Loop-Erasure of Planar Brownian Motion

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Abstract

We use a coupling technique to prove that there exists a loop-erasure of the time-reversal of a planar Brownian motion stopped on exiting a simply connected domain, and that the loop-erased curve is a radial SLE_2 curve. This result extends to Brownian motions and Brownian excursions under certain conditioning in a finitely connected plane domain, and the loop-erased curve is a continuous LERW curve.

1 Introduction

In this paper we will derive the existence of a loop-erasure of the time-reversal of a planar Brownian motion up to some finite stopping time. It is well-known that simple random walks on a regular lattice such as $\delta\mathbb{Z}^2$ converge to planar Brownian motions as the mesh $\delta \to 0$. The loop-erasure of a simple random walk is called a loop-erased random walk (LERW). Lawler, Schramm, and Werner proved [8] that the LERW on the discrete approximation of a simply connected domain converges to the Schramm-Loewner evolution (SLE) [4] with parameter $\kappa = 2$, i.e., SLE_2, when the mesh tends to 0. So it is reasonable to conjecture that a planar Brownian motion in a simply connected domain a.s. has a unique (up to equivalence) loop-erasure, which is an SLE_2 curve. In this paper we will prove the existence. The uniqueness is still open to the author. In addition, we expect that there exists a deterministic algorithm to erase the loops on the Brownian motion. This is also not solved in this paper. The result in this paper extends naturally to finitely connected domains. For simplicity, we will only deal with simply connected domains, and work on the time-reversal of the Brownian motion.

From [1], the Hausdorff dimension of SLE_2 curve is $5/4$. The result of this paper implies that the percolation dimension ([2]) of a planar Brownian motion (the minimal Hausdorff dimension of a subpath of a Brownian path) is no more than $5/4$. This value is strictly less than the boundary dimension of planar Brownian motion, which is equal to $4/3$ ([5][6]).

In [7], Lawler, Schramm, and Werner proved that, by adding Brownian bubbles to a chordal SLE_2 curve and filling the holes, one obtains a Brownian excursion in a simply connected domain from one boundary point to another boundary point with holes filled in. Their result gives an evidence that a loop-erasure of a planar Brownian motion should exist.

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We will use the coupling technique introduced in [14] to prove the existence of the loop-erasure. The coupling technique is used to create a coupling of a conditional planar Brownian motion with a radial SLE_2 curve in a simply connected domain such that, for every t in the definition domain of the radial SLE_2 curve, say β, the first hitting point of the planar Brownian motion at the set $\beta[0,t]$ is the tip point: $\beta(t)$. Corollary 2.1 will then be applied.

2 Preliminary

In [3], the loop-erasure of a finite path on a graph is defined as follows. Let $X=(X_0,X_1,\dots,X_n)$ be a finite path. Let $w(0)=0$ and $\tau=0$. If $X_{w(\tau)}\ne X_n$, let $w(\tau+1)=\sup\{k:X_k=X_{w(\tau)}\}+1$, let the value of τ be incremented by 1, and repeat this process; if $X_{w(\tau)}=X_n$, stop. In the end, we get integer numbers $\tau\ge 0$ and $0=w(0)<w(1)<\cdots<w(\tau)$. Then the lattice path $Y_k=X_{w(k)}$, $0\le k\le\tau$, is called the loop-erasure of X. It is easy to see that every vertex of Y lies on X, Y is a simple lattice path, and has the same initial and final vertices as X.

From the definition, it is clear that a path $Y=(Y_0,Y_1,\dots,Y_\tau)$ is the loop-erasure of another path $X=(X_0,X_1,\dots,X_n)$ if and only if there is an increasing function $w:\{0,1,\dots,\tau\}\to\{0,1,\dots,n\}$ such that $w(0)=0$, $Y_k=X_{w(k)}$ for $0\le k\le\tau$, $Y_\tau=X_n$, and for $0\le k\le\tau-1$, the path $(Y_0,Y_1,\dots,Y_k)$ is disjoint from the path $(X_{w(k+1)},X_{w(k+1)+1},\dots,X_n)$. From this observation, we may extend the definition of loop-erasure to (continuous) curves.

Definition 2.1 *We say a continuous curve $Y(t)$, $c\le t\le d$, is a loop-erasure of another continuous curve $X(t)$, $a\le t\le b$, if $Y(c)=X(a)$, $Y(d)=X(b)$, and there is an increasing function w from $[c,d]$ into $[a,b]$ such that $Y(t)=X(w(t))$ for $c\le t\le d$, and for any $t_1<t_2\in[c,d]$, $Y[c,t_1]\cap X[w(t_2),b]=\emptyset$.*

It is easy to see that Y must be a simple curve. In fact, we have an equivalent definition.

Lemma 2.1 *A simple curve $Y(t)$, $c\le t\le d$, is a loop-erasure of a curve $X(t)$, $a\le t\le b$, if and only if $Y(c)=X(a)$, $Y(d)=X(b)$, and for any $T\in(c,d)$, the biggest $s\in[a,b]$ such that $X(s)\in Y[c,T]$ satisfies that $X(s)=Y(T)$.*

Proof. First, suppose Y is a loop-erasure of X, and let w be as in the definition. Fix $T\in[c,d)$. For any $t\in(T,d]$, we have $Y[c,T]\cap X[w(t),d]=\emptyset$. Thus, $X(s)\notin Y[c,T]$ if $s>w_+(T)$, where $w_+(T)$ is the right-hand limit of w at T. On the other hand, for any $t\in(T,d]$, we have $Y(t)=X(w(t))$. By letting $t\to T^+$, we conclude that $Y(T)=X(w_+(T))$. So the biggest s such that $X(s)\in Y[c,T]$ is $w_+(T)$, and $X(w_+(T))=Y(T)$.

Now we prove the other direction. For $c\le t\le d$, let $w(t)$ be the biggest $s\in[a,b]$ such that $X(s)\in Y[c,t]$. Then w is an increasing function, and from the assumption, $Y(t)=X(w(t))$ for $c<t<d$. The equality holds for $t=c$ since in that case $Y[c,t]$ is a single point $Y(c)$. It also holds for $t=d$ because $X(b)=Y(d)$ implies that $w(d)=b$, and so we have $Y(d)=X(b)=X(w(d))$. Since Y is simple, so w is strictly increasing. For $t_1<t_2\in[c,d]$, we have $w(t_2)>w(t_1)$, so from the definition of $w(t_1)$ we have $X[w(t_2),b]\cap Y[c,t_1]=\emptyset$. $\square$

Corollary 2.1 *A simple curve $Y(t)$, $c \le t \le d$, is a loop-erasure of the time-reversal of a curve $X(t)$, $a \le t \le b$, if and only if $Y(c) = X(b)$, $Y(d) = X(a)$, and for any $T \in (c, d)$, the first s such that $X(s) \in Y[c, T]$ satisfies that $X(s) = Y(T)$.*

Two loop-erasures of a curve X are called equivalent if they have the same image. Given a curve $X(t)$, $a \le t \le b$, there may exist more than one loop-erasures, which are not equivalent. For example, in the compact space S obtained by adding $+\infty$ and $-\infty$ to the strip $\{z \in \mathbb{C} : 0 \le \operatorname{Im} z \le 1\}$, there is a curve, which starts from $-\infty$, ends at $+\infty$, and travels through the line segments in the following order: $\dots, [n, n+1], [n+1, n+i], [n+i, n+1+i], [n+1+i, n+1], [n+1, n+2], \dots$, where $n \in \mathbb{Z}$. Such curve has at least two loop-erasures: one has image $\mathbb{R} \cup \{+\infty, -\infty\}$, the other has image $(\mathbb{R} + i) \cup \{+\infty, -\infty\}$, which can not be equivalent.

3 Planar Brownian Motion in Simply Connected Domains

We identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$, and use the convention that a standard real Brownian motion starts from 0, and has variance t at time t for $t \ge 0$, and that a standard complex Brownian motion is a complex valued random process whose real part and imaginary part are two independent standard real Brownian motions. Suppose $B_{\mathbb{C}}(t)$ is a standard complex Brownian motion, and $D \subsetneqq \mathbb{C}$ is a simply connected domain containing 0. Let $\tau = \tau_D$ be the first time that $B_{\mathbb{C}}(t) \notin D$. Then τ is an a.s. finite stopping time. We will focus on the loop-erasures of the time-reversal of $B_{\mathbb{C}}(t)$, $0 \le t \le \tau$. From the remarks in the Section 7, we will see that $B_{\mathbb{C}}(t)$, $0 \le t \le \tau$, itself has a loop-erasure, which is a disc SLE_2 curve. The following is the main theorem in this paper.

Theorem 3.1 *Almost surely there is a loop-erasure of the time-reversal of $B_{\mathbb{C}}(t)$, $0 \le t \le \tau$, which is a radial SLE_2 curve that grows in D towards 0 from a random boundary point of D, whose distribution is the harmonic measure in D seen from 0.*

From Riemann Mapping Theorem and conformal invariance (up to time-change) of complex Brownian motion [9], SLE, and harmonic measure, we suffice to consider the special case that $D = \mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. For $\rho \in \mathbb{T}$, let $B^{\rho}_{\mathbb{C}}(t)$ be $B_{\mathbb{C}}(t)$ conditioned to exit $\mathbb{D}$ at ρ. The explicit definition of $B^{\rho}_{\mathbb{C}}(t)$ will be given below. From the two lemmas below in this section and the rotation symmetry of both $B^{\rho}_{\mathbb{C}}$ and radial SLE in $\mathbb{D}$ from ρ to 0, to prove Theorem 3.1, we suffice to show the following theorem.

Theorem 3.2 *Almost surely there is a loop-erasure of the time-reversal of $B^{1}_{\mathbb{C}}(t)$, $0 \le t \le \tau_1$, which is a radial SLE_2 curve that grows in $\mathbb{D}$ from 1 towards 0.*

Let $P(z) = \operatorname{Re} \frac{1+z}{1-z}$ and $P_\rho(z) = P(z/\rho)$ for $\rho \in \mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$. Then P_ρ is harmonic and positive in $\mathbb{D}$; vanishes on $\mathbb{T}$ except at ρ; and $P_\rho(0) = 1$. We call P_ρ the normalized (by its value at 0) Poisson kernel in $\mathbb{D}$ with the pole at ρ. Let $\delta_\rho(t)$, $0 \le t < \tau_\rho$, be a complex valued function that solves the ODE

$$\delta'_\rho(t) = \frac{2\partial_{\bar z} P_\rho(\delta_\rho(t) + B_{\mathbb{C}}(t))}{P_\rho(\delta_\rho(t) + B_{\mathbb{C}}(t))}, \quad \delta_\rho(0) = 0;$$

and suppose that the solution can not be extended beyond τ_ρ. Here $2\partial_{\overline{z}} = \partial_x + i\partial_y$. Let $B^\rho_{\mathbb{C}}(t) = B_{\mathbb{C}}(t) + \delta_\rho(t)$, $0 \le t < \tau_\rho$. Then $B^\rho_{\mathbb{C}}(t)$ starts from 0 and satisfies the SDE

$$dB^\rho_{\mathbb{C}}(t) = dB_{\mathbb{C}}(t) + \frac{2\partial_{\overline{z}} P_\rho(B^\rho_{\mathbb{C}})}{P_\rho(B^\rho_{\mathbb{C}})} dt, \quad 0 \le t < \tau_\rho. \tag{3.1}$$

This is a complex SDE, and its real stochastic part and imaginary stochastic part are two independent standard Brownian motions. If f is an analytic function, then from Itô's formula [9] for real valued functions, the process $f(B^\rho_{\mathbb{C}}(t))$ satisfies the complex SDE:

$$df(B^\rho_{\mathbb{C}}(t)) = |f'(B^\rho_{\mathbb{C}}(t))| d\widetilde{B}_{\mathbb{C}}(t) + f'(B^\rho_{\mathbb{C}}(t)) \frac{2\partial_{\overline{z}} P_\rho(B^\rho_{\mathbb{C}})}{P_\rho(B^\rho_{\mathbb{C}})} dt, \tag{3.2}$$

where $\widetilde{B}_{\mathbb{C}}(t) := \frac{f'(B_{\mathbb{C}}(t))}{|f'(B^\rho_{\mathbb{C}}(t))|} B_{\mathbb{C}}(t)$ has the same distribution as $B_{\mathbb{C}}(t)$. There is no drift term coming from the second derivatives of f because $\Delta f \equiv 0$.

The process $B^\rho_{\mathbb{C}}(t)$ satisfies rotation symmetry, which means that for $a \in \mathbb{T}$, $(R_a(B^\rho_{\mathbb{C}}(t)))$ has the same distribution as $(B^{a\rho}_{\mathbb{C}}(t))$, where $R_a(z) := az$. This follows easily from (3.2) with $f = R_a$. Note that for any $z \in \mathbb{C}$, $a \frac{2\partial_{\overline{z}} P_\rho(z)}{P_\rho(z)} = \frac{2\partial_{\overline{z}} P_{a\rho}(az)}{P_{a\rho}(az)}$.

There is no compact set $K \subset D$ such that $B^\rho_{\mathbb{C}}(t) \in K$ for $0 \le t < \tau_\rho$. For otherwise, the solution $\delta_\rho(t)$ could be extended beyond τ_ρ. The next two lemmas give the reason why $B^\rho_{\mathbb{C}}(t)$ is viewed as $B_{\mathbb{C}}(t)$ conditioned to exit $\mathbb{D}$ at ρ.

Lemma 3.1 *Let ν denote the distribution of $(B_{\mathbb{C}}(t) : 0 \le t < \tau)$. For every $\rho \in \mathbb{T}$, let $\mu(\rho, \cdot)$ denote the distribution of $(B^\rho_{\mathbb{C}}(t) : 0 \le t < \tau_\rho)$. Then $\nu = \int_{\mathbb{T}} \mu(\rho, \cdot) d\lambda(\rho)$, where λ is the uniform probability measure on $\mathbb{T}$.*

Proof. From Itô's formula, the process $M_\rho(t) := P_\rho(B_{\mathbb{C}}(t))$, $0 \le t < \tau$, is a positive local martingale. So if σ is any Jordan curve in $\mathbb{D}$ surrounding 0, and τ_σ is the first time that $B_{\mathbb{C}}(t)$ visits σ, then $\mathbf{E}[M_\rho(\tau_\sigma)] = M_\rho(0) = 1$. From Girsanov Theorem, it is easy to check that the distribution of $(B^\rho_{\mathbb{C}}(t) : 0 \le t < \tau_\sigma)$ is absolutely continuous w.r.t. that of $(B_{\mathbb{C}}(t) : 0 \le t < \tau_\sigma)$, and the Radon-Nikodym derivative is $M_\rho(\tau_\sigma)$.

We are considering probability measures on the space of curves $\gamma(t)$, $0 \le t < T$, in $\mathbb{D}$, started from 0. Let $(\mathcal{F}_t)$ denote the natural filtration generated by the curves. For each $n \in \mathbb{N}$, let τ_n denote the first time when $|\gamma(t)| \ge 1 - 1/n$. Then each τ_n is an $(\mathcal{F}_t)$-stopping time, and the whole sigma-algebra $\mathcal{F}$ is generated by the union $\bigcup_{n\in\mathbb{N}} \mathcal{F}_{\tau_n}$. For each $n \in \mathbb{N}$ and $\rho \in \mathbb{T}$, $\mu(\rho, \cdot)$ is absolutely continuous w.r.t. ν on $\mathcal{F}_{\tau_n}$, and the Radon-Nikodym derivative is $P_\rho(B_{\mathbb{C}}(\tau_n))$. We have that $\rho \mapsto P_\rho(B_{\mathbb{C}}(\tau_n))$ is continuous, and $\int_{\mathbb{T}} P_\rho(B_{\mathbb{C}}(\tau_n)) d\lambda(\rho) = 1$. Thus, $\nu = \int_{\mathbb{T}} \mu(\rho, \cdot) d\lambda(\rho)$ on $\mathcal{F}_{\tau_n}$. Finally, since $\mathcal{F}$ is the σ-algebra generated by the union $\bigcup_{n\in\mathbb{N}} \mathcal{F}_{\tau_n}$, which is an algebra, so the proof is finished by Monotone Class Theorem. $\square$

Lemma 3.2 *Almost surely $\lim_{t\to\tau_\rho^-} B^\rho_{\mathbb{C}}(t) = \rho$.*

Proof. Let $W_\rho(z) = \frac{\rho+z}{\rho-z}$, which maps $\mathbb{D}$ conformally onto the right half plane $\{\operatorname{Re} z > 0\}$, and maps ρ to ∞. We have $P_\rho = \operatorname{Re} W_\rho$, so $2\partial_{\bar z} P_\rho = \overline{W'_\rho}$. Let $Z_\rho(t) = W_\rho(B^\rho_{\mathbb{C}}(t))$. Then $P_\rho(B^\rho_{\mathbb{C}}(t)) = \operatorname{Re} Z_\rho(t)$. From (3.2), there is another standard complex Brownian motion $\widetilde{B}_{\mathbb{C}}(t)$ such that $Z_\rho(t)$ satisfies the SDE:

$$dZ_\rho(t) = |W'_\rho(B^\rho_{\mathbb{C}}(t))| d\widetilde{B}_{\mathbb{C}}(t) + \frac{|W'_\rho(B^\rho_{\mathbb{C}}(t))|^2}{\operatorname{Re} Z_\rho(t)} dt, \quad 0 \le t < \tau_\rho.$$

Let $u_\rho(t) = \int_0^t |W'_\rho(B^\rho_{\mathbb{C}}(s))|^2 ds$, $0 \le t < \tau_\rho$. Then u_ρ is continuous and increasing, and maps $[0, \tau_\rho)$ onto $[0, S_\rho)$ for some $S_\rho \in (0, \infty]$. Let $Z^u_\rho(t) = Z_\rho(u_\rho^{-1}(t))$, $0 \le t < S_\rho$. Then there is another standard complex Brownian motion $\widehat{B}_{\mathbb{C}}(t)$ such that $Z^u_\rho(t)$ satisfies the SDE:

$$dZ^u_\rho(t) = d\widehat{B}_{\mathbb{C}}(t) + \frac{1}{\operatorname{Re} Z^u_\rho(t)} dt, \quad 0 \le t < S_\rho. \tag{3.3}$$

Since the curve $B^\rho_{\mathbb{C}}(t)$, $0 \le t < \tau_\rho$, is not contained in any compact subset of $\mathbb{D}$, so $Z^u_\rho(t)$, $0 \le t < S_\rho$, is not contained in any compact subset of $\{\operatorname{Re} z > 0\}$. Thus, $S_\rho = \infty$. From (3.3), $\operatorname{Re} Z^u_\rho$ is a Bessel process of dimension 3 started from 1. Since $|Z^u_\rho(t)| \ge \operatorname{Re} Z^u_\rho(t)$, and $S_\rho = \infty$, so a.s. $\lim_{t\to\infty} |Z^u_\rho(t)| = \infty$. Since $\lim_{z\to\infty} W_\rho^{-1}(z) = \rho$, so we derive the conclusion. $\Box$

4 Schramm-Loewner Evolution

Schramm-Loewner evolution (SLE) was introduced by Oded Schramm [11] to study the scaling limits of 2-dimensional statistical lattice model at criticality, where the conformal invariance property appears in the limit. It is very successful in giving mathematical proofs of the conjectures proposed by physicists. The definition of SLE combines the Loewner's differential equation with a stochastic input. For the completeness of this paper, we now give a brief introduction of radial SLE, which is one of the major versions of SLE. The reader may refer to [10] and [4] for more properties of SLE.

Let $B(t)$ be a standard real Brownian motion. Let $\kappa > 0$ be a parameter. Let $\xi(t) = \sqrt{\kappa} B(t)$, $t \ge 0$. The following differential equation is called the radial Loewner equation driven by ξ.

$$\partial_t g_t(z) = g_t(z) \frac{e^{i\xi(t)} + g_t(z)}{e^{i\xi(t)} - g_t(z)}, \quad g_0(z) = z. \tag{4.1}$$

It turns out that there is a decreasing family of domains $(D_t : 0 \le t < \infty)$ with $D_0 = \mathbb{D}$ and $0 \in D_t$ for all $t \ge 0$, such that each g_t is defined on D_t, maps D_t conformally onto $\mathbb{D}$, and satisfies $g_t(0) = 0$ and $g'_t(0) = e^t$. Moreover, almost surely

$$\beta(t) := \lim_{\mathbb{D} \ni z \to e^{i\xi(t)}} g_t^{-1}(z) \tag{4.2}$$

exists for $0 \le t < \infty$, and $\beta(t)$, $0 \le t < \infty$, is a continuous curve in $\overline{\mathbb{D}}$ with $\beta(0) = 1$ and $\lim_{t\to\infty} \beta(t) = 0$. Such β is called a standard radial SLE$_\kappa$ curve. The radial SLE$_\kappa$ curve in a general simply connected domain which grows from a boundary point to an interior point is defined as the image of such β under a conformal map from $\mathbb{D}$ onto this domain, which takes 1 and 0 to the initial and end points, respectively. If $\kappa \in (0,4]$, β is a simple curve, intersects $\mathbb{T}$ only at its initial point, and for each $t \ge 0$, $D_t = \mathbb{D} \setminus \beta((0,t])$; if $\kappa > 4$, β is no longer a simple curve, and for each $t \ge 0$, D_t is the connected component of $\mathbb{D} \setminus \beta((0,t])$ which contains 0. In this paper we are mostly interested in the case $\kappa = 2$, so β is a simple curve.

There is an interesting local martingale associated with radial SLE_2, which was used to prove the convergence of LERW to SLE_2 [8]. Recall that $P_{e^{i\xi(t)}}$ is the normalized Poisson kernel in $\mathbb{D}$ with the pole at $e^{i\xi(t)}$. Since g_t^{-1} maps $\mathbb{D}$ conformally onto $D_t = \mathbb{D} \setminus \beta(0,t]$, fixes 0, and has continuous extension to $\overline{\mathbb{D}}$, which maps $e^{i\xi(t)}$ to $\beta(t)$, so $Q_t := P_{e^{i\xi(t)}} \circ g_t$ is the normalized ($Q_t(0) = 1$) Poisson kernel in D_t with the pole at $\beta(t)$. We have the following proposition.

Proposition 4.1 *Let $\kappa = 2$. Then for any $z \in \mathbb{D}$, $(Q_t(z) : 0 \le t < T_z)$ is a local martingale, where $T_z \in (0,\infty]$ is such that $[0,T_z)$ is the maximal interval with $z \in D_t$ for $t \in [0,T_z)$*

5 Local Martingale in Two Time Variables

Theorem 3.2 will be proved by constructing a coupling of the process $B^1_{\mathbb{C}}(t)$, $0 \le t < \tau_1$, with a standard radial SLE_2 curve $\beta(t)$, $0 \le t < \infty$, such that conditioned on β up to a finite stopping time T, the part of $B^1_{\mathbb{C}}$ before hitting $\beta[0,T]$ is a complex Brownian motion in $D \setminus \beta[0,T]$ conditioned to hit $\beta(T)$. In this section, we will first construct a local coupling.

First we suppose that the conditional complex Brownian motion $B^1_{\mathbb{C}}(t_1)$, $0 \le t_1 < \tau_1$, and the standard radial SLE_2 curve $\beta(t_2)$, $0 \le t_2 < \infty$, are independent. This is a trivial coupling of the above two processes. Let $\xi(t_2) = \sqrt{2}B(t_2)$ be the driving function of β, and let g_t denote the radial Loewner maps. Let $(\mathcal{F}^1_{t_1})$ and $(\mathcal{F}^2_{t_2})$ be the natural filtrations generated by $B^1_{\mathbb{C}}(t_1)$ and $(\xi(t_2))$, respectively. Then $(\beta(t_2))$ and (g_{t_2}) are $(\mathcal{F}^2_{t_2})$-adapted. Let

$$\mathcal{D} = \{(t_1,t_2) \in [0,\tau_1) \times [0,\infty) : B^1_{\mathbb{C}}[0,t_1] \cap \beta[0,t_2] = \emptyset\}.$$

For every $t_2 \in [0,\infty)$, let $\mathcal{T}_1(t_2)$ be the maximal number such that $(t_1,t_2) \in \mathcal{D}$ for $t_1 \in [0,\mathcal{T}_1(t_2))$; for every $t_1 \in [0,\tau_1)$, let $\mathcal{T}_2(t_1)$ be the maximal number such that $(t_1,t_2) \in \mathcal{D}$ for $t_2 \in [0,\mathcal{T}_2(t_1))$. If $\bar{t}_2 < \infty$ is an $(\mathcal{F}^2_{t_2})$-stopping time, then $\mathcal{T}_1(\bar{t}_2)$ is an $(\mathcal{F}^1_{t_1} \times \mathcal{F}^2_{\bar{t}_2})$-stopping time; if $\bar{t}_1 < \tau_1$ is an $(\mathcal{F}^1_{t_1})$-stopping time, then $\mathcal{T}_2(\bar{t}_1)$ is an $(\mathcal{F}^1_{\bar{t}_1} \times \mathcal{F}^2_{t_2})$-stopping time.

Let $Q_{t_2} = P_{e^{i\xi(t_2)}} \circ g_{t_2}$ be as in Proposition 4.1. Since $g_0 = \mathrm{id}$ and $\xi(0) = 0$, so $Q_0(z) = P_1(z) = \frac{1+z}{1-z}$. Define M on $\mathcal{D}$ such that

$$M(t_1,t_2) = \frac{Q_{t_2}(B^1_{\mathbb{C}}(t_1))}{Q_0(B^1_{\mathbb{C}}(t_1))}.$$

It is clear that $M(t_1,0) = 1$ for any $0 \le t_1 < \tau_1$. Since $B^1_{\mathbb{C}}(0) = 0$ and $Q_{t_2}(0) \equiv 1$, so $M(0,t_2) = 1$ for any $0 \le t_2 < \infty$.

Lemma 5.1 *(a) For any $(\mathcal{F}^1_{t_1})$-stopping time $\bar t_1 < \tau_1$, $M(\bar t_1, t_2)$, $0 \le t_2 < \mathcal{T}_2(\bar t_1)$, is an $(\mathcal{F}^1_{\bar t_1} \times \mathcal{F}^2_{t_2})$-local martingale. (b) For any $(\mathcal{F}^2_{t_2})$-stopping time $\bar t_2 < \infty$, $M(t_1, \bar t_2)$, $0 \le t_1 < \mathcal{T}_1(\bar t_2)$, is an $(\mathcal{F}^1_{t_1} \times \mathcal{F}^2_{\bar t_2})$-local martingale.*

Proof. (a) This part follows immediately from Proposition 4.1.
(b) Let $f_{\bar t_2} = Q_{\bar t_2}/Q_0$. Then $f_{\bar t_2}$ is $\mathcal{F}^2_{\bar t_2}$-measurable, and $M(t_1, \bar t_2) = f_{\bar t_2}(B^1_{\mathbb{C}}(t_1))$. Recall that $Q_0 = P_1$. From (3.1) ($\rho = 1$) and Itô's formula, we see that $M(t_1, \bar t_2)$, $0 \le t_1 < \mathcal{T}_1(\bar t_2)$, satisfies the $(\mathcal{F}^1_{t_1} \times \mathcal{F}^2_{\bar t_2})$-adapted SDE:

$$d_1 M(t_1, \bar t_2) = \mathrm{Re}[2\partial_z f_{\bar t_2}(B^1_{\mathbb{C}}(t_1)) dB_{\mathbb{C}}(t_1)] + \mathrm{Re}[2\partial_z f_{\bar t_2}(B^1_{\mathbb{C}}(t_1)) \frac{2\partial_{\bar z} Q_0(B^1_{\mathbb{C}})}{Q_0(B^1_{\mathbb{C}})}] dt_1$$

$$+ \frac{1}{2}\Delta f_{\bar t_2}(B^1_{\mathbb{C}}(t_1)) dt_1.$$

We have $f_{\bar t_2} Q_0 = Q_{\bar t_2}$, and both Q_0 and $Q_{\bar t_2}$ are harmonic. So

$$0 = \Delta Q_{\bar t_2} = 4\partial_z \partial_{\bar z}(f_{\bar t_2} Q_0) = f_{\bar t_2}\Delta Q_0 + Q_0 \Delta f_{\bar t_2} + 4\partial_z f_{\bar t_2} \partial_{\bar z} Q_0 + 4\partial_{\bar z} f_{\bar t_2} \partial_z Q_0$$

$$= Q_0 \Delta f_{\bar t_2} + 8\,\mathrm{Re}[\partial_z f_{\bar t_2} \partial_{\bar z} Q_0].$$

So we have

$$d_1 M(t_1, \bar t_2) = \mathrm{Re}[2\partial_z f_{\bar t_2}(B^1_{\mathbb{C}}(t_1)) dB_{\mathbb{C}}(t_1)]. \tag{5.1}$$

Thus, $M(t_1, \bar t_2)$, $0 \le t_1 < \mathcal{T}_1(\bar t_2)$, is an $(\mathcal{F}^1_{t_1} \times \mathcal{F}^2_{\bar t_2})$-local martingale. $\square$

Let J denote the set of Jordan curves in $\mathbb{D} \setminus \{0\}$ that surround 0. For every $\sigma \in$ J, let T^1_σ be the first time that $B^1_{\mathbb{C}}(t_1)$ hits σ; let T^2_σ be the first time that $\beta(t_2)$ hits σ. Then T^j_σ is an $(\mathcal{F}^j_{t_j})$-stopping time, $j = 1, 2$. Let JP denote the set of $(\sigma_1, \sigma_2) \in \mathrm{J}^2$ such that $\sigma_1 \cap \sigma_2 = \emptyset$, and σ_2 surrounds σ_1. Then for any $(\sigma_1, \sigma_2) \in$ JP, $[0, T^1_{\sigma_1}] \times [0, T^2_{\sigma_2}] \subset \mathcal{D}$.

Lemma 5.2 *For any $(\sigma_1, \sigma_2) \in$ JP, $|\ln(M)|$ is bounded on $[0, T^1_{\sigma_1}] \times [0, T^2_{\sigma_2}]$ by a constant depending only on σ_1 and σ_2.*

Proof. Fix $(\sigma_1, \sigma_2) \in$ JP. In this proof, a uniform constant means a constant depending only on σ_1 and σ_2; and we say a variable is uniformly bounded if its absolute value is bounded by a uniform constant. Let $N(t_1, t_2) = Q_{t_2}(B^1_{\mathbb{C}}(t_1))$. Since $M(t_1, t_2) = N(t_1, t_2)/N(t_1, 0)$, so we suffice to show that $\ln(N)$ is uniformly bounded on $[0, T^1_{\sigma_1}] \times [0, T^2_{\sigma_2}]$. Fix $t_1 \in [0, T^1_{\sigma_1}]$ and $t_2 \in [0, T^2_{\sigma_2}]$. Let E_{σ_j} denote the domain bounded by σ_j, $j = 1, 2$. Let $\Omega = E_{\sigma_2} \setminus \overline{E_{\sigma_1}}$ and $\Omega_{t_2} = D_{t_2} \setminus \overline{E_{\sigma_1}}$ for $t_2 \in [0, T^2_{\sigma_2}]$. Recall that $D_{t_2} = \mathbb{D} \setminus \beta((0, t_2])$. Let m and m_{t_2} denote the moduli of the above doubly connected domains, respectively. Then m is a uniform constant, and $m \le m_{t_2}$. Since g_{t_2} maps D_{t_2} conformally onto $\mathbb{D}$, so it maps Ω_{t_2} onto $\mathbb{D} \setminus g_{t_2}(\overline{E_{\sigma_1}})$, which must have modulus $m_{t_2} \ge m$. Since $0 \in \overline{E_{\sigma_1}}$ and $g_{t_2}(0) = 0$, so $0 \in g_{t_2}(\overline{E_{\sigma_1}})$. There is uniform constant $r_m \in (0, 1)$ such that the modulus of $\mathbb{D} \setminus [0, r_m]$ equals m. It is known that, for connected compact sets $K \subset \mathbb{D}$ with $0 \in K$ and the modulus of $\mathbb{D} \setminus K$ being at least m, the maximum of $r(K) := \sup_{z\in K}|z|$ is attained when $K=[0,r_m]$. Now $g_{t_2}(\overline{E_{\sigma_1}})$ satisfies the property of K, so $g_{t_2}(\overline{E_{\sigma_1}}) \subset \{|z|\le r_m\}$. Since $B^1_{\mathbb{C}}(t_1)\in E_{\sigma_1}$, so $|g_{t_2}(B^1_{\mathbb{C}}(t_1))|\le r_m$. Since $N(t_1,t_2)=Q_{t_2}(B^1_{\mathbb{C}}(t_1))=P(g_{t_2}(B^1_{\mathbb{C}}(t_1))/e^{i\xi_2(t_2)})$, where $P(z)=\operatorname{Re}\frac{1+z}{1-z}$, so $\frac{1-r_m}{1+r_m}\le N(t_1,t_2)\le \frac{1+r_m}{1-r_m}$. Thus, $|\ln(N)|\le \ln(\frac{1+r_m}{1-r_m})$, which is a uniform constant. $\square$

The stochastic process $M(t_1,t_2)$ valued at certain pair of times (T_1,T_2) will be used as a Radon-Nikodym derivative to weight some simple probability distribution to get a somehow complicated distribution. Here are the details. Fix $(\sigma_1,\sigma_2)\in \mathrm{JP}$. Let μ denote the joint distribution of $B^1_{\mathbb{C}}(t_1)$, $0\le t_1<\tau_1$, with $\beta(t_2)$, $0\le t_2<\infty$, which are independent to each other. From Lemma 5.1 and Lemma 5.2, we have $\int M(T^1_{\sigma_1},T^2_{\sigma_2})d\mu = M(0,0)=1$. Define ν_{σ_1,σ_2} such that $d\nu_{\sigma_1,\sigma_2}/d\mu = M(T^1_{\sigma_1},T^2_{\sigma_2})$. Then ν_{σ_1,σ_2} is also a probability measure. Now suppose the joint distribution of the above two random curves is ν_{σ_1,σ_2} instead of μ. Since $M=1$ when either t_1 or t_2 equals 0, so the marginal distributions of ν_{σ_1,σ_2} agree with those of μ. Thus, ν_{σ_1,σ_2} is also a coupling measure of $B^1_{\mathbb{C}}(t_1)$, $0\le t_1<\tau_1$, with $\beta(t_2)$, $0\le t_2<\infty$. We now look at the behavior of the sub-curves $B^1_{\mathbb{C}}(t_1)$, $0\le t_1\le T^1_{\sigma_1}$, and $\beta(t_2)$, $0\le t_2\le T^2_{\sigma_2}$. Fix any $(\mathcal{F}^2_{t_2})$-stopping time $\bar t_2\le T^2_{\sigma_2}$. From (3.1), (5.1), and Girsanov Theorem, under ν_{σ_1,σ_2}, there is an $(\mathcal{F}^1_{t_1}\times\mathcal{F}^2_{\bar t_2})$-standard complex Brownian motion $\widetilde B_{\mathbb{C}}(t_1)$ such that $B^1_{\mathbb{C}}(t_1)$, $0\le t_1\le T^1_{\sigma_1}$, satisfies the $(\mathcal{F}^1_{t_1}\times\mathcal{F}^2_{\bar t_2})$-adapted SDE:

$$\begin{aligned} dB^1_{\mathbb{C}}(t_1) &= d\widetilde B_{\mathbb{C}}(t_1) + \frac{2\partial_{\bar z}P_1(B^1_{\mathbb{C}})}{P_1(B^1_{\mathbb{C}})}dt_1 + \frac{2\partial_{\bar z}f_{\bar t_2}(B^1_{\mathbb{C}})}{f_{\bar t_2}(B^1_{\mathbb{C}})}dt_1 \\ &= d\widetilde B_{\mathbb{C}}(t_1) + \frac{2\partial_{\bar z}Q_{\bar t_2}(B^1_{\mathbb{C}})}{Q_{\bar t_2}(B^1_{\mathbb{C}})}dt_1, \end{aligned} \tag{5.2}$$

where the second equality holds because $P_1 f_{\bar t_2}=Q_0 f_{\bar t_2}=Q_{\bar t_2}$.

6 Coupling Measures

Let M be as in the last section. Then we have the following proposition.

Proposition 6.1 *For any finite collection (σ_1^m,σ_2^m), $1\le m\le n$, in* JP*, there is an a.s. continuous stochastic process M_* defined on $[0,\infty]^2$, which satisfies the following properties:*

(i) $M_*=M$ on $[0,T^1_{\sigma_1^m}]\times[0,T^2_{\sigma_2^m}]$, $1\le m\le n$;

(ii) $M_*(t,0)=M_*(0,t)=1$ for any $t\in[0,\infty]$;

(iii) There are constants $C_2>C_1>0$ depending only on (σ_1^m,σ_2^m), $1\le m\le n$, such that $C_1\le M_*(t_1,t_2)\le C_2$ on $[0,\infty]^2$;

(iv) For any $(\mathcal{F}^2_{t_2})$-stopping time $\bar t_2$, $M_*(t_1,\bar t_2)$, $0\le t_1\le\infty$, is an $(\mathcal{F}^1_{t_1}\times\mathcal{F}^2_{\bar t_2})$-martingale;

(v) For any $(\mathcal{F}^1_{t_1})$-stopping time $\bar t_1$, $M_*(\bar t_1,t_2)$, $0\le t_2\le\infty$, is an $(\mathcal{F}^1_{\bar t_1}\times\mathcal{F}^2_{t_2})$-martingale.

For the proof, we may first define M_* on $[0,\infty]\times\{0\}\cup\{0\}\times[0,\infty]\cup\bigcup_{m=1}^n[0,T^1_{\sigma_1^m}]\times[0,T^2_{\sigma_2^m}]$ by (i) and (ii), and then extend M_* to $[0,\infty]^2$ in such a way that: if R is a rectangle obtained by dividing $[0,\infty]^2$ using the lines $\{t_1=T^1_{\sigma_1^m}\}$ and $\{t_2=T^2_{\sigma_2^m}\}$, $1\le m\le n$, and R is not contained in any $[0,T^1_{\sigma_1^m}]\times[0,T^2_{\sigma_2^m}]$, then there are functions $f_1^R(t_1)$ and $f_2^R(t_2)$ such that $M_*(t_1,t_2)=f_1^R(t_1)f_2^R(t_2)$ on R. Such M_* is well constructed, and is unique. Property (iii) follows from Lemma 5.2. Property (iv) and (v) follow from the local martingale property of M. The reader may refer to [14] (Theorem 6.1) for the explicit formula of M_* and a detailed proof of a similar proposition.

Let JP_* be the set of $(\sigma_1,\sigma_2)\in\mathrm{JP}$ such that both σ_1 and σ_2 are polygonal curves whose vertices have rational coordinates. Then JP_* is countable. Let (σ_1^m,σ_2^m), $m\in\mathbb{N}$, be an enumeration of JP_*. For each $n\in\mathbb{N}$, let M_*^n be the M_* given by the above proposition for (σ_1^m,σ_2^m), $1\le m\le n$, in the above enumeration. Let the probability μ be as in the last section. For each $n\in\mathbb{N}$, define ν^n such that $d\nu^n=M_*^n(\infty,\infty)d\mu$. From the property of M_*, $\int M_*^n(\infty,\infty)d\mu=M_*^n(0,0)=1$, so ν^n is a probability measure. Since $M_*^n=1$ when either t_1 or t_2 equals 0, so ν^n is also a coupling measure of $B^1_{\mathbb{C}}(t_1)$, $0\le t_1<\tau_1$, with $\beta(t_2)$, $0\le t_2<\infty$.

Fix any $m\in\mathbb{N}$. If $n\ge m$, from the martingale property of M_*^n, we have

$$\mathbf{E}\left[M_*^n(\infty,\infty)|\mathcal{F}^1_{T^1_{\sigma_1^m}}\times\mathcal{F}^2_{T^2_{\sigma_2^m}}\right]=M_*^n(T^1_{\sigma_1^m},T^2_{\sigma_2^m})=M(T^1_{\sigma_1^m},T^2_{\sigma_2^m}).$$

Thus, on $\mathcal{F}^1_{T^1_{\sigma_1^m}}\times\mathcal{F}^2_{T^2_{\sigma_2^m}}$, ν^n equals $\nu_{\sigma_1^m,\sigma_2^m}$ defined in the last section. We want to construct a coupling measure ν^∞ of $B^1_{\mathbb{C}}(t_1)$, $0\le t_1<\tau_1$, with $\beta(t_2)$, $0\le t_2<\infty$, such that for any $m\in\mathbb{N}$, ν^∞ equals $\nu_{\sigma_1^m,\sigma_2^m}$ on $\mathcal{F}^1_{T^1_{\sigma_1^m}}\times\mathcal{F}^2_{T^2_{\sigma_2^m}}$. Such ν^∞ could be defined as a subsequential weak limit of (ν^n) in some suitable topology as follows.

Let $\mathcal{C}:=\cup_{T\in[0,\infty]}C([0,T],\overline{\mathbb{D}})$. Extend $B^1_{\mathbb{C}}$ to $[0,\tau_1]$ such that $B^1_{\mathbb{C}}(\tau_1)=1$, and extend β to $[0,\infty]$ such that $\beta(\infty)=0$. Then both $B^1_{\mathbb{C}}$ and β are random elements in $\mathcal{C}$. Let μ_1 and μ_2 be their distributions, respectively. We view them as probability measures on $\mathcal{C}$, where the σ-algebra is generated by the events $\{T\ge a, f(a)\in A\}$, where $0\le a\le\infty$. So $\mu=\mu_1\times\mu_2$ is a probability measure on $\mathcal{C}\times\mathcal{C}$.

Let Γ denote the space of nonempty compact subsets of $[0,\infty]\times\overline{\mathbb{D}}$ endowed with Hausdorff metric. Then Γ is a compact metric space. Define $G:\mathcal{C}\to\Gamma$ such that $G(f)$ is the graph of f. Then G is a one-to-one map. Let $I_G=G(\mathcal{C})$. One may check that G and G^{-1} (defined on I_G) are both measurable. This is also true for $G\times G$ and $G^{-1}\times G^{-1}$.

For $n\in\mathbb{N}$, $\bar\nu^n:=(G\times G)_*(\nu^n)$ is a probability measure on Γ^2. Since Γ^2 is compact, so $(\bar\nu^n)$ has a subsequence $(\bar\nu^{n_k})$ that converges weakly to some probability measure $\bar\nu^\infty$ on $\Gamma\times\Gamma$. Let $\nu_j^{n_k}$ and ν_j^∞, $j=1,2$, denote the marginal distributions of ν^{n_k} and ν^∞. Then for $j=1,2$, $\bar\nu_j^{n_k}\to\bar\nu_j^\infty$ weakly. For $n\in\mathbb{N}$ and $j=1,2$, since $\nu_j^n=\mu_j$, $\bar\nu_j^n=G_*(\mu_j)$. Thus, $\bar\nu_j^\infty=G_*(\mu_j)$, $j=1,2$. So $\bar\nu^\infty$ is supported by I_G^2. Let $\nu^\infty=(G^{-1}\times G^{-1})_*(\bar\nu^\infty)$ be a probability measure on $\mathcal{C}^2$. For $j=1,2$, we have $\nu_j^\infty=(G^{-1})_*(\bar\nu_j^\infty)=\mu_j$. So ν^∞ is also a coupling measure of μ_1 and μ_2.

It remains to check that for any $m \in \mathbb{N}$, ν^∞ equals $\nu_{\sigma_1^m,\sigma_2^m}$ on $\mathcal{F}^1_{T^1_{\sigma_1^m}} \times \mathcal{F}^2_{T^2_{\sigma_2^m}}$. For any $\sigma \in \mathrm{J}$, define a truncate map P_σ from $\mathcal{C}$ onto itself such that $P_\sigma(f)$ is the restriction of f to $[0,\tau_\sigma]$, where τ_σ is the first time that $f(t) \in \sigma$. Fix $m \in \mathbb{N}$. Then ν^∞ equals $\nu_{\sigma_1^m,\sigma_2^m}$ on $\mathcal{F}^1_{T^1_{\sigma_1^m}} \times \mathcal{F}^2_{T^2_{\sigma_2^m}}$ iff

$$(P_{\sigma_1^m} \times P_{\sigma_2^m})_*(\nu^\infty) = (P_{\sigma_1^m} \times P_{\sigma_2^m})_*(\nu_{\sigma_1^m,\sigma_2^m}). \tag{6.1}$$

From an earlier observation, (6.1) holds if ν^∞ is replaced by ν^n with $n \ge m$.

Let $D : (f,g) \mapsto (f,g,f,g)$ be a diagonal map from $\mathcal{C}^2$ to $\mathcal{C}^4$. For $n \in \mathbb{N}$, let

$$\bar\lambda^n = [((G \circ P_{\sigma_1^m}) \times (G \circ P_{\sigma_2^m})) \times (G \times G)]_* \circ D_*(\nu^n).$$

Then $\bar\lambda^n$ is a probability measure on $\Gamma^4 = \Gamma^2 \times \Gamma^2$. It is a coupling of $(G\times G)_* \circ (P_{\sigma_1^m} \times P_{\sigma_2^m})_*(\nu^n)$ and $(G \times G)_*(\nu^n)$, and is supported by

$$\mathcal{G} := \{(F_1, F_2, F_3, F_4) \in \Gamma^4 : (0,0) \in F_1 \subset F_3, (0,1) \in F_2 \subset F_4\}.$$

Here we use the facts that $B^1_{\mathbb{C}}(0) = 0$ and $\beta(0) = 1$.

Since Γ^4 is a compact space, the sequence $(\bar\lambda^{n_k})$ has a subsequence, say $(\bar\lambda^{n_{k_j}})$, which converges weakly to a probability measure $\bar\lambda^\infty$ on $\Gamma^2 \times \Gamma^2$. Then $\bar\lambda^\infty$ is also supported by $\mathcal{G}$. Let $\bar\lambda^\infty_1$ and $\bar\lambda^\infty_2$ be the marginal distributions of $\bar\lambda^\infty$ on the first two variables and the last two variables, respectively. Then we have $(G \times G)_* \circ (P_{\sigma_1^m} \times P_{\sigma_2^m})_*(\nu^{n_{k_j}}) \to \bar\lambda^\infty_1$ and $(G \times G)_*(\nu^{n_{k_j}}) \to \bar\lambda^\infty_2$. Since (6.1) holds with ν^∞ replaced by $\nu^{n_{k_j}}$ if $n_{k_j} \ge m$, so $\bar\lambda^\infty_1 = (G\times G)_* \circ (P_{\sigma_1^m} \times P_{\sigma_2^m})_*(\nu_{\sigma_1^m,\sigma_2^m})$. Since $(G\times G)_*(\nu^{n_k}) \to (G\times G)_*(\nu^\infty)$, so $\bar\lambda^\infty_2 = (G\times G)_*(\nu^\infty)$. Let $\lambda^\infty = (G^{-1} \times G^{-1} \times G^{-1} \times G^{-1})_*(\bar\lambda^\infty)$, and let λ^∞_1 and λ^∞_2 be its marginal distributions. Then we have $\lambda^\infty_1 = (P_{\sigma_1^m} \times P_{\sigma_2^m})_*(\nu_{\sigma_1^m,\sigma_2^m})$ and $\lambda^\infty_2 = \nu^\infty$. Since $\bar\lambda^\infty$ is supported by $\mathcal{G}$, so λ^∞ is supported by the set of $(f_1, f_2, f_3, f_4) \in \mathcal{C}^4$ such that f_1 and f_2 are subcurves of f_3 and f_4, respectively. From the property of $(P_{\sigma_1^m} \times P_{\sigma_2^m})_*(\nu_{\sigma_1^m,\sigma_2^m})$, we can further conclude that λ^∞ is supported by $\{(f_1, f_2, f_3, f_4) \in \mathcal{C}^4 : f_1 = P_{\sigma_1^m}(f_3), f_2 = P_{\sigma_2^m}(f_4)\}$. So we obtain (6.1). The reader may refer to Lemma 4.1 in [15] for a more detailed argument.

Now for each $m \in \mathbb{N}$, $\nu^\infty = \nu_{\sigma_1^m,\sigma_2^m}$ on $\mathcal{F}^1_{T^1_{\sigma_1^m}} \times \mathcal{F}^2_{T^2_{\sigma_2^m}}$. Let $\bar t_2$ be an $(\mathcal{F}^2_{t_2})$-stopping time with $\bar t_2 \le T^2_{\sigma_2^m}$. From the discussion at the end of the last section, we see that $B^1_{\mathbb{C}}(t_1)$, $0 \le t_1 \le T^1_{\sigma_1^m}$, satisfies (5.2) for some $(\mathcal{F}^1_{t_1} \times \mathcal{F}^2_{\bar t_2})$-standard complex Brownian motion $\widetilde B_{\mathbb{C}}(t_1)$.

Fix $t_2 \in (0,\infty)$. For $n \in \mathbb{N}$, define

$$R_n = \sup\{T^1_{\sigma_1^m} : 1 \le m \le n, T^2_{\sigma_2^m} \ge t_2\}.$$

Fix $n \in \mathbb{N}$. Then for any $1 \le m \le n$, if $t_2 \le T^2_{\sigma_2^m}$, then $B^1_{\mathbb{C}}(t_1)$, $0 \le t_1 \le T^1_{\sigma_1^m}$, satisfies (5.2). So $B^1_{\mathbb{C}}(t_1)$, $0 \le t_1 \le R_n$, should also satisfy (5.2).

From the definition, $\mathcal{T}_1(t_2)$ is the maximal number such that $B^1_{\mathbb{C}}(t_1)$ is disjoint from $\beta[0,t_2]$ for $0 \le t_1 < \mathcal{T}_1(t_2)$. It is easy to check that $\mathcal{T}_1(t_2) = \sup_{n\in\mathbb{N}} R_n$. Thus, $B^1_{\mathbb{C}}(t_1)$, $0 \le t_1 < \mathcal{T}_1(t_2)$, should also satisfy (5.2). Let $W_{t_2}(z) = \frac{e^{i\xi(t_2)} + g_{t_2}(z)}{e^{i\xi(t_2)} - g_{t_2}(z)}$. Then $Q_{t_2} = \mathrm{Re}\, W_{t_2}$; W_{t_2} maps D_{t_2}

conformally onto the right half plane, and maps $\beta(t_2)$ to ∞. The argument in the proof of Lemma 3.2 can be used here to show that a.s. $\lim_{t_1\to\mathcal{T}_1(t_2)} B^1_{\mathbb{C}}(t_1)=\beta(t_2)$. Thus, $B^1_{\mathbb{C}}(\mathcal{T}_1(t_2))=\beta(t_2)$. In fact, we may view $B^1_{\mathbb{C}}(t_1)$, $0\le t_1<\mathcal{T}_1(t_2)$, as the complex Brownian motion $B_{\mathbb{C}}(t)$ conditioned to leave D_{t_2} at $\beta(t_2)$. This result holds for every $t_2\in(0,\infty)$. So a.s. for every $t_2\in\mathbb{Q}\cap(0,\infty)$, we have $B^1_{\mathbb{C}}(\mathcal{T}_1(t_2))=\beta(t_2)$.

From the definition, it is clear that $\mathcal{T}_1$ as a function of t_2 is decreasing, and $B^1_{\mathbb{C}}[0,\mathcal{T}_1(t_2))$ is disjoint from $\beta[0,t_2]$ for any $t_2\in[0,\infty)$. For any $a\in\mathbb{R}$, it is easy to check that $\{t_2:\mathcal{T}_1(t_2)>a\}$ is an open subset of $[0,\infty)$. So $t_2\mapsto\mathcal{T}_1(t_2)$ is right-continuous. Since both $B^1_{\mathbb{C}}$ and β are continuous, and $\mathbb{Q}\cap(0,\infty)$ is dense in $(0,\infty)$, so a.s. for any $t_2\in(0,\infty)$, $B^1_{\mathbb{C}}(\mathcal{T}_1(t_2))=\beta(t_2)$. From Corollary 2.1, we see that β is a loop-erasure of the reversal of $B^1_{\mathbb{C}}$.

7 Some Remarks

1. One can prove that, under the new coupling measure ν^∞, for any $(\mathcal{F}^1_{t_1})$-stopping time $\bar t_1<\tau_1$, the curve $\beta(t_2)$, $0\le t_2<\mathcal{T}_2(\bar t_1)$, is a radial SLE_2 curve in $\mathbb{D}$ started from 1 aimed at $B^1_{\mathbb{C}}(\bar t_2)$, which stops on hitting $B^1_{\mathbb{C}}[0,\bar t_1]$. In general, β may not visit $B^1_{\mathbb{C}}(\bar t_2)$.

2. Theorem 3.2 can be extended to finitely connected plane domains. Let γ_1 be a Brownian motion started from an interior point z_0 in a finitely connected domain D, stopped on exiting D, and conditioned to hit ∂D at z_1. The process satisfies SDE (3.1) with P_ρ replaced by the Poisson kernel function in D with the pole at z_2. Then the time-reversal of γ_1 has a loop-erasure, which is a continuous LERW in D growing from z_1 to z_0 ([13]).

3. Let γ_2 be the Brownian excursion in D from one boundary point z_1 to another boundary point z_2. The process starts from z_1, and after the initial time, it becomes a Brownian motion in D conditioned to exit D at z_2. We can conclude that the time-reversal of γ_2 has a loop-erasure, which is a continuous LERW in D from z_2 to z_1. For the proof, we may use the coupling technique to construct a coupling of γ_2 with a continuous LERW β in D from z_2 to z_1 such that conditioned on the part of β up to a finite stopping time T, the part of γ_2 before hitting $\beta[0,T]$ is a Brownian excursion in $D\setminus\beta[0,T]$ from z_1 to $\beta(T)$. It is well known that the time-reversal of γ_2 is the Brownian excursion in D from z_2 to z_1. So γ_2 itself has a loop-erasure, which is a continuous LERW in D from z_1 to z_2. Especially, if D is simply connected, then the Brownian excursion from z_1 to z_2 has a loop-erasure, which is a chordal SLE_2 curve in D from z_1 to z_2.

4. Let γ_3 be the Brownian motion in D started from an interior point z_0 and conditioned to hit another interior point z_3. The process satisfies SDE (3.1) with P_ρ replaced by $G_D(z_1,\cdot)$, where $G_D(\cdot,\cdot)$ is the Green function in D. Using the coupling technique, we can conclude that the time-reversal of γ_3 has a loop-erasure, which is a continuous LERW in D from z_3 to z_0 ([16]). It is well known that the time-reversal of γ_3 is the Brownian motion in D started from z_3 conditioned to hit z_0. So γ_3 itself has a loop-erasure, which is a continuous LERW in D from z_0 to z_3.

5. Let γ_4 be a Brownian excursion in D started from a boundary point z_1 and conditioned to hit an interior point z_0. The process starts from z_1, and after the initial time, it becomes the Brownian motion in D conditioned to hit z_0. We can conclude that its time-reversal has a loop-erasure, which is a continuous LERW in D from z_0 to z_1. It is well known that γ_4 is the time-reversal of the γ_1 in Remark 2. So γ_1 has a loop-erasure, which is a continuous LERW in D from z_0 to z_1; and γ_4 has a loop-erasure, which is a continuous LERW in D from z_1 to z_0. In particular, if D is a simply connected domain, a continuous LERW from an interior point z_0 to a random boundary point with harmonic measure distribution is a disc SLE_2 curve ([12]) in D started from z_0. So we conclude that the $B_{\mathbb{C}}(t)$, $0 \le t \le \tau$, in Theorem 3.1 has a loop-erasure, which is a disc SLE_2 curve in D started from z_0.

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