler and Zhou ([16]) that, if U is a pre-compact measurable subset of $\mathbb{H}$, i.e., $\overline{U}$ is a compact subset of $\mathbb{H}$, then $(\Psi_t(U))$ is of class $\mathcal{D}$, i.e., $\{\Psi_T(U) : T \text{ is a finite stopping time}\}$ is uniformly integrable. In this case they can apply the Doob-Meyer decomposition theorem to get a unique continuous increasing $(\mathcal{F}_t^B)$-adapted process $\Theta_t(U)$ such that $\Theta_0(U) = 0$ and $M_t(U) := \Psi_t(U) + \Theta_t(U)$ is a uniformly integrable $(\mathcal{F}_t^B)$-martingale. This means that $M_\infty(U) := \lim_{t\to\infty} M_t(U)$ a.s. exists, and $\mathbb{E}_\kappa[M_\infty(U)|\mathcal{F}_t^B] = M_t(U)$, $0 \le t < \infty$. From Lemma A.2, we know that $\Psi_\infty(U) = 0$. Thus,

$$\mathbb{E}[\Theta_\infty(U)|\mathcal{F}_t^B] - \Theta_t(U) = \Psi_t(U), \quad 0 \le t < \infty. \tag{4.3}$$

The process $(\Theta_t(U))$ determines a $\mathbb{P}_\kappa$-kernel $d\Theta(U)$. Then we may define the killing $\mathcal{K}_{d\Theta(U)}(\mathbb{P}_\kappa)$. From (2.1), we have

$$\mathbb{P}_\kappa - a.s., \quad \frac{d\mathcal{K}_{d\Theta(U)}(\mathbb{P}_\kappa)|_{\mathcal{F}_t^0\cap\Sigma_t}}{d\mathbb{P}_\kappa|_{\mathcal{F}_t^0\cap\Sigma_t}} = \mathbb{E}[\Theta_\infty(U)|\mathcal{F}_t^B] - \Theta_t(U), \quad 0 \le t < \infty. \tag{4.4}$$

Combining (4.2), (4.3) and (4.4), we see that,

$$\mathbb{P}_U = \mathcal{K}_{d\Theta(U)}(\mathbb{P}_\kappa). \tag{4.5}$$

It is easy to check that, if U_1 and U_2 are disjoint pre-compact measurable subsets of $\mathbb{H}$, then a.s. $\Theta_t(U_1 \cup U_2) = \Theta_t(U_1) + \Theta_t(U_2)$ for $0 \le t < \infty$. Thus, the kernel $d\Theta_t(U)$ (and also $\Theta_t(U)$) is increasing in U. We may extend the definition of $\Theta_t(U)$ to any measurable subset U of $\overline{\mathbb{H}}$ as follows. Let (U_n) be a sequence of compact subsets of $\mathbb{H}$ such that U_n is contained in the interior of U_{n+1} for every $n \in \mathbb{N}$, and $\mathbb{H} = \bigcup U_n$. For any measurable subset U of $\overline{\mathbb{H}}$, we first define the kernel $\mu_U = \lim_{n\to\infty} d\Theta(U \cap U_n)$, then let $\Theta_t(U) = \mu_U([0,t])$, $t \ge 0$ (so $d\Theta(U) = \mu_U$). The definition does not depend on the choice of (U_n). In particular, $\Theta_t := \Theta_t(\overline{\mathbb{H}})$ is called the natural parametrization of γ.

The $\Theta_t(U)$ is understood as the total time that the chordal SLE_κ curve γ spends in U in the natural parametrization before time t. It is proved in [13] that the kernel $\mathcal{M}_U$ from $\mathcal{L}(\Sigma^{\mathcal{L}})$ to $\overline{\mathbb{H}}$ defined by

$$\mathcal{M}_U(\gamma, \cdot) := \gamma_*(d\Theta(U)) \tag{4.6}$$

agrees with the d-dimensional Minkowski content measure of $\gamma \cap U$, and $\mathcal{M}_{\mathbb{H}}(\gamma, \cdot)$ is the d-dimensional Minkowski content measure of γ.

Theorem 4.1. *Let $\kappa \in (0,8)$. Let U be any measurable subset of $\overline{\mathbb{H}}$. Then we have*

$$\mathbb{P}_U \widehat{\oplus} \mathbb{P}_\kappa = \mathbb{P}_\kappa \otimes d\Theta(U), \tag{4.7}$$

$$\mathcal{L}_*(\mathbb{P}_z^{\kappa,\kappa-8} \oplus \mathbb{P}_\kappa) \otimes \mathbf{1}_U G(z) dA(z) = \mathcal{L}_*(\mathbb{P}_\kappa) \otimes \mathcal{M}_U, \tag{4.8}$$

where $\mathcal{M}_U$ is the d-dimensional Minkowski content measure of $\gamma \cap U$.

Proof. If U is pre-compact in $\mathbb{H}$, then (4.7) follows from (4.5) and Proposition 2.4. For general measurable $U \subset \mathbb{H}$, (4.7) follows from the above special case and a limiting procedure.

We now apply the extended Loewner map $\widehat{\mathcal{L}}(\lambda, t) = (\mathcal{L}(\lambda), \mathcal{L}(\lambda)(t))$ to (4.7). We observe that $\widehat{\mathcal{L}}_*(\mathbb{P}_z^{\kappa,\kappa-8} \widehat{\oplus} \mathbb{P}_\kappa) = \mathcal{L}_*(\mathbb{P}_z^{\kappa,\kappa-8} \oplus \mathbb{P}_\kappa) \otimes \delta_z$, where δ_z is the Dirac measure at z. Integrating the equality over z against the measure $\mathbf{1}_U G(z) dA(z) = \mathbf{1}_{U \cap \mathbb{H}} G(z) dA(z)$, we get

$$\widehat{\mathcal{L}}_*(\mathbb{P}_U^{\kappa,\kappa-8} \widehat{\oplus} \mathbb{P}_\kappa) = \mathcal{L}_*(\mathbb{P}_z^{\kappa,\kappa-8} \oplus \mathbb{P}_\kappa) \otimes \mathbf{1}_U G(z) dA(z). \tag{4.9}$$

Using (4.6) and the definition of $\widehat{\mathcal{L}}$, we find that

$$\widehat{\mathcal{L}}_*(\mathbb{P}_\kappa \otimes d\Theta(U)) = \mathcal{L}_*(\mathbb{P}_\kappa) \otimes \mathcal{M}_U. \tag{4.10}$$

Then (4.7), (4.9) and (4.10) together imply (4.8). □

Corollary 4.1. *Let $\kappa \in (0,8)$. Let U be a measurable subset of $\mathbb{H}$ with $\int_U G(z) dA(z) < \infty$. If we integrate the laws of two-sided radial SLE_κ curves through z against the measure $\mathbf{1}_U G(z) dA(z)$, then we get a bounded measure on curves, which is absolutely continuous w.r.t. the law of SLE_κ curve, and the Radon-Nikodym derivative is the d-dimensional Minkowski content of $\gamma \cap U$.*

Proof. This follows from projecting both sides of (4.8) to Σ^{C} and that $\mathbb{E}_\kappa[\mathcal{M}_U(\gamma, \cdot)] = \mathbb{E}_\kappa[\Theta_\infty(U)] = \int_U G(z) dA(z) < \infty$. □

Corollary 4.2. *Let $\kappa \in (0,8)$. Suppose (γ, z) is a $C([0,\infty),\mathbb{C}) \times \overline{\mathbb{H}}$-valued random variable with the properties that γ has the law of an SLE$_\kappa$ curve, and given γ, the law of z is absolutely continuous w.r.t. the d-dimensional Minkowski content measure of γ. Then the law of z is absolutely continuous w.r.t. $\mathbf{1}_{\mathbb{H}} dA(z)$, and the law of γ given z is absolutely continuous w.r.t. the law of a two-sided radial SLE$_\kappa$ curve through z.*

Proof. From the assumption, we see that the law of (γ, z) is absolutely continuous w.r.t. the measure in either side of (4.8) with $U = \mathbb{H}$. □

Remarks.

1. Roughly speaking, the meaning of (4.8) is that the following two methods generate the same measure on the space of curve-point pairs:
   (i) first sample a point according to the measure $\mathbf{1}_U G(z) dA(z)$, and then sample a two-sided radial SLE$_\kappa$ curve γ through z;
   (ii) first sample an SLE$_\kappa$ curve γ, and then sample a point z on γ according to the d-dimensional Minkowski content measure of $\gamma \cap U$. Here the measure of γ is changed after sampling z because $\mathcal{M}_U$ is not a probability kernel.
2. Corollary 4.2 means that, if we sample a random point z on a chordal SLE$_\kappa$ curve γ according to a law that is absolutely continuous w.r.t. the d-dimensional Minkowski content measure, and then condition on z, then the curve γ looks "similar" to a two-sided radial SLE$_\kappa$ curve through z.

Corollary 4.3. *Let D be a simply connected domain with two distinct prime ends a and b. Let $\mathbb{P}^\kappa_{D;a,b}$ denote the law of an SLE$_\kappa$ curve in D from a to b. For any $z \in D$, let $\mathbb{P}^\kappa_{D;a,b;z}$ denote the law of a two-sided radial SLE$_\kappa$ curve in D from a to b through z. For any measurable set $U \subset D$, let $\mathcal{M}_U(\gamma, \cdot)$ be the d-dimensional Minkowski content measure on $\gamma \cap U$. Then*

$$\mathbb{P}^\kappa_{D;a,b;z} \otimes \mathbf{1}_U G_{D;a,b}(z) dA(z) = \mathbb{P}^\kappa_{D;a,b} \otimes \mathcal{M}_U. \tag{4.11}$$

Moreover, if $\int_U G_{D;a,b}(z) dA(z) < \infty$, then $\int_U \mathbb{P}^\kappa_{D;a,b;z} dA(z) \ll \mathbb{P}^\kappa_{D;a,b}$, and the Radon-Nikodym derivative is the d-dimensional Minkowski content of $\gamma \cap U$.

Proof. Formula (4.11) follows immediately from (4.8), the conformal covariance of SLE$_\kappa$ Green function, and the transformation rule of d-dimensional Minkowski content measure under confomal maps. The rest follows from projecting the measures in (4.11) to $\Sigma^{\mathbb{C}}$. □

Remark. The above corollary in the case $\kappa \le 4$ was proved earlier by Laurie Field in [7], where he assumed that the domain D is bounded and has analytic boundary.

Inspired by Theorem 4.1 and its corollaries, we make the following definition.

Definition 4.1. *Let $\kappa > 0$ and $\rho \le \frac{\kappa}{2} - 4$. We say that SLE$_\kappa$ admits an SLE$_\kappa(\rho)$ decomposition if (4.7) holds with $G(z)$ replaced by $G^{\kappa,\rho}(z)$ and $\mathbb{P}_U$ replaced by $\mathbb{P}^{\kappa,\rho}_U$, for some increasing adapted process $\Theta_t(U)$, which then implies (4.8) with $\mathbb{P}^{\kappa,\kappa-8}_z$ replaced by $\mathbb{P}^{\kappa,\rho}_z$ and $\mathcal{M}_U$ defined by (4.6) for the new $\Theta_t(U)$.*

We have shown that SLE$_\kappa$ admits an SLE$_\kappa(\kappa-8)$ decomposition for $\kappa \in (0,8)$. From the argument, we see that, one approach to show that SLE$_\kappa$ admits an SLE$_\kappa(\rho)$ decomposition is to show that for any pre-compact measurable subset U of $\mathbb{H}$, $\Psi^{\kappa,\rho}_t(U)$ defined in (3.7) is of

class $\mathcal{D}$, and then apply the Doob-Meyer decomposition theorem. But this is very difficult in general (c.f. [14, 16]). In the next section, we will show that SLE_κ admits an $\mathrm{SLE}_\kappa(-8)$ decomposition for any $\kappa > 0$ using a different approach.

5 Capacity Parametrization

Fix $\kappa > 0$. We say that a chordal SLE_κ curve γ (with the reparametrization given by its definition) has capacity parametrization because the hull K_t determined by $\gamma([0,t])$ has half-plane capacity $2t$ for each $t \geq 0$. In this section we consider $\mathrm{SLE}_\kappa(\rho)$ processes with $\rho = -8$, which turns out to be closely related to the capacity parametrization.

For any $z \in \mathbb{H}$ and $N > 0$, let $\mathbb{P}^{\kappa,-8}_{z;N}$ be the measure on $(\Sigma, \mathcal{F}^0)$ defined by $\mathbb{P}^{\kappa,-8}_{z;N}(E) = \mathbb{P}^{\kappa,-8}_z(E \setminus \Sigma_N)$. Note that $\mathbb{P}^{\kappa,-8}_{z;N}$ is not a probability measure. Then we have $\mathbb{P}^{\kappa,-8}_{z;N} \ll \mathbb{P}^{\kappa,-8}_z$, and so $\mathbb{P}^{\kappa,-8}_{z;N} \lhd \mathbb{P}^{\kappa,-8}_z$. Since $\mathbb{P}^{\kappa,-8}_z \lhd \mathbb{P}_\kappa$, we get $\mathbb{P}^{\kappa,-8}_{z;N} \lhd \mathbb{P}_\kappa$. We now calculate the local Radon-Nikodym derivatives of $\mathbb{P}^{\kappa,-8}_{z;N}$ w.r.t. $\mathbb{P}_\kappa$.

Let $\lambda_t = \sqrt{\kappa} B_t$, and g_t the Loewner maps driven by λ. Let $M_t(z) = M^{\kappa,-8}_t(z)$ and $G(z) = G^{\kappa,-8}(z)$ be as defined by (3.4)-(3.6). Then we have $G(z) = (\mathrm{Im}\, z/|z|)^{8/\kappa}$; $M_t(z) = G(g_t(z) - \lambda_t)|g_t'(z)|^2$, $0 \leq t < T_z$; and $M_t(z) = 0$, $T_z \leq t < \infty$.

Let $t \in [0, \infty)$ and $E \in \mathcal{F}^0_t \cap \Sigma_t$. If $t \geq N$, then $\mathbb{P}^{\kappa,-8}_{z;N}(E) = 0$ because $E \subset \Sigma_t \subset \Sigma_N$. If $t < N$, from (3.8), we get

$$\mathbb{P}^{\kappa,-8}_{z;N}(E) = \mathbb{P}^{\kappa,-8}_z(E \setminus \Sigma_N) = \mathbb{P}^{\kappa,-8}_z(E) - \mathbb{P}^{\kappa,-8}_z(E \cap \Sigma_N)$$

$$= \int_E \frac{M_t(z)}{G(z)} d\mathbb{P}_\kappa - \int_{E \cap \Sigma_N} \frac{M_N(z)}{G(z)} d\mathbb{P}_\kappa = \frac{1}{G(z)} \int_E (M_t(z) - \mathbb{E}_\kappa[M_N(z)|\mathcal{F}^0_t]) d\mathbb{P}_\kappa.$$

Here the third "=" holds because $E \in \mathcal{F}^0_t \cap \Sigma_t$ and $E \cap \Sigma_N \in \mathcal{F}^0_N \cap \Sigma_N$, and the fourth "=" holds because $\mathbb{P}_\kappa$ is supported by $\Sigma_\infty \subset \Sigma_N$. Thus,

$$\frac{d\mathbb{P}^{\kappa,-8}_{z;N}|_{\mathcal{F}^0_t \cap \Sigma_t}}{d\mathbb{P}_\kappa|_{\mathcal{F}^0_t \cap \Sigma_t}} = \begin{cases} \frac{1}{G(z)}(M_t(z) - \mathbb{E}_\kappa[M_N(z)|\mathcal{F}^0_t]), & 0 \leq t < N; \\ 0, & N \leq t < \infty. \end{cases} \tag{5.1}$$

For $t \geq 0$, define G_t on $\mathbb{H}$ by

$$G_t(z) = M_0(z) - \mathbb{E}_\kappa[M_t(z)] = G(z) - \mathbb{E}_\kappa[M_t(z)],$$

and let $C_{\kappa,t} = \int_{\mathbb{H}} G_t(z) dA(z)$. From (3.8), we get

$$\frac{G_t(z)}{G(z)} = 1 - \int_{\Sigma_t} \frac{M_t(z)}{G(z)} d\mathbb{P}_\kappa = 1 - \mathbb{P}^{\kappa,\rho}_z[\Sigma_t] = \mathbb{P}^{\kappa,\rho}_z[T_z \leq t]. \tag{5.2}$$

Thus, $G_0 \equiv 0$, $G_t \geq 0$ for $t > 0$, and

$$\lim_{t \to \infty} G_t(z) = G(z) \mathbb{P}^{\kappa,\rho}_z[T_z < \infty] = G(z). \tag{5.3}$$

From the scaling property of $\mathrm{SLE}_\kappa(-8)$ process, we get $\frac{G_{a^2 t}(az)}{G(az)} = \frac{G_t(z)}{G(z)}$ for any $a > 0$. Since $G(az) = G(z)$, we get $G_{a^2 t}(az) = G_t(z)$. Especially, we have

$$G_t(z) = G_1(\frac{z}{\sqrt{t}}), \quad t > 0. \tag{5.4}$$

Lemma 5.1. *We have $C_{\kappa,t} = tC_{\kappa,1}$ for every $t \geq 0$, and $C_{\kappa,1} \in (0,\infty)$.*

Proof. From (5.4) and that $G_0 \equiv 0$, we get $C_{\kappa,t} = tC_{\kappa,1}$, $t \geq 0$. From (5.3) and (5.4), we get $C_{\kappa,1} > 0$. Now we show $C_{\kappa,1} < \infty$. Fix $z_0 = x_0 + iy_0 \in \mathbb{H}$ and consider an $\mathrm{SLE}_\kappa(-8)$ process started from 0 with force point z_0. We will use some results in Appendix A. We may express T_{z_0} by (A.4), where V_s is a diffusion process that satisfies the SDE (A.3) with initial value $V_0 = \operatorname{arcsinh}(x_0/y_0)$. So we immediately have $T_{z_0} \geq y_0^2 \int_0^\infty e^{-4s} ds = y_0^2/4$. Thus, $T_{z_0} > 1$ if $y_0 > 2$. From (5.2) we see that $G_1 \equiv 0$ on $\{\operatorname{Im} z > 2\}$. From the left-right symmetry, we see that $G_1(x+iy) = G_1(-x+iy)$. We also know that $|G_1(z)| \leq |G(z)| \leq 1$ for all $z \in \mathbb{H}$. Thus, to show that $C_{\kappa,1} = \int_{\mathbb{H}} G_1(z) dA(z) < \infty$, it suffices to prove that there is $M > 0$ such that $\int_M^\infty \int_0^2 G_1(x+iy) dy dx < \infty$.

Suppose $|\sqrt{\kappa} B_s| \leq 1 + bs$ for every $s \geq 0$, where $b > 0$ is to be determined. From (A.3) and that $|\tanh(x)| \leq 1$, we have $V_s \geq V_0 - 1 - bs - (\frac{\kappa}{2} + 4)s$, $s \geq 0$. Thus,

$$\cosh^2(V_2) \geq \frac{1}{4} e^{2V_s} \geq \frac{1}{4} e^{2V_0} \cdot e^{-(2b+\kappa+8)s-2} \geq e^{-(2b+\kappa+8)s-2} \sinh^2(V_0) = \frac{x_0^2}{y_0^2} e^{-(2b+\kappa+8)s-2}.$$

From (A.4) we then get $T_{z_0} \geq \frac{e^{-2} x_0^2}{\kappa+12+2b}$, which then implies $T_{z_0} > 1$ if $2b < e^{-2} x_0^2 - \kappa - 12$. From (A.5) we know that $\mathbb{P}[\sqrt{\kappa}|B_s| \leq 1 + bs, \forall s \geq 0] \geq 1 - 2e^{-\frac{2}{\kappa} b}$. Thus, by taking $2b = e^{-2} x_0^2 - \kappa - 12 - \varepsilon$ and letting $\varepsilon \to 0^+$, we get

$$\mathbb{P}_{z_0}^{\kappa,-8}[T_{z_0} \leq 1] \leq e^{-\frac{1}{\kappa}(e^{-2} x_0^2 - \kappa - 12)}, \quad \text{if } e^{-2} x_0^2 > \kappa + 12.$$

Let $M = \sqrt{e^2(\kappa+12)}$. From (5.2) we get

$$G_1(z_0) \leq G(z_0) \mathbb{P}_{z_0}^{\kappa,-8}[T_{z_0} \leq 1] \leq \mathbb{P}_{z_0}^{\kappa,-8}[T_{z_0} \leq 1] \leq e^{-\frac{1}{\kappa}(e^{-2} x_0^2 - \kappa - 12)}, \quad x_0 > M.$$

So we get $\int_M^\infty \int_0^2 G_1(z_0) dy dx < \infty$, as desired. □

Let m be the Lebesgue measure on $\mathbb{R}$. For any measurable set $U \subset \overline{\mathbb{H}}$, define

$$\Theta_t^{\kappa,-8}(U) = C_{\kappa,1}\, \mathrm{m}(\gamma^{-1}(U) \cap [0,t]), \quad t \geq 0.$$

Especially, we have $\Theta_t^{\kappa,-8} := \Theta_t^{\kappa,-8}(\overline{\mathbb{H}}) = C_{\kappa,1} t$. We can now state the following theorem, which is similar to Theorem 4.1.

Theorem 5.1. *For any measurable set $U \subset \overline{\mathbb{H}}$,*

$$\mathbb{P}_U^{\kappa,-8} \widehat{\oplus} \mathbb{P}_\kappa = \mathbb{P}_\kappa \otimes d\Theta_\cdot^{\kappa,-8}(U), \tag{5.5}$$

$$\mathcal{L}_*(\mathbb{P}_z^{\kappa,-8} \oplus \mathbb{P}_\kappa) \otimes \mathbf{1}_U G^{\kappa,-8}(z) dA(z) = \mathcal{L}_*(\mathbb{P}_\kappa) \otimes \mathcal{M}_U^{\kappa,-8}, \tag{5.6}$$

where $\mathcal{M}_U^{\kappa,-8}$ is the kernel from $\mathcal{L}(\Sigma^{\mathcal{L}})$ to $\overline{\mathbb{H}}$ defined by $\mathcal{M}_U^{\kappa,-8}(\gamma, \cdot) = \gamma_*(d\Theta_\cdot^{\kappa,-8}(U))$.

Proof. From the renewal property of Loewner's equation and the Markov property of $\sqrt{\kappa} B_t$, we see that, if $t \leq N$, then $M_t(z) - \mathbb{E}_\kappa[M_N(z)|\mathcal{F}_t^0] = \mathbf{1}_{T_z > t} |g_t'(z)|^2 G_{N-t}(g_t(z) - \lambda_t)$. Thus,

$$\int_{\mathbb{H}} (M_t(z) - \mathbb{E}_\kappa[M_N(z)|\mathcal{F}_t^0]) dA(z) = \int_{\mathbb{H} \setminus K_t} G_{N-t}(g_t(z) - \lambda_t) |g_t'(z)|^2 dA(z)$$

$$= \int_{\mathbb{H}} G_{N-t}(w)dA(w) = C_{\kappa,N-t} = (N-t)C_{\kappa,1}.$$

Here in the second "=" we use $w = g_t(z) - \lambda_t$ and the fact that g_t maps $\mathbb{H} \setminus K_t$ conformally onto $\mathbb{H}$. The above formula together with Proposition 2.1 and (5.1) implies that $\int_{\mathbb{H}} \mathbb{P}^{\kappa,-8}_{z;N} G(z)dA(z) \lhd \mathbb{P}_\kappa$, and the local Radon-Nikodym derivative is $C_{\kappa,1}(N-t) \vee 0$. From Proposition 2.2 we get

$$\int_{\mathbb{H}} \mathbb{P}^{\kappa,-8}_{z;N} G(z)dA(z) = \mathcal{K}_{C_{\kappa,1}d(t\wedge N)}(\mathbb{P}_\kappa).$$

Letting $N \to \infty$, we get

$$\mathbb{P}^{\kappa,-8}_{\mathbb{H}} = \mathcal{K}_{C_{\kappa,1}dt}(\mathbb{P}_\kappa). \tag{5.7}$$

Using Proposition 2.4, we get

$$\mathbb{P}^{\kappa,-8}_{\mathbb{H}} \widehat{\oplus} \mathbb{P}_\kappa = \mathbb{P}_\kappa \otimes C_{\kappa,1}\, \mathrm{m}\,|_{[0,\infty)}.$$

Applying the extended Loewner map $\widehat{\mathcal{L}}(\lambda, t) = (\mathcal{L}(\lambda), \mathcal{L}(\lambda)(t))$, we get

$$\mathcal{L}_*(\mathbb{P}^{\kappa,-8}_z \oplus \mathbb{P}_\kappa) \otimes G(z)dA(z) = \mathcal{L}_*(\mathbb{P}_\kappa) \otimes \mathcal{M}^{\kappa,-8}, \tag{5.8}$$

where $\mathcal{M}^{\kappa,-8}$ is the kernel from $\mathcal{L}(\Sigma^{\mathcal{L}})$ to $\overline{\mathbb{H}}$ defined by $\mathcal{M}^{\kappa,-8}_U(\gamma, \cdot) = \gamma_*(C_{\kappa,1}\, \mathrm{m}\,|_{[0,\infty)})$. Note that $\mathcal{M}^{\kappa,-8}_U(\gamma, \cdot)$ is the restriction of $\mathcal{M}^{\kappa,-8}(\gamma, \cdot)$ to U. So we get (5.6) by restricting both sides of (5.8) to $\Sigma^{\mathbb{C}} \times U$. Finally, we get (5.5) by applying $\widehat{\mathcal{L}}^{-1}$. □

The following two corollaries are similar to Corollaries 4.1 and 4.2.

Corollary 5.1. *Let U be a measurable subset of $\mathbb{H}$ with $\int_U G^{\kappa,-8}(z)dA(z) < \infty$. If we integrate the laws of extended SLE$_\kappa(-8)$ curve started from 0 with force point at z against the measure $\mathbf{1}_U G^{\kappa,-8}(z)dA(z)$, then we get a bounded measure on curves, which is absolutely continuous w.r.t. the law of an SLE$_\kappa$ curve, and the Radon-Nikodym derivative is $|\mathcal{M}^{\kappa,-8}_U(\gamma, \cdot)| = C_{\kappa,1}\, \mathrm{m}(\gamma^{-1}(U))$.*

Corollary 5.2. *Suppose (γ, t) is a $C([0,\infty), \mathbb{C}) \times [0,\infty)$-valued random variable with the properties that γ has the law of an SLE$_\kappa$ curve, and given γ, the law of t is absolutely continuous w.r.t. the Lebesgue measure on $[0,\infty)$. Then the law of $\gamma(t)$ is absolutely continuous w.r.t. $\mathbf{1}_{\mathbb{H}} dA(z)$, and the law of γ conditioned on $z = \gamma(t)$ is absolutely continuous w.r.t. the law of an extended SLE$_\kappa(-8)$ curve started from 0 with force point at z.*

Remark. From Corollary 5.2 we see that, if we sample a point on an SLE$_\kappa$ curve according to a law that is absolutely continuous w.r.t. the capacity parametrization, and stop the SLE$_\kappa$ curve at that point, then the law of the stopped curve conditioned on that point is absolutely continuous w.r.t. the law of SLE$_\kappa(-8)$ curve. This extends a result in [28], whose argument used the symmetry of backward SLE$_\kappa$ welding for $\kappa \in (0, 4]$ (c.f. [21]) and the conformal removability of SLE$_\kappa$ curves for $\kappa \in (0, 4)$ (c.f. [9, 20]).

More specifically, from [28, Remark 2 after Theorem 6.6], we know that, if $\kappa \in (0, 4)$, then the above conditional stopped curve is the conformal image of an initial segment of a whole-plane SLE$_\kappa(\kappa+2)$ curve, which is also an end segment of a whole-plane SLE$_\kappa(\kappa+2)$ curve, thanks to the reversibility of whole-plane SLE$_\kappa(\rho)$ curve (c.f. [18]). From Proposition B.1, we know that an end segment of a whole-plane SLE$_\kappa(\kappa+2)$ curve can be mapped

conformally to an end segment of a chordal $\mathrm{SLE}_\kappa(-8)$ curve. Thus, Corollary 5.2 extends a weaker version of [28, Remark 2 after Theorem 6.6] from $\kappa \in (0,4)$ to $\kappa \in (0,\infty)$. The result here is weaker because we can not conclude that the conditional stopped curve is exactly the conformal image of an $\mathrm{SLE}_\kappa(-8)$ curve, but can only say that its law is *absolutely continuous* w.r.t. the law of an $\mathrm{SLE}_\kappa(-8)$ curve.

Corollary 5.3. *Let γ be an SLE_κ curve. Then for any measurable set $U \subset \overline{\mathbb{H}}$, we have $\mathbb{E}[\mathrm{m}(\gamma^{-1}(U))] = \frac{1}{C_{\kappa,1}} \int_U G^{\kappa,-8}(z)dA(z)$. Especially, we have a.s. $\mathrm{m}(\gamma^{-1}(\mathbb{R})) = 0$.*

Proof. The first statement follows from computing the total mass of the measures in (5.6) and the fact that $|\mathcal{M}_U^{\kappa,-8}| = C_{\kappa,1}\,\mathrm{m}(\gamma^{-1}(U))$. The second statement follows from taking $U = \mathbb{R}$ and the fact that $\int_R G^{\kappa,-8}(z)dA(z) = 0$. □

Remark. The above corollary says that $\frac{1}{C_{\kappa,1}} G^{\kappa,-8}(z)$ is the density of SLE_κ curve in capacity parametrization, and so may be called the capacity Green function for SLE_κ.

Corollary 5.4. *Let γ be an SLE_κ curve. Then there is a random conformal map W from $\mathbb{H}$ into $\widehat{\mathbb{C}}$ such that $W(\gamma(1)) = \infty$, and the law of $W(\gamma(t))$, $0 \le t \le 1$, is absolutely continuous w.r.t. an end segment of a whole-plane $SLE_\kappa(\kappa+2)$ curve, up to a reparametrization.*

Proof. Let $\widetilde{\gamma}$ be an SLE_κ curve independent of γ. Let $\widetilde{\lambda}_t$ and $\widetilde{g}_t$ be the driving function and Loewner maps associated with $\widetilde{\gamma}$. Let ξ be a random variable with law $\mathbf{1}_{[0,1]}dx$ that is independent of $\widetilde{\gamma}$ and γ. Define $\widehat{\gamma}$ such that $\widehat{\gamma}(t) = \widetilde{\gamma}(t)$ for $0 \le t \le \xi$ and $\widehat{\gamma}(t) = \widetilde{f}_\xi(\widetilde{\lambda}_\xi + \gamma(t-\xi))$ for $\xi \le t < \infty$, where $\widetilde{f}_\xi$ is the continuation of $\widetilde{g}_\xi^{-1}$ to $\overline{\mathbb{H}}$. From the domain Markov property of SLE_κ, we see that $\widehat{\gamma}$ is an SLE_κ curve independent of ξ. Since $\xi+1$ has the law $\mathbf{1}_{[1,2]}dx$, from Corollary 5.2, we see that, conditioned on $z_0 = \widehat{\gamma}(\xi+1)$, the law of $\widehat{\gamma}_t$, $0 \le t \le \xi+1$, is absolutely continuous w.r.t. the law of an $\mathrm{SLE}_\kappa(-8)$ curve started from 0 with force point at z_0. Let W_1 be a conformal map from $\mathbb{H}$ onto $\mathbb{D} := \{|z| < 1\}$ such that $W_1(0) = 1$ and $W_1(z_0) = 0$. From Proposition B.1, we know that W_1 maps an $\mathrm{SLE}_\kappa(-8)$ curve started from 0 with force point at z_0 to a radial $\mathrm{SLE}_\kappa(\kappa+2)$ curve in $\mathbb{D}$ started from 1 with force point at $W_1(\infty)$, up to a reparametrization.

From [10], there is a conformal map W_2 from $\mathbb{D}$ into $\widehat{\mathbb{C}}$ such that $W_2(0) = \infty$, and W_2 maps the radial $\mathrm{SLE}_\kappa(\kappa+2)$ curve to an end segment of a whole-plane $\mathrm{SLE}_\kappa(\kappa+2)$ curve. Let $W_0(z) = \widetilde{f}_\xi(\widetilde{\lambda}_\xi + z)$, which is a conformal map from $\mathbb{H}$ into $\mathbb{H}$. Then $W := W_2 \circ W_1 \circ W_0$ is the conformal map we are looking for. □

Remark. Corollary 5.4 extends a weaker version of [28, Theorem 5.3], which states that, for $\kappa \in (0,4)$, there is a random conformal map W from $\mathbb{H}$ into $\widehat{\mathbb{C}}$ such that $W(\gamma(1)) = 0$, and $W(\gamma(t))$, $0 \le t \le 1$, is an *initial* segment of a whole-plane $\mathrm{SLE}_\kappa(\kappa+2)$ curve, up to a reparametrization.

Corollary 5.5. *If $\int_U G^{\kappa,-8}(z)dA(z) < \infty$, then $M_t^{\kappa,-8}(U) := \Psi_t^{\kappa,-8}(U) + \Theta_t^{\kappa,-8}(U)$ is a uniformly integrable $(\mathcal{F}_t^0)$-martingale.*

Proof. It suffices to show that $M_t^{\kappa,-8}(U) = \mathbb{E}_\kappa[C_{\kappa,1}\,\mathrm{m}(\gamma^{-1}(U))|\mathcal{F}_t^0]$ for every $t \ge 0$. We have $C_{\kappa,1}\,\mathrm{m}(\gamma^{-1}(U)) = C_{\kappa,1}\,\mathrm{m}(\gamma^{-1}(U) \cap [0,t]) + C_{\kappa,1}\,\mathrm{m}(\gamma^{-1}(U) \cap [t,\infty])$, and $C_{\kappa,1}\,\mathrm{m}(\gamma^{-1}(U) \cap [0,t]) = \Theta_t^{\kappa,-8}(U)$ is $\mathcal{F}_t^0$-measurable. Thus, it remains to show that

$$\mathbb{E}_\kappa[C_{\kappa,1}\,\mathrm{m}(\gamma^{-1}(U) \cap [t_0,\infty])|\mathcal{F}_{t_0}^0] = \Psi_{t_0}^{\kappa,-8}(U), \quad t_0 \ge 0.$$

Fix $t_0 \geq 0$. From the domain Markov Property of SLE_κ, there is an SLE_κ curve $\widehat{\gamma}$ independent of $\mathcal{F}^0_{t_0}$ such that $\gamma(t_0+t) = f_{t_0}(\lambda_{t_0} + \widehat{\gamma}(t))$ for all $t \geq 0$, where f_{t_0} is the continuation of $g_{t_0}^{-1}$ to $\overline{\mathbb{H}}$, and λ_t and g_t are the driving function and Loewner maps associated with γ. Let $\widetilde{g}_{t_0}(z) = g_{t_0}(z) - \lambda_{t_0}$ and $\widetilde{f}_{t_0}(z) = f_{t_0}(\lambda_{t_0} + z)$. Conditioned on $\mathcal{F}^0_{t_0}$, we have

$$\mathrm{m}(\gamma^{-1}(U) \cap [t_0, \infty]) = \mathrm{m}(\widehat{\gamma}^{-1}(\widetilde{f}_{t_0}^{-1}(U))) = \mathrm{m}(\widehat{\gamma}^{-1}(\widetilde{f}_{t_0}^{-1}(U) \cap \mathbb{H}))$$

$$= \mathrm{m}(\widehat{\gamma}^{-1}(\widetilde{g}_{t_0}(U \cap (\mathbb{H} \setminus K_{t_0})))$$

From Corollary 5.3, we get

$$\mathbb{E}_\kappa[\mathrm{m}(\gamma^{-1}(U) \cap [t_0, \infty])|\mathcal{F}^0_{t_0}] = \int_{\widetilde{g}_{t_0}(U \cap (\mathbb{H} \setminus K_{t_0}))} G^{\kappa,-8}(z) dA(z)$$

$$= \int_{U \cap (\mathbb{H} \setminus K_{t_0})} G^{\kappa,-8}(g_{t_0}(z) - \lambda_{t_0})|g'_{t_0}(z)|^2 dA(z) = \int_U M^{\kappa,-8}_{t_0}(z) dA(z) = \Psi^{\kappa,-8}_{t_0}(U).$$

This finishes the proof. □

6 Intersection of SLE with the boundary

In this section, we decompose SLE_κ into $\mathrm{SLE}_\kappa(\rho)$ processes with the force point lying on $\mathbb{R}$. An $\mathrm{SLE}_\kappa(\rho)$ process started from $a_0 \in \mathbb{R}$ with force point at $x_0 \in \mathbb{R} \setminus \{a_0\}$ is the solution of the Loewner equation driven by λ_t, $0 \leq t < T_{x_0}$, which is the solution of the SDE:

$$d\lambda_t = \sqrt{\kappa} dB_t + \frac{\rho}{\lambda_t - g^\lambda_t(x_0)}, \quad \lambda_0 = a_0.$$

The Loewner curve driven by λ, which a.s. exists, is called an $\mathrm{SLE}_\kappa(\rho)$ curve started from a_0 with force point at x_0.

Let $\mathbb{P}^{\kappa,\rho}_{x_0}$ denote the law of the driving $\mathrm{SLE}_\kappa(\rho)$ process started from 0 with force point at $x_0 \in \mathbb{R} \setminus \{0\}$. From Proposition 2.3, we know that $\mathbb{P}^{\kappa,\rho}_{x_0} \lhd \mathbb{P}_\kappa$. We now derive the local Radon-Nikodym derivative. Let $\lambda_t = \sqrt{\kappa} B_t$ and g_t be the Loewner maps driven by λ. Define

$$M^{\kappa,\rho}_t(x_0) = |g_t(x_0) - \lambda_t|^{\frac{\rho}{\kappa}} \cdot g'_t(x_0)^{\frac{\rho}{\kappa}(1 - \frac{\kappa}{4} + \frac{\rho}{4})}, \quad 0 \leq t < T_{x_0},$$

and

$$G^{\kappa,\rho}(x_0) = M^{\kappa,\rho}_0(x_0) = |x_0|^{\frac{\rho}{\kappa}}, \quad z \in \mathbb{H}.$$

Direct calculation using Itô's formula shows that $(M^{\kappa,\rho}_t(x_0))$ is an $(\mathcal{F}^0_t)$-adapted continuous local martingale, and satisfies the SDE:

$$dM^{\kappa,\rho}_t(x_0) = M^{\kappa,\rho}_t(x_0) \cdot \frac{\rho/\sqrt{\kappa}}{\lambda_t - g_t(x_0)} dB_t, \quad 0 \leq t < T_{x_0}.$$

We further define $M^{\kappa,\rho}_t(x_0) = 0$ for $t \in [T_{x_0}, \infty)$. From Proposition 2.3 we know that

$$\frac{d\mathbb{P}^{\kappa,\rho}_{x_0}|_{\mathcal{F}^0_t \cap \Sigma_t}}{d\mathbb{P}_\kappa|_{\mathcal{F}^0_t \cap \Sigma_t}} = \frac{M^{\kappa,\rho}_t(x_0)}{G^{\kappa,\rho}(x_0)}, \quad 0 \leq t < \infty.$$

For a measurable subset U of $\mathbb{R}$, define

$$\Psi_t^{\kappa,\rho}(U) = \int_U M_t^{\kappa,\rho}(x)dx, \quad \mathbb{P}_U^{\kappa,\rho} = \int_U \mathbb{P}_x^{\kappa,\rho} G^{\kappa,\rho}(x)dx.$$

For each $x \in \mathbb{R}\backslash\{0\}$, $(M_t^{\kappa,\rho}(x))$ is a supermartingale because it is a positive local martingale. Thus, if $\int_U G^{\kappa,\rho}(x)dx < \infty$, then $(\Psi_t^{\kappa,\rho}(U))$ is also a supermartingale, which has to be $\mathbb{P}_\kappa$-a.s. finite. From Proposition 2.1, we have

$$\frac{d\mathbb{P}_U^{\kappa,\rho}|_{\mathcal{F}_t^0\cap\Sigma_t}}{d\mathbb{P}_\kappa|_{\mathcal{F}_t^0\cap\Sigma_t}} = \Psi_t^{\kappa,\rho}(U), \quad 0 \le t < \infty.$$

It is known that (c.f. [26]), if $\rho \le \frac{\kappa}{2} - 4$, then for an $\mathrm{SLE}_\kappa(\rho)$ curve γ with force point x_0, we have a.s. $\lim_{t\to T_{x_0}^-} \gamma(t) = x_0$, which then implies that $T_{x_0} < \infty$, and $\lim_{t\to T_{x_0}^-} \lambda_t$ converges, where λ is the corresponding driving function. This means that $\mathbb{P}_{x_0}^{\kappa,\rho}$ is supported by $\Sigma^\oplus$, and we may define $\mathbb{P}_{x_0}^{\kappa,\rho} \oplus \mathbb{P}_\kappa$. The pushforward measure $\mathcal{L}_*(\mathbb{P}_{x_0}^{\kappa,\rho} \oplus \mathbb{P}_\kappa)$ is called the law of an extended $\mathrm{SLE}_\kappa(\rho)$ curve started from 0 with force point at x_0.

For $\kappa \in (0,8)$ and $\rho = \kappa - 8$, an extended $\mathrm{SLE}_\kappa(\rho)$ curve with force point at x_0 is also called a two-sided chordal SLE_κ curve through x_0. It can be understood as an SLE_κ curve conditioned to pass though x_0. To make this rigorous, one may condition an SLE_κ curve γ on the event that $\mathrm{dist}(x_0, \gamma) < r$, and then pass the limit $r \to 0$. A two-sided chordal SLE_κ curve also satisfies reversibility, which is similar to that of a two-sided radial SLE_κ curve.

It is proved in [2] that, for $\kappa \in (4,8)$, if U is a pre-compact measurable subset of $\mathbb{R}\setminus\{0\}$, then $(\Psi_t^{\kappa,\kappa-8}(U))$ is of class $\mathcal{D}$, and Doob-Meyer decomposition theorem can be applied to get a unique continuous increasing function $\Theta_t(U)$ such that $\Theta_0(U) = 0$ and $\Psi_t(U) + \Theta_t(U)$ is a uniformly integrable martingale. We may then define $\Theta_t(U)$ for any measurable subset U of $\mathbb{R}$ using a limiting procedure. It is conjectured (c.f. [2] and [12]) that $\Theta_t(U)$ agrees up to a multiplicative constant with the d'-dimensional Minkowski content of $\gamma([0,t]) \cap U$, where $d' := 2 - \frac{8}{\kappa}$ is the Hausdorff dimension of $\gamma \cap \mathbb{R}$ (c.f. [3]). Using the argument in Section 4, we can obtain the following theorem and corollaries.

Theorem 6.1. *Let $\kappa \in (4,8)$. Let U be any measurable subset of $\mathbb{R}$. Then we have*

$$\mathbb{P}_U^{\kappa,\kappa-8} \widehat{\oplus} \mathbb{P}_\kappa = \mathbb{P}_\kappa \otimes d\Theta(U),$$

$$\mathcal{L}_*(\mathbb{P}_x^{\kappa,\kappa-8} \oplus \mathbb{P}_\kappa) \otimes \mathbf{1}_U G^{\kappa,\kappa-8}(x)dx = \mathcal{L}_*(\mathbb{P}_\kappa) \otimes \mathcal{M}_U,$$

where $\mathcal{M}_U$ is the $\mathcal{L}_*(\mathbb{P}_\kappa)$-kernels from $\mathcal{L}(\Sigma^{\mathcal{L}})$ to $\overline{\mathbb{H}}$ defined by $\mathcal{M}_U(\gamma,\cdot) = \gamma_*(d\Theta(U))$.

Corollary 6.1. *Let $\kappa \in (4,8)$. Let U be a measurable subset of $\mathbb{R}$ with $\int_U G^{\kappa,\kappa-8}(x)dx < \infty$. If we integrate the laws of two-sided chordal SLE_κ through x against the measure $\mathbf{1}_U G^{\kappa,\kappa-8}(x)dx$, then we get a bounded measure on curves, which is absolutely continuous w.r.t. the law of an SLE_κ curve, and the Radon-Nikodym derivative is $|\mathcal{M}_U(\gamma,\cdot)| = \Theta_\infty(U)(\mathcal{L}^{-1}(\gamma))$.*

Corollary 6.2. *Let $\kappa \in (4,8)$. Suppose (γ, x) is a $C([0,\infty),\mathbb{C}) \times \mathbb{R}$-valued random variable with the properties that γ has the law of an SLE_κ curve, and given γ, the law of x is absolutely continuous w.r.t. $\mathcal{M}_{\mathbb{R}}(\gamma,\cdot)$. Then the law of x is absolutely continuous w.r.t.* m, *and the law of γ given x is absolutely continuous w.r.t. the law of a two-sided chordal SLE_κ curve through x.*

Remark. Since an SLE_κ curve for $\kappa \in (0,4]$ does not visit any point on $\mathbb{R}\setminus\{0\}$, Corollary 6.1 does not hold in the case $\kappa \in (0,4]$, and neither does the main result in [2].

7 Decomposition of Planar Brownian motion

We now use the argument in the proof of Theorem 4.1 to decompose a planar Brownian motion. We modify the definition of Σ such that the continuous functions take values in $\mathbb{R}^2$. Let $D \subset \mathbb{R}^2$ be a bounded simply connected domain that contains $0 = (0,0)$. Let B_t be a planar Brownian motion that starts from 0, and τ_D the first time that B_t exists D. Let $G_D(\cdot,\cdot)$ denote the Green function in D. Fix $z_0 = (x_0, y_0) \in D \setminus \{0\}$. Consider the Doob's h-transform of B_t with $h = G_D(\cdot, z_0)$. This a diffusion process that satisfies the SDE:

$$dZ_t^{z_0} = dB_t + \frac{\nabla G_D(\cdot, z_0)|_{Z_t^{z_0}}}{G_D(Z_t^{z_0}, z_0)}\, dt, \quad Z_0^{z_0} = 0.$$

We may view $Z_t^{z_0}$ as B_t conditioned to visit z_0 before exiting D. Let $\mathbb{P}$ and $\mathbb{P}_{z_0}^D$ denote the laws of (B_t) and $(Z_t^{z_0})$, respectively. Then $\mathbb{P}_{z_0}^D \lhd \mathbb{P}$, and

$$\frac{d\mathbb{P}_{z_0}^D|_{\mathcal{F}_t^0 \cap \Sigma_t}}{d\mathbb{P}|_{\mathcal{F}_t^0 \cap \Sigma_t}} = \frac{G_D(B_{t\wedge\tau_D}, z_0)}{G_D(0, z_0)}, \quad 0 \le t < \infty.$$

Let $f(w) = \int_D G_D(w,z)dA(z)$. Then f is the solution of the Dirichlet problem $\Delta f \equiv -1$ in D and $f \equiv 0$ on ∂D. Define

$$\mathbb{P}^D = \int_D \mathbb{P}_z^D G_D(0,z)dA(z), \quad \Psi_t = \int_D G_D(B_{t\wedge\tau_D}, z)dA(z) = f(B_{t\wedge\tau_D}).$$

From Proposition 2.1, we see that $\mathbb{P}^D \lhd \mathbb{P}$ and

$$\frac{d\mathbb{P}^D|_{\mathcal{F}_t^0 \cap \Sigma_t}}{d\mathbb{P}|_{\mathcal{F}_t^0 \cap \Sigma_t}} = \Psi_t, \quad 0 \le t < \infty. \tag{7.1}$$

From Itô's formula and the PDE for f, we see that $M_t := \Psi_t + \frac{t}{2}$, $0 \le t < \tau_D$, is a continuous local martingale, and $\lim_{t\to\tau_D^-} M_t = \frac{1}{2}\tau_D$. We further define $M_t = \frac{1}{2}\tau_D$ for $t \ge \tau_D$. Since D is bounded, the probability $\mathbb{P}[\tau_D > N]$ decays exponentially as $N \to \infty$. Also note that f is bounded. Thus, (M_t) is a uniformly integrable martingale, and we have

$$\mathbb{E}[\frac{\tau_D}{2}|\mathcal{F}_t^0] = \mathbb{E}[M_\infty|\mathcal{F}_t^0] = M_t = \Psi_t + \frac{t\wedge\tau_D}{2}, \quad 0 \le t < \infty.$$

Define a process (θ_t) such that $\theta_t = \frac{t\wedge\tau_D}{2}$. Then we have

$$\mathbb{E}[\theta_\infty|\mathcal{F}_t^0] - \theta_t = \Psi_t, \quad 0 \le t < \infty. \tag{7.2}$$

From Proposition 2.2, (7.1) and (7.2), we see that

$$\mathbb{P}^D = \mathcal{K}_{d\theta}(\mathbb{P}). \tag{7.3}$$

Apply first the operation $\widehat{\oplus}\mathbb{P}$ and then the pushforward by the map $(B,t) \mapsto (B, B(t))$ to both sides of (7.3). Using a variation of Proposition 2.4, we obtain the following theorem.

Theorem 7.1. *Let* m *and* dA *denote the Lebesgue measures on* $\mathbb{R}$ *and* $\mathbb{R}^2$*, respectively. Then we have*

$$(\mathbb{P}_z^D \oplus \mathbb{P}) \otimes G_D(z,0)dA(z) = \mathbb{P} \otimes \mathcal{M},$$

where $\mathcal{M}$ *is a kernel from* Σ *to* D *defined by* $\mathcal{M}(B,\cdot) = \frac{1}{2}B_*(\mathrm{m}|_{[0,\tau_D]})$.

Appendices

A Boundedness of $\mathrm{SLE}_\kappa(\rho)$

In this section, we prove Proposition 3.1 (i). Fix $\kappa > 0$ and $\rho \le \frac{\kappa}{2} - 4$. Fix $z_0 = x_0 + iy_0 \in \mathbb{H}$. Let γ be an $\mathrm{SLE}_\kappa(\rho)$ curve with force point at z_0. Let $\lambda(t)$, $0 \le t < T_{z_0}$, be the driving function, and g_t be the corresponding Loewner maps. Define X_t, Y_t, D_t using (3.1). Then the ODEs (3.2) still hold, and now X_t satisfies the SDE:

$$dX_t = -\sqrt{\kappa} dB_t + \frac{(\rho+2)X_t}{X_t^2 + Y_t^2} dt. \tag{A.1}$$

Let $u(t) = \int_0^t \frac{1}{X_s^2 + Y_s^2} ds$. Let $\widehat{X}_s = X_{u^{-1}(s)}$, $\widehat{Y}_s = Y_{u^{-1}(s)}$, and $\widehat{R}_s = \frac{\widehat{X}_s}{\widehat{Y}_s}$. Using (3.2) and (A.1), we find that $\widehat{Y}_s = y_0 e^{-2s}$ and there is a standard Brownian motion $\widehat{B}_s$ such that

$$d\widehat{R}_s = \sqrt{\kappa}\sqrt{1 + \widehat{R}_s^2} d\widehat{B}_s + (\rho + 4)\widehat{R}_s ds. \tag{A.2}$$

Let $V_s = \operatorname{arcsinh}(\widehat{R}_s)$. From Itô's formula, we get

$$dV_s = \sqrt{\kappa} d\widehat{B}_s + (\rho + 4 - \frac{\kappa}{2}) \tanh(V_s) ds. \tag{A.3}$$

From $u(t) = \int_0^t \frac{1}{X_s^2 + Y_s^2} ds$, $\widehat{Y}_s = y_0 e^{-2s}$, and $V_s = \operatorname{arcsinh}(\widehat{R}_s)$, we get

$$T_{z_0} = \int_0^\infty \widehat{Y}_s^2 (1 + \widehat{R}_s^2) ds = y_0^2 \int_0^\infty e^{-4s} \cosh^2(V_s) ds. \tag{A.4}$$

Lemma A.1. *Suppose (V_s) solves (A.3) with $\rho \le \frac{\kappa}{2} - 4$. Then (V_s) can be coupled with a standard Brownian motion $\widetilde{B}_s$ such that $|V_s| \le |V_0 + \sqrt{\kappa}\widetilde{B}_s|$ for all s.*

Proof. From Itô-Tanaka's formula, $|V_s|$ satisfies the SDE:

$$d|V_s| = \operatorname{sign}(V_s)\sqrt{\kappa} d\widehat{B}_s + (\rho + 4 - \frac{\kappa}{2}) \tanh(|V_s|) ds + dL_s^V,$$

where $L_\cdot^V$ is continuous and increasing, and stays constant on the intervals on which $|V_\cdot| > 0$. On the other hand, if $\widetilde{B}_s$ is a standard Brownian motion, then $|V_0 + \sqrt{\kappa}\widetilde{B}_s|$ satisfies the SDE:

$$d|V_0 + \sqrt{\kappa}\widetilde{B}_s| = \operatorname{sign}(V_0 + \sqrt{\kappa}\widetilde{B}_s)\sqrt{\kappa} d\widetilde{B}_s + dL_s^B,$$

where $L_\cdot^B$ is continuous and increasing. Define two standard Brownian motions $B_t^{(1)}$ and $B_t^{(2)}$ such that $B_t^{(1)} = \int_0^t \operatorname{sign}(V_s) d\widehat{B}_s$ and $B_t^{(2)} = \int_0^t \operatorname{sign}(V_0 + \sqrt{\kappa}\widetilde{B}_s) d\widetilde{B}_s$. We may couple $\widetilde{B}_s$ with $\widehat{B}_s$, and so with V_s, such that $B_\cdot^{(1)} = B_\cdot^{(2)}$. Since $\rho + 4 - \frac{\kappa}{2} \le 0$, from the SDE for $|V_s|$ and $|V_0 + \sqrt{\kappa}\widetilde{B}_s|$, we find that $|V_s| - \sqrt{\kappa}B_s^{(1)} - L_s^V$ is decreasing, and $|V_0 + \sqrt{\kappa}\widetilde{B}_s| - \sqrt{\kappa}B_s^{(1)}$ is increasing. Suppose that there is $s_0 \ge 0$ such that $|V_{s_0}| > |V_0 + \sqrt{\kappa}\widetilde{B}_{s_0}|$. Then $s_0 > 0$. Let $s_0' \in (0, s_0)$ be such that $|V_{s_0'}| = |V_0 + \sqrt{\kappa}\widetilde{B}_{s_0'}|$ and $|V_s| > |V_0 + \sqrt{\kappa}\widetilde{B}_s|$ for $s \in (s_0', s_0]$. Then $|V_s| > 0$ on $(s_0', s_0]$. So L_s^V stays constant on $[s_0', s_0]$, which implies that $|V_s| - \sqrt{\kappa}B_s^{(1)}$ is decreasing on $[s_0', s_0]$. Since $|V_0 + \sqrt{\kappa}\widetilde{B}_s| - \sqrt{\kappa}B_s^{(1)}$ is increasing, we conclude that $|V_s| - |V_0 + \sqrt{\kappa}\widetilde{B}_s|$ is decreasing on $[s_0', s_0]$, which contradicts that $|V_{s_0}| > |V_0 + \sqrt{\kappa}\widetilde{B}_{s_0}|$ and $|V_{s_0'}| = |V_0 + \sqrt{\kappa}\widetilde{B}_{s_0'}|$. Thus, $|V_s| \le |V_0 + \sqrt{\kappa}\widetilde{B}_s|$ for all $s \ge 0$. $\square$

We will use the following well-known inequalities about Brownian motions:

$$\mathbb{P}[|\sqrt{\kappa}B_t| \le at+b, \forall t \in [0,\infty)] \ge 1-2e^{-\frac{2ab}{\kappa}}, \quad a,b>0. \tag{A.5}$$

Proof of Proposition 3.1 (i). From (A.4) we get $T_{z_0} \le y_0^2 \int_0^\infty e^{-4s}e^{2|V_s|}ds$. From Lemma A.1, (V_s) may be coupled with a standard Brownian motion $\widetilde{B}_s$ such that $|V_s| \le |V_0+\sqrt{\kappa}\widetilde{B}_s|$. Since $e^{2|V_0|} \le 4\cosh^2(V_0) = 4(1+\widehat{R}_0^2) = 4\frac{|z_0|^2}{y_0^2}$, we get

$$T_{z_0} \le y_0^2 \int_0^\infty e^{-4s}e^{2|V_0|+2\sqrt{\kappa}|\widetilde{B}_s|}ds \le 4|z_0|^2 \int_0^\infty e^{2\sqrt{\kappa}|\widetilde{B}_s|-4s}ds.$$

From (A.5) we see that the probability that $\sqrt{\kappa}|\widetilde{B}_s| \le s+b$ for all $s \ge 0$ is at least $1-2e^{-\frac{2}{\kappa}b}$, and on this event, from the above formula we get $T_{z_0} \le 2|z_0|^2e^{2b}$. Thus,

$$\mathbb{P}_{z_0}^{\kappa,\rho}[T_{z_0} \le 2|z_0|^2e^{2b}] \ge 1-2e^{-\frac{2}{\kappa}b}, \quad b>0. \tag{A.6}$$

This implies that $\mathbb{P}_{z_0}^{\kappa,\rho}$-a.s., $T_{z_0}<\infty$.

Recall that $\lambda_t = \sqrt{\kappa}B_t + \int_0^t \frac{-\rho X_r}{X_r^2+Y_r^2}dr$, $0 \le t < T_{z_0}$. The finiteness of T_{z_0} implies that a.s. $\lim_{t\to T_{z_0}^-} B_t \in \mathbb{R}$. For the other term, consider

$$\int_0^{T_{z_0}} \frac{|X_t|}{X_t^2+Y_t^2}dt = \int_0^\infty |\widehat{X}_s|ds = \int_0^\infty \widehat{Y}_s|\sinh(V_s)|ds \le y_0 \int_0^\infty e^{|V_s|-2s}ds.$$

Coupling (V_s) with $(\widetilde{B}_s)$ using Lemma A.1 and then using (A.5), we find that $\mathbb{P}_{z_0}^{\kappa,\rho}$-a.s., $\sup_{s\ge 0}(|V_s|-s)<\infty$. Thus, $\mathbb{P}_{z_0}^{\kappa,\rho}$-a.s., $\int_0^{T_{z_0}} \frac{|X_t|}{X_t^2+Y_t^2}dt<\infty$, which implies that the limit $\lim_{t\to T_{z_0}^-}\int_0^t \frac{X_r}{X_r^2+Y_r^2}dr$ exists and is finite. Thus, $\mathbb{P}_{z_0}^{\kappa,\rho}$-a.s., $\lim_{t\to T_{z_0}^-}\lambda_t \in \mathbb{R}$. □

Corollary A.1. *Almost surely $\gamma([0,T_{z_0}))$ is bounded.*

Proof. This follows from Proposition 3.1 (i) and Lemma 4.1 in [10]. □

Lemma A.2. *For any bounded measurable $U \subset \overline{\mathbb{H}}$, $\mathbb{P}_\kappa$-a.s., $\Psi_t^{\kappa,\rho}(U) \to 0$ as $t\to\infty$.*

Proof. Taking $e^{2b} = t/(2|z_0|^2)$ in (A.6) for some $t>0$, we get

$$\mathbb{P}_{z_0}^{\kappa,\rho}[T_{z_0}>t] \le 2^{1+1/\kappa}(t/|z_0|^2)^{-1/\kappa}, \quad t>0.$$

Using (3.8), we get

$$\mathbb{E}_\kappa[M_t^{\kappa,\rho}(z_0)] = G^{\kappa,\rho}(z_0)\mathbb{P}_{z_0}^{\kappa,\rho}[T_{z_0}>t] \le 2^{1+\frac{1}{\kappa}}|z_0|^{\frac{\rho+2}{\kappa}+\frac{\rho^2}{8\kappa}}\cdot t^{-\frac{1}{\kappa}}, \quad t>0.$$

Suppose $|z| \le R$ for every $z \in U$. Then

$$\mathbb{E}_\kappa[\Psi_t^{\kappa,\rho}(U)] = \int_U \mathbb{E}_\kappa[M_t^{\kappa,\rho}(z)]dA(z) \le \pi 2^{\frac{1}{\kappa}}R^{2+\frac{\rho+2}{\kappa}+\frac{\rho^2}{8\kappa}}\cdot t^{-\frac{1}{\kappa}}, \quad t>0.$$

Thus, $\lim_{t\to\infty}\mathbb{E}_\kappa[\Psi_t^{\kappa,\rho}(U)]=0$. Since $(\Psi_t^{\kappa,\rho}(U))$ is a positive supermartingale, from Doob's martingale convergence theorem, we see that $\mathbb{P}_\kappa$-a.s., $\lim_{t\to\infty}\Psi_t^{\kappa,\rho}(U)$ exists. From Fatou's lemma, $\mathbb{E}_\kappa[\lim_{t\to\infty}\Psi_t^{\kappa,\rho}(U)] \le \lim_{t\to\infty}\mathbb{E}_\kappa[\Psi_t^{\kappa,\rho}(U)]=0$. So we get the conclusion. □

Remark. Proposition 3.1 (i) and Lemma A.2 also hold if $\frac{\kappa}{2}-4<\rho<\frac{\kappa}{2}-2$. In that case we may use the estimate $|V_s| \le |V_0|+\sqrt{\kappa}|\widehat{B}_s|+(\rho+4-\frac{\kappa}{2})s$, which follows from (A.3) and that $|\tanh(x)| \le 1$ for $x \in \mathbb{R}$. Then we may apply (A.5) with $a \in (0, \frac{\kappa}{2}-2-\rho)$.

B Transience of $\mathrm{SLE}_\kappa(\rho)$

In this section, we prove Proposition 3.1 (ii). We will use radial $\mathrm{SLE}_\kappa(\rho)$ processes. Let's review the radial Loewner equation and radial $\mathrm{SLE}_\kappa(\rho)$ processes. The radial Loewner equation driven by $\lambda \in C([0,T))$ is the equation

$$\partial_t g_t(z) = g_t(z)\frac{e^{i\lambda_t} + g_t(z)}{e^{i\lambda_t} - g_t(z)}, \quad g_0(z) = z. \tag{B.1}$$

For $t \geq 0$, let K_t be the set of $z \in \mathbb{D} := \{|z| < 1\}$ such that $s \mapsto g_s(z)$ blows up before or at t. Then each g_t maps $\mathbb{D} \setminus K_t$ conformally onto $\mathbb{D}$ with $g_t(0) = 0$ and $g_t'(0) = e^t$. We call g_t and K_t radial Loewner maps and hulls, respectively, driven by λ.

Let $w_1, w_2 \in \mathbb{T} := \partial\mathbb{D}$. Pick any $x_1, x_2 \in \mathbb{R}$ such that $w_j = e^{ix_j}$, $j = 1, 2$. Let B_t be a standard Brownian motion. Let λ_t and q_t, $0 \leq t < T_0$, be the solution with maximal interval of the following system of equations:

$$\begin{cases} d\lambda_t = \sqrt{\kappa}dB_t + \frac{\rho}{2}\cot_2(\lambda_t - q_t), & \lambda_0 = x_1; \\ dq_t = \cot_2(q_t - \lambda_t)dt. & q_0 = x_2. \end{cases} \tag{B.2}$$

Here we use $\cot_2(x) := \cot(x/2)$. Then the solution of the radial Loewner equation driven by λ_t, $0 \leq t < T_0$, is called a (standard) radial $\mathrm{SLE}_\kappa(\rho)$ process started from w_1 with force point at w_2. The definition does not depend on the choices of x_1 and x_2. Suppose g_t are the radial Loewner maps driven by λ. Then for $t \in [0, T_0)$, K_t has positive distance from w_2, and $g_t(w_2)$ is defined and equal to e^{iq_t}. From Proposition B.1 below we know that a radial Loewner curve driven by λ a.s. exists. That means that a.s. for each $0 \leq t < T_0$, g_t^{-1} extends continuously to $\overline{\mathbb{D}}$, and $\gamma(t) := f_t(e^{i\lambda_t})$, $0 \leq t < T_0$, is a continuous curve in $\overline{\mathbb{D}}$, where f_t is the continuation of g_t^{-1} from $\overline{\mathbb{D}}$ into $\overline{\mathbb{D}}$. Moreover, for each $0 \leq t < T_0$, $\mathbb{D} \setminus K_t$ is the connected component of $\mathbb{D} \setminus \gamma([0,t])$ that contains 0.

It is easy to check that the above radial $\mathrm{SLE}_\kappa(\rho)$ curve γ satisfy rotation and domain Markov properties. The rotation property says that, for any $C \in \mathbb{T}$, $C \cdot \gamma$ is a radial $\mathrm{SLE}_\kappa(\rho)$ curve started from $C \cdot w_1$ with force point at $C \cdot w_2$. The domain Markov property says that, if τ is a stopping time with $\tau < T_0$, then conditioned on $\gamma(t)$, $t \leq \tau$, there is a radial $\mathrm{SLE}_\kappa(\rho)$ curve $\widehat{\gamma}$ started from $e^{i\lambda_T}$ with force point at e^{iq_T} such that $\gamma(\tau + t) = f_\tau(\widehat{\gamma}(t))$, $0 \leq t < T_0 - \tau$.

The following proposition about the conformal equivalence between chordal $\mathrm{SLE}_\kappa(\rho)$ processes and radial $\mathrm{SLE}_\kappa(\rho)$ processes is a special case of [23].

Proposition B.1. *Let γ be a chordal $SLE_\kappa(\rho)$ curve started from $a_0 \in \mathbb{R}$ with force point at $z_0 \in \mathbb{H}$. Let W be a conformal map from $\mathbb{H}$ onto $\mathbb{D}$ such that $W(z_0) = 0$. Then, after a time change, $W \circ \gamma$ is a radial $SLE_\kappa(\kappa - 6 - \rho)$ curve started from $W(a_0)$ with force point at $W(\infty)$, and stopped whenever the force point is disconnected from 0 by $W \circ \gamma$.*

Lemma B.1. *Let γ be a radial $SLE_\kappa(\rho)$ curve with force point w_2. If $\rho \geq \frac{\kappa}{2} - 2$, then a.s. the force point $w_2 \notin \overline{\gamma}$.*

Proof. This follows immediately from Corollary A.1 and Proposition B.1. □

Lemma B.2. *Let $\gamma(t)$, $0 \leq t < T_0$, be a radial $SLE_\kappa(\rho)$ curve with force point at $w_2 \in \mathbb{T}$. Let g_t and K_t, $0 \leq t < T_0$, be the corresponding maps and hulls. Let f_t be the continuation of g_t^{-1} from $\mathbb{D}$ to $\overline{\mathbb{D}}$. Suppose τ is a stopping time for γ with $0 < \tau < T_0$. Then a.s. $I_\tau := \mathbb{T} \setminus \overline{K}_\tau$ is an open arcs on $\mathbb{T}$ that contains w_2, and $f_\tau(\overline{\mathbb{D}} \setminus \overline{g_\tau(I_\tau)}) \subset \mathbb{D}$.*

Proof. Since $\tau < T_0$, $g_t(w_2)$ does not blow up at τ, so $\mathrm{dist}(w_2, K_\tau) > 0$, which implies that $\mathbb{T} \setminus \overline{K_\tau}$ contains an open arc containing w_2. To conclude the proof, it suffices to show that $\overline{K}_\tau$ is connected and has no cut point on $\mathbb{T}$. Using Proposition B.1, it suffices to show that, if L_t, $0 \le t < S_0$, are the hulls generated by a chordal $\mathrm{SLE}_\kappa(\kappa - 6 - \rho)$ process, and τ' is a stopping time for (L_t) with $0 < \tau' < S_0$, then $L_{\tau'}$ is connected and has no cut point on $\mathbb{R}$. From Girsanov's Theorem, it suffices to show the above result for chordal SLE_κ hulls L_t. This is clearly true if $\kappa \in (0, 4]$, since each L_t is the image of a simple curve that intersects $\mathbb{R}$ at only one point. For $\kappa > 4$, the result follows from [27, Theorem 6.1]. $\square$

From (B.2), we see that $L_t := q_t - \lambda_t$ satisfies the SDE

$$dL_t = -\sqrt{\kappa} dB_t + (\frac{\rho}{2} + 1) \cot_2(L_t) dt, \quad 0 \le t < T_0. \tag{B.3}$$

We may pick x_1 and x_2 such that $L_0 = x_2 - x_1 \in (0, 2\pi)$. We see that if $T_0 < \infty$, then $\lim_{t \to T_0^-} L_t$ exists and is either 0 or 2π. From now on, assume that $\rho \ge \frac{\kappa}{2} - 2$. Define a differentiable function f on $(0, 2\pi)$ such that $f'(x) = \sin(x/2)^{-\frac{2}{\kappa}(\rho+2)}$ for $x \in (0, 2\pi)$. Since $\rho \ge \frac{\kappa}{2} - 2$, we see that $f((0, 2\pi)) = \mathbb{R}$. From Itô's formula, we see that $f(L_t)$ is a continuous local martingale that satisfies the SDE $df(L_t) = -\sin(L_t/2)^{-\frac{2}{\kappa}(\rho+2)} \sqrt{\kappa} dB_t$. Define u such that $u(0) = 0$ and $u'(t) = \kappa \sin(L_t/2)^{-\frac{4}{\kappa}(\rho+2)}$, $0 \le t < T_0$. Let $R_0 = u(T_0^-)$ and let $v : [0, R_0) \to [0, T_0)$ be the inverse of u. Then $f(L_{v(t)})$, $0 \le t < R_0$, is a standard Brownian motion stopped at R_0. We claim that a.s. this Brownian motion does not stop, i.e., $R_0 = \infty$. In fact, if $R_0 < \infty$, then a.s. $\lim_{t \to R_0^-} f(L_{v(t)}) \in \mathbb{R}$, which then implies that $\lim_{t \to T_0^-} L_t \in (0, 2\pi)$. Since $\lim_{t \to T_0^-} L_t$ does not equal to 0 or 2π, we have $T_0 = \infty$. From $\lim_{t \to \infty} L_t \in (0, 2\pi)$, we then get $\lim_{t \to \infty} u'(t) \in (0, \infty)$, which contradicts that $R_0 = \int_0^{T_0} u'(t) dt < \infty$. Thus, a.s. $R_0 = \infty$. From the property of Brownian motion, we know that a.s. $\limsup_{t \to T_0^-} f(L_t) = \infty$ and $\liminf_{t \to T_0^-} f(L_t) = -\infty$, which then implies that $\limsup_{t \to T_0^-} L_t = 2\pi$ and $\liminf_{t \to T_0^-} L_t = 0$, i.e., (L_t) is recurrent. Since L_t does not tend to 0 or 2π as $t \to T_0^-$, we can conclude that $T_0 = \infty$.

The following theorem is about the transience of radial $\mathrm{SLE}_\kappa(\rho)$ processes, which is equivalent to the transience of whole-plane $\mathrm{SLE}_\kappa(\rho)$ processes. It was proved in [18] using GFF/SLE coupling that whole-plane $\mathrm{SLE}_\kappa(\rho)$ satisfies transience for all $\kappa > 0$ and $\rho > -2$. A special case when $\kappa \in (0, 8)$ and $\rho = 2$ (transience of two-sided radial SLE_κ) was proved earlier in [11]. We now give a different proof of the transience property in the case that $\rho \ge \frac{\kappa}{2} - 2$, in which the force point is never swallowed.

Proposition B.2. *Let $\kappa > 0$ and $\rho \ge \frac{\kappa}{2} - 2$. Let $w_1 \ne w_2 \in \mathbb{T}$. Let $\gamma(t)$, $0 \le t < \infty$, be a radial $\mathrm{SLE}_\kappa(\rho)$ curve started from w_1 with force point at w_2. Then a.s. $\lim_{t \to \infty} \gamma(t) = 0$.*

Proof. From rotation property, we may assume that $w_1 = 1$. Let $x_1 = 0$ and $x_2 \in (0, 2\pi)$ be such that $w_j = e^{ix_j}$, $j = 1, 2$. Let λ_t and q_t, $0 \le t < \infty$, be the solution of (B.2) such that λ is the driving function for γ. Let g_t be the radial Loewner maps driven by λ. Let $L_t = q_t - \lambda_t$, $0 \le t < \infty$. Then (L_t) is a recurrent process that solves (B.3). Define $S = \bigcap_{t \ge 0} \overline{\gamma([t, \infty))}$. Then S is a nonempty compact subset of $\overline{\mathbb{D}}$ composed of all subsequential limits of $\gamma(t)$ as $t \to \infty$. It suffices to show that a.s. $S = \{0\}$.

Since each g_t maps $\mathbb{D} \setminus K_t$ conformally onto $\mathbb{D}$ with $g_t(0) = 0$ and $g_t'(0) = e^t$, we see that the conformal radius of 0 in $\mathbb{D} \setminus K_t$ is e^{-t}. From Schwarz lemma, the distance from 0 to $K_t \cup \mathbb{T}$ is no more than e^{-t}. For $t \ge 0$, let ρ_t be the connected component of $\{|z| = e^{-1}\} \cap (\mathbb{D} \setminus K_t)$

that disconnect 0 from $\mathbb{T}\setminus\overline{K_t}$ in $\mathbb{D}\setminus K_t$. From Beurling estimate (c.f. Theorem 3.69 of [10]), there is a positive constant c such that for any $t\geq 0$, the probability that a planar Brownian motion started from 0 hits ρ_t before K_t is no more than $ce^{-t/2}$, which then implies that the harmonic measure of $g_t(\rho_t)$ in $\mathbb{D}\setminus g_t(\rho_t)$ viewed from 0 is no more than $ce^{-t/2}$. Suppose $t>1$. Then ρ_t is a crosscut in $\mathbb{D}\setminus K_t$. So $g_t(\rho_t)$ is a crosscut in $\mathbb{D}$. The harmonic measure estimate then implies that the diameter of $g_t(\rho_t)$ is no more than $c'e^{-t/2}$ for another positive constant c'.

Fix $\varepsilon>0$. From Lemma B.1 and that $S\subset\overline{\gamma}$, we can find $r>0$ such that the probability that $\mathrm{dist}(w_2,S)<r$ is less than ε. Choose $t_0>1$ such that $c'e^{-t_0/2}<r$. Let τ be the first $t\geq t_0$ such that $L_t=L_0$. Since (L_t) is recurrent, τ is a finite stopping time. From the rotation and domain Markov properties of γ, we know that there is a random curve $\widehat{\gamma}$, which has the same law as γ, and is independent of $\gamma(t)$, $t\leq\tau$, such that $\gamma(\tau+t)=\widehat{f}_\tau(\widehat{\gamma}(\tau+t))$, $t\geq 0$, where $\widehat{f}_\tau(z)=f_\tau(e^{i\lambda_\tau}\cdot z)$. Recall that f_τ is the continuation of g_τ^{-1} from $\overline{\mathbb{D}}$ into $\overline{\mathbb{D}}$. Let $\widehat{S}=\bigcap_{t\geq 0}\overline{\widehat{\gamma}([t,\infty))}$. Then $\widehat{S}$ has the same law as S, and $S=\widehat{f}_\tau(\widehat{S})$. Since g_τ fixes 0 and maps $w_2=e^{iq_0}$ to e^{iq_τ}, we see that $\widehat{f}_\tau$ fixes both 0 and $w_2=e^{iq_0}=e^{iL_0}=e^{iL_\tau}$.

Since $\tau\geq t_0>1$, $\widehat{f}_\tau^{-1}(\rho_\tau)=e^{-i\lambda_\tau}\cdot g_\tau(\rho_\tau)$ is a crosscut in $\mathbb{D}$ with diameter no more than $c'e^{-\tau/2}\leq c'e^{-t_0/2}<r$. Since ρ_τ disconnects the arc $\mathbb{T}\setminus\overline{K}_\tau$ from 0 in $\mathbb{D}\setminus K_\tau$, $\widehat{f}_\tau^{-1}(\rho_\tau)$ disconnects the arc $\widehat{f}_\tau^{-1}(\mathbb{T}\setminus\overline{K}_\tau)$ from 0 in $\mathbb{D}$. Since $\mathbb{T}\setminus\overline{K}_\tau$ contains w_2, so does $\widehat{f}_\tau^{-1}(\mathbb{T}\setminus\overline{K}_\tau)$. From Lemma B.2, we see that, a.s. for any $z\in\overline{\mathbb{D}}\setminus\widehat{f}_\tau^{-1}(\mathbb{T}\setminus\overline{K}_\tau)$, $\widehat{f}_\tau(z)\in\mathbb{D}$. Since $\widehat{S}$ has the same law as S, from the choice of r, we see that the probability that $\mathrm{dist}(w_2,\widehat{S})\geq r$ is at least $1-\varepsilon$. Suppose $\mathrm{dist}(w_2,\widehat{S})\geq r$. Since $\widehat{f}_\tau^{-1}(\rho_\tau)$ is a crosscut in $\mathbb{D}$ of diameter less than r that disconnects 0 from the arc $\widehat{f}_\tau^{-1}(\mathbb{T}\setminus\overline{K}_\tau)$, which contains w_2, we have $\widehat{S}\cap\widehat{f}_\tau^{-1}(\mathbb{T}\setminus\overline{K}_\tau)=\emptyset$, which implies that $S=\widehat{f}_\tau(\widehat{S})\subset\mathbb{D}$. So the probability that $S\subset\mathbb{D}$ is at least $1-\varepsilon$. Since $\varepsilon>0$ is arbitrary, we conclude that a.s. $S\subset\mathbb{D}$ and $\widehat{S}=\widehat{f}_\tau^{-1}(S)\subset\mathbb{D}$. From (B.1), we see that $|\widehat{f}_\tau(z)|=|g_\tau^{-1}(z)|\leq|z|$ for any $z\in\overline{\mathbb{D}}$, and $|\widehat{f}_\tau(z)|<|z|$ if $z\in\mathbb{D}\setminus\{0\}$. Let $r_S=\max\{|z|:z\in S\}$ and $r_{\widehat{S}}=\max\{|z|:z\in\widehat{S}\}$. Since $\widehat{S}\subset\overline{\mathbb{D}}$, and $S=\widehat{f}_\tau(\widehat{S})$, we have $r_S\leq r_{\widehat{S}}$, and $r_S<r_{\widehat{S}}$ whenever $\widehat{S}\subset\mathbb{D}$ and $\widehat{S}\not\subset\{0\}$. Since S and $\widehat{S}$ have the same law and a.s. $\widehat{S}\subset\mathbb{D}$, we conclude that a.s. $\widehat{S}=\{0\}$, and so does S. □

Proof of Proposition 3.1 (ii). This follows from Propositions B.1 and B.2. □

Remark. The argument in Sections 4 and 5 do not really rely on Proposition 3.1 because the proposition in the case $\rho=\kappa-8$ and $\rho=-8$ follows immediately from (4.5) and (5.7), respectively.

References

[1] L. V. Ahlfors. *Conformal invariants: topics in geometric function theory.* McGraw-Hill Book Co., New York, 1973.

[2] T. Alberts, S. Sheffield. The Covariant Measure of SLE on the Boundary, *Probab. Theory Rel.*, **149**:331-371, 2011.

[3] T. Alberts, S. Sheffield. Hausdorff dimension of the SLE curve intersected with the real line, *Electron. J. Probab.*, **40**:1166-1188, 2008.

[4] V. Beffara. The dimension of the SLE curves. *Ann. Probab.*, **36**(4):1421-1452, 2008.

[5] C. Benes, F. J. Viklund, G. Lawler. Conformal invariance of the loop-erased random walk intensity. arXiv:1402.7345, 2014.

[6] F. Camia, C. Newman. Critical Percolation Exploration Path and SLE(6): a Proof of Convergence. *Probab. Theory Rel.*, **139**:473-520, 2007.

[7] L. Field. Two-sided radial SLE and length-biased chordal SLE. In preprint, 2015.

[8] L. Field and G. Lawler. Escape probability and transience for SLE. Electron. J. Probab., **20**(10):1-14, 2015.

[9] P. Jones and S. Smirnov. Removability theorems for Sobolev functions and quasiconformal maps. *Ark. Mat.*, **38**(2):263-279, 2000.

[10] G. Lawler. *Conformally Invariant Processes in the Plane*, Amer. Math. Soc., 2005.

[11] G. Lawler. Continuity of radial and two-sided radial SLE at the terminal point. In the *Tradition of Ahlfors-Bers. VI. Contemp. Math.*, **590**:101-124. Amer. Math. Soc., Providence, RI, 2013.

[12] G. Lalwer. Minkowski content of the intersection of a Schramm-Loewner evolution (SLE) curve with the real line. In preprint, 2014.

[13] G. Lawler and M. Rezaei. Minkowski content and natural parameterization for the Schramm-Loewner evolution. *Ann. Probab.*, **43**:1082-1120, 2015.

[14] G. Lawler and S. Sheffield. A natural parametrization for the Schramm-Loewner evolution. *Ann. Probab.*, **39**,:1896-1937, 2011.

[15] G. Lawler. O. Schramm, and W. Werner. Conformal invariance of planar loop-erased random walks and uniform spanning trees, *Ann. Probab.*, **32**:939–995, 2004.

[16] G. Lawler and W. Zhou, SLE curves and natural parametrization. *Ann. Probab.*, **41**:1556-1584, 2013.

[17] J. Miller and S. Sheffield. Imaginary Geometry III: reversibility of SLE_κ for $\kappa \in (4, 8)$, arXiv:1201.1498.

[18] J. Miller and S. Sheffield. Imaginary Geometry IV: interior rays, whole-plane reversibility, and space-filling trees, arXiv:1302.4738.

[19] D. Revuz and M. Yor. *Continuous Martingales and Brownian Motion.* Springer, Berlin, 1991.

[20] S. Rohde and O. Schramm. Basic properties of SLE, *Ann. Math.*, **161**:879–920, 2005.

[21] S. Rohde and D. Zhan. Backward SLE and the symmetry of the welding. Available at arXiv:1307.2532. To appear in *Probab. Theory Rel.*.

[22] O. Schramm. Scaling limits of loop-erased random walks and uniform spanning trees. *Israel J. Math.*, **118**:221-288, 2000.

[23] O. Schramm and D. B. Wilson. SLE coordinate changes. *New York Journal of Mathematics*, **11**:659-669, 2005.

[24] S. Smirnov. Critical percolation in the plane: Conformal invariance, Cardy's formula, scaling limits. *C. R. Acad. Sci. Paris Ser. I Math.* **333**:239–244, 2001.

[25] D. Zhan. Reversibility of chordal SLE. *Ann. Probab.*, **36**(4):1472-1494, 2008.

[26] D. Zhan. Duality of chordal SLE. *Invent. Math.*, **174**(2):309-353, 2008.

[27] D. Zhan. Duality of chordal SLE, II. *Ann. I. H. Poincare-Pr.*, **46**(3):740-759, 2010.

[28] D. Zhan. Ergodicity of the tip of an SLE curve. Available at arXiv:1310.2573. To appear in *Probab. Theory Rel.*.