Green's function for chordal SLE curves

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Abstract

For a chordal SLE_κ ($\kappa \in (0,8)$) curve in a domain D, the n-point Green's function valued at distinct points $z_1, \dots, z_n \in D$ is defined to be

$$G(z_1, \dots, z_n) = \lim_{r_1, \dots, r_n \downarrow 0} \prod_{k=1}^{n} r_k^{d-2} \mathbb{P}[\mathrm{dist}(\gamma, z_k) < r_k, 1 \le k \le n],$$

provided that the limit converges. In this paper, we will show that such Green's functions exist for any finite number of points. Along the way we provide the rate of convergence and modulus of continuity for Green's functions as well. Finally, we give up-to-constant bounds for them.

Contents

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1 Introduction

The Schramm-Loewner evolution (SLE) is a measure on the space of curves which was defined in the groundbreaking work of Schramm [19]. It is the main universal object emerging as the scaling limit of many models from statistical physics. Since then the geometry of SLE curves has been studied extensively. See [17, 8] for definition and properties of SLE.

One of the most important functions associated to SLE (in general any random process) is the Green's function. Roughly, it can be defined as the normalized probability that SLE curve hits a set of $n \geq 1$ given points in its domain. See equation (1.1) for precise definition. For $n = 1$, the existence of Green's function for chordal SLE was given in [9] where conformal radius was used instead of Euclidean distance. For $n = 2$, the existence was proved in [15] (again for conformal radius instead of Euclidean distance) following a method initiated by Beffara [4]. Finally in [12] the authors showed that Green's function as defined here (using Euclidean distance) exists for $n = 1, 2$ by modifying proofs in the above mentioned papers. To the best of our knowledge, existence of Green's function for $n > 2$ has not been proved so far. Our main goal in this paper is to show that Green's function exists for all $n \geq 2$. In addition we find convergence rate and modulus of continuity of the Green's function, and provide sharp bounds for it.

Chordal SLE$_\kappa$ ($\kappa > 0$) in a simply connected domain D is a probability measure on curves in $\overline{D}$ from one marked boundary point (or prime end) a to another marked boundary point (or prime end) b. It is first defined in the upper half plane $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ using chordal Loewner equation, and then extended to other domains by conformal maps. For $\kappa \geq 8$, the curve is space filling ([17]), i.e., it visits every point in the domain. In this paper we only consider SLE$_\kappa$ for $\kappa \in (0, 8)$ and fix κ throughout. It is known ([4]) that SLE$_\kappa$ has Hausdorff dimension $d = 1 + \frac{\kappa}{8}$. Let $z_1, \dots, z_n \in D$ be n distinct points. The n-point Green's function for SLE$_\kappa$ (in D from a to b) at $z_1, \dots, z_n$ is defined by

$$G_{(D;a,b)}(z_1, \dots, z_n) = \lim_{r_1,\dots,r_n \downarrow 0} \prod_{k=1}^{n} r_k^{d-2}\, \mathbb{P}\Big[\bigcap_{k=1}^{n} \{\operatorname{dist}(z_k, \gamma) \leq r_k\}\Big], \tag{1.1}$$

provided the limit exists. By conformal invariance of SLE, we easily see that the Green's function satisfies conformal covariance. That is, if $G_{(\mathbb{H};0,\infty)}$ exists, then $G_{(D;a,b)}$ exists for any triple $(D;a,b)$, and if g is a conformal map from $(D;a,b)$ onto $(\mathbb{H};0,\infty)$, then

$$G_{(D;a,b)}(z_1,\dots,z_n)=\prod_{k=1}^{n}|g'(z_j)|^{2-d}G_{(\mathbb{H};0,\infty)}(g(z_1),\dots,g(z_n)).$$

Thus, it suffices to prove the existence of $G_{(\mathbb{H};0,\infty)}$, which we write as G. As we mentioned above, the one-point Green's function $G(z)$ has a closed-form formula:

$$G(z)=\hat{c}(\operatorname{Im} z)^{d-2+\alpha}|z|^{-\alpha}. \tag{1.2}$$

where $\alpha=\frac{8}{\kappa}-1$ is the boundary exponent, and $\hat{c}$ is a positive constant.

Now we can state the main result of the paper.

Theorem 1.1. *For any $n\in\mathbb{N}$, $G(z_1,\dots,z_n)$ exists and is locally Hölder continuous. Also there is an explicit function $F(z_1,\dots,z_n)$ (defined in (2.5)) such that for any distinct points $z_1,\dots,z_n\in\mathbb{H}$, $G(z_1,\dots,z_n)\asymp F(z_1,\dots,z_n)$, where the constant depends only on κ and n.*

We prove stronger results than Theorem 1.1. Specifically we provide a rate of convergence in the limit (1.1). See Theorem 4.1. The function $F(z_1,\dots,z_n)$ appeared implicitly in [18] and we define it explicitly here. The upper bound for Green's function (assuming existence of G) was proved in [18, Theorem 1.1] but the lower bound is new.

Our result will shed light on the study of some random lattice paths, e.g., loop-erased random walk (LERW), which are known to converge to SLE ([14, 21]). More specifically, combining the convergence rate of LERW to SLE_2 ([5]) with our convergence rate of the rescaled visiting probability to Green's function for SLE, one may get a good estimate on the probability that a number of small discs be visited by LERW.

We may also work on the Green's function when some points lie on the boundary. In order to have a non-trivial limit, the exponent $d-2$ in the definition (1.1) for these points should be replaced by $-\alpha$. For $\kappa=8/3$, the existence of boundary Green's function for any n follows from the restriction property ([6]). The existence and exact formulas of boundary Green's functions when $n=1,2$ were provided in [11]. In [7] the authors found closed-form formulas of boundary Green's functions of up to 4 points assuming their existence. Since our upper bound (Proposition 2.3) and lower bound (Theorem 4.3) are about the probability that SLE visits discs, where the centers are allowed to lie on the boundary, we immediately have sharp bounds of the boundary or mixed type Green's functions assuming their existence, which may be proved using the main technique here.

It is also interesting to study the Green's functions for other types of SLE such as radial SLE, $\mathrm{SLE}_\kappa(\rho)$, or stopped SLE. In [3], the authors proved the existence of the conformal radius version of one-point Green's function for radial SLE.

The rest of the paper is organized as the following. In section 2 we go over basic definitions and tools that we need from complex analysis and SLE theory. Then in section 3 we describe the main estimates that we need to show convergence, continuity and lower bound. One of them is a generalization of the main result in [18] which quantifies the probability that SLE can go back and forth between a set of points. We prove this estimate in the Appendix. In section 4 we state our main results in the form that we prove them. After that in section 5 we use estimates provided in section 3 to show existence and continuity of the Green's function following a method initiated in [15]. We prove these two theorems together by induction on the number of the points. Finally in section 6 we prove the sharp lower bound for Green's function given in Theorem 4.3.

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2 Preliminaries

2.1 Notation and Definitions

We fix $\kappa \in (0,8)$ and set (Hausdorff dimension and boundary exponent)

$$d = 1 + \frac{\kappa}{8}, \qquad \alpha = \frac{8}{\kappa} - 1.$$

Note that $d \in (0,2)$ and $\alpha > 2 - d$. Throughout, a constant (such as d or α) depends only on κ and a variable $n \in \mathbb{N}$ (number of points), unless otherwise specified. We write $X \lesssim Y$ or $Y \gtrsim X$ if there is a constant $C > 0$ such that $X \le CY$. We write $X \asymp Y$ if $X \lesssim Y$ and $X \gtrsim Y$. We write $X = O(Y)$ if there are two constants $\delta, C > 0$ such that if $|Y| < \delta$, then $|X| \le C|Y|$. Note that this is slightly weaker than $|X| \lesssim |Y|$.

For $y \ge 0$ define P_y on $[0,\infty)$ by

$$P_y(x) = \begin{cases} y^{\alpha-(2-d)} x^{2-d}, & x \le y; \\ x^{\alpha}, & x \ge y. \end{cases}$$

we will frequently use the following lemmas without reference.

Lemma 2.1. *For $0 \le x_1 < x_2$, $0 \le y_1 \le y_2$, $0 < x$, and $0 \le y$, we have*

$$\frac{P_{y_1}(x_1)}{P_{y_1}(x_2)} \le \frac{P_{y_2}(x_1)}{P_{y_2}(x_2)};$$

$$\Big(\frac{x_1}{x_2}\Big)^{\alpha} \le \frac{P_y(x_1)}{P_y(x_2)} \le \Big(\frac{x_1}{x_2}\Big)^{2-d} = \frac{P_{x_2}(x_1)}{P_{x_2}(x_2)};$$

$$\Big(\frac{y_1}{y_2}\Big)^{\alpha-(2-d)} \le \frac{P_{y_1}(x)}{P_{y_2}(x)} \le 1.$$

Proof. For the first formula, one may first prove that it holds in the following special cases: $y_1 \le y_2 \in [0, x_1]$; $y_1 \le y_2 \in [x_1, x_2]$; and $y_1 \le y_2 \in [x_2, \infty]$. The formula in the general case then easily follows. The second formula follows from the first by first setting $y_1 = 0$ and $y_2 = y$ and then $y_1 = y$ and $y_2 = x_2 \vee y$. The third formula can be proved by considering the following cases one by one: $x \in (0, y_1]$; $x \in [y_1, y_2]$; and $x \in [y_2, \infty)$. $\square$

Lemma 2.2. *Let $z_1, \dots, z_n$ be distinct points in $\overline{\mathbb{H}}$. Let S be a nonempty set in $\mathbb{C}$ with positive distance from $\{z_1, \dots, z_n\}$. Then for any permutation σ of $\{1, \dots, n\}$,*

$$\prod_{k=1}^{n} P_{\operatorname{Im} z_{\sigma(k)}}(\operatorname{dist}(z_{\sigma(k)}, S \cup \{z_{\sigma(j)} : j < k\})) \asymp \prod_{k=1}^{n} P_{\operatorname{Im} z_k}(\operatorname{dist}(z_k, S \cup \{z_j : j < k\})). \tag{2.1}$$

Proof. It suffices to prove the lemma for $\sigma = (k_0, k_0 + 1)$. In this case, the factors on the LHS of (2.1) for $k \neq k_0, k_0 + 1$ agree with the corresponding factors on the RHS of (2.1). So we only need to focus on the factors for $k = k_0, k_0 + 1$. Let $w_1 = z_{k_0}$, $w_2 = z_{k_0+1}$, $u_j = \operatorname{Im} w_j$, $L_j = \operatorname{dist}(w_j, S \cup \{z_k : k < k_0\})$, $j = 1, 2$. Then it suffices to show that

$$P_{u_2}(L_2) P_{u_1}(L_1 \wedge |w_1 - w_2|) \asymp P_{u_1}(L_1) P_{u_2}(L_2 \wedge |w_2 - w_1|). \tag{2.2}$$

Let $r = |w_1 - w_2|$. Note that $|u_1 - u_2|, |L_1 - L_2| \leq r$. We consider several cases. First, suppose $L_1 \leq r$. Then $L_2 \leq 2r$, and we get $L_1 \wedge r = L_1$ and $L_2/2 \leq L_2 \wedge r \leq L_2$. From the above lemma, we immediately get (2.2). Second, suppose $L_2 \leq r$. This case is similar to the first case. Third, suppose $L_1, L_2 \geq r$. In this case, $L_1 \wedge r = L_2 \wedge r = r$, and $L_1 \asymp L_2$. Now we consider subcases. First, suppose $u_1 \leq r$. Then $u_2 \leq 2r$. Since $r \leq u_2 \leq 2r$, from the above lemma, we get $\frac{P_{u_2}(L_2)}{P_{u_2}(r)} \asymp \frac{P_r(L_2)}{P_r(r)} = (\frac{L_2}{r})^\alpha$. Since $u_1 \leq r$, we have $\frac{P_{u_1}(L_1)}{P_{u_1}(r)} = (\frac{L_1}{r})^\alpha$. Since $L_1 \asymp L_2$, we get (2.2) in the first subcase. Second, suppose $u_2 \leq r$. This is similar to the first subcase. Third, suppose $u_1, u_2 \geq r$. Then we get $\frac{P_{u_j}(L_j)}{P_{u_j}(r)} = (\frac{L_j}{r})^{2-d}$, $j = 1, 2$. Using $L_1 \asymp L_2$, we get (2.2) in the last subcase. $\square$

For (ordered) set of distinct points $z_1, \dots, z_n \in \overline{\mathbb{H}} \setminus \{0\}$, we let $z_0 = 0$ and define for $1 \leq k \leq n$,

$$l_k = \min_{0 \leq j \leq k-1} \{|z_k - z_j|\}, \quad d_k = \min_{0 \leq j \leq n, j \neq k} \{|z_k - z_j|\}, \quad y_k = \operatorname{Im} z_k, \quad R_k = d_k \wedge y_k. \tag{2.3}$$

Also Set

$$Q = \max_{1 \leq k \leq n} \frac{|z_k|}{d_k} \geq 1. \tag{2.4}$$

Note that we have

$$R_k \leq d_k \leq l_k.$$

For $r_1, \dots, r_n > 0$, define

$$F(z_1, \dots, z_n; r_1, \dots, r_n) = \prod_{k=1}^{n} \frac{P_{y_k}(r_k)}{P_{y_k}(l_k)};$$

$$F(z_1, \dots, z_n) = \lim_{r_1, \dots, r_n \to 0^+} \prod_{k=1}^{n} r_k^{d-2} F(z_1, \dots, z_n; r_1, \dots, r_n) = \prod_{k=1}^{n} \frac{y_k^{\alpha-(2-d)}}{P_{y_k}(l_k)}. \tag{2.5}$$

This is the function F in Theorem 1.1. When it is clear from the context, we write F for $F(z_1,\dots,z_n)$. From Lemma 2.1 we see that

$$F(z_1,\dots,z_n;r_1,\dots,r_n)\le F(z_1,\dots,z_n)\prod_{k=1}^n r_k^{2-d},\quad \text{if } r_k\le l_k, 1\le k\le n. \tag{2.6}$$

Applying Lemma 2.2 with $S=\{0\}$, we see that for any permutation σ of $\{1,\dots,n\}$,

$$F(z_1,\dots,z_n;r_1,\dots,r_n)\asymp F(z_{\sigma(1)},\dots,z_{\sigma(n)};r_{\sigma(1)},\dots,r_{\sigma(n)}), \tag{2.7}$$

and

$$F(z_1,\dots,z_n)\asymp F(z_{\sigma(1)},\dots,z_{\sigma(n)}).$$

Let D be a simply connected domain with two distinct prime ends w_0 and w_∞. We define

$$F_{(D;w_0,w_\infty)}(z_1,\dots,z_n)=\prod_{j=1}^n |g'(z_j)|^{2-d}\cdot F(g(z_1),\dots,g(z_n)),$$

where g is any conformal map from $(D;w_0,w_\infty)$ onto $(\mathbb{H};0,\infty)$. Although such g is not unique, the value of $F_{(D;w_0,w_\infty)}$ does not depend on the choice of g.

Throughout, we use γ to denote a (random) chordal Loewner curve, use (U_t) to denote its driving function, and (g_t) and (K_t) the chordal Loewner maps and hulls driven by U_t). This means that γ is a continuous curve in $\overline{\mathbb{H}}$ starting from a point on $\mathbb{R}$; for each t, $H_t:=\mathbb{H}\setminus K_t$ is the unbounded component of $\mathbb{H}\setminus\gamma[0,t]$, whose boundary contains $\gamma(t)$; and g_t is a conformal map from $(H_t;\gamma(t),\infty)$ onto $(\mathbb{H};0,\infty)$ that solves the chordal Loewner equation

$$\partial_t g_t(z)=\frac{2}{g_t(z)-U_t},\quad g_0(z)=z. \tag{2.8}$$

Let $Z_t=g_t-U_t$ denote the centered Loewner map, which is a conformal map from $(H_t;\gamma(t),\infty)$ onto $(\mathbb{H};0,\infty)$. See [8] for more on Loewner curves.

When γ is fixed, for any set S, τ_S is used to denote the infimum of the times that γ visits S, and is set to be ∞ if such times do not exist. We write $\tau_r^{z_0}$ for $\tau_{\{|z-z_0|\le r\}}$, and T_{z_0} for $\tau_0^{z_0}=\tau_{\{z_0\}}$. So another way to say that $\operatorname{dist}(\gamma,z_0)\le r$ is $\tau_r^{z_0}<\infty$.

Let $\mathbb{P}$ denote the law of a chordal SLE$_\kappa$ curve in $\mathbb{H}$ from 0 to ∞, and $\mathbb{E}$ the corresponding expectation. Then $\mathbb{P}$ is a probability measure on the space of chordal Loewner curves such that the driving function (U_t) has the law of $\sqrt{\kappa}$ times a standard Brownian motion. In fact, chordal SLE$_\kappa$ is defined by solving (2.8) with $U_t=\sqrt{\kappa}B_t$.

As we mentioned the upper bound in Theorem 1.1 is not new. We now state [18, Theorem 1.1] using the notation just defined.

Proposition 2.3. *Let $z_1,\dots,z_n$ be distinct points in $\overline{\mathbb{H}}\setminus\{0\}$. Let $d_1,\dots,d_n$ be defined by (2.3). Let $r_j\in(0,d_j)$, $1\le j\le n$. Then we have*

$$\mathbb{P}[\tau_{r_j}^{z_j}<\infty, 1\le j\le n]\lesssim F(z_1,\dots,z_n;r_1,\dots,r_n).$$

2.2 Lemmas on $\mathbb{H}$-hulls

We will need some results on $\mathbb{H}$-hulls. A relatively closed bounded subset K of $\mathbb{H}$ is called an $\mathbb{H}$-hull if $\mathbb{H}\setminus K$ is simply connected. Given an $\mathbb{H}$-hull K, we use g_K to denote the unique conformal map from $\mathbb{H}\setminus K$ onto $\mathbb{H}$ that satisfies $g_K(z)=z+O(|z|^{-1})$ as $z\to\infty$. The half-plane capacity of K is $\mathrm{hcap}(K):=\lim_{z\to\infty} z(g_K(z)-z)$. Let $f_K=g_K^{-1}$. If $K=\emptyset$, then $g_K=f_K=\mathrm{id}$, and $\mathrm{hcap}(K)=0$. Now suppose $K\neq\emptyset$. Let $a_K=\min(\overline{K}\cap\mathbb{R})$ and $b_K=\max(\overline{K}\cap\mathbb{R})$. Let $K^{\mathrm{doub}}=K\cup[a_K,b_K]\cup\{\overline{z}:z\in K\}$. By Schwarz reflection principle, g_K extends to a conformal map from $\mathbb{C}\setminus K^{\mathrm{doub}}$ onto $\mathbb{C}\setminus[c_K,d_K]$ for some $c_K<d_K\in\mathbb{R}$, and satisfies $g_K(\overline{z})=\overline{g_K(z)}$. In this paper, we write S_K for $[c_K,d_K]$.

Examples

- For $x_0\in\mathbb{R}$ and $r>0$, let $\overline{D}^+_{x_0,r}$ denote semi-disc $\{z\in\mathbb{H}:|z-x_0|\le r\}$, which is an $\mathbb{H}$-hull. It is straightforward to check that $g_{\overline{D}^+_{x_0,r}}(z)=z+\frac{r^2}{z-x_0}$, $\mathrm{hcap}(\overline{D}^+_{x_0,r})=r^2$, and $S_{\overline{D}^+_{x_0,r}}=[x_0-2r,x_0+2r]$.
- Each K_t associated with a chordal Loewner curve γ is an $\mathbb{H}$-hull with $\mathrm{hcap}(K_t)=2t$. Since $\gamma(t)\in\partial K_t$ and $g_t(\gamma(t))=U_t$, we have $U_t\in S_{K_t}$.

Lemma 2.4. *For any nonempty $\mathbb{H}$-hull K, there is a positive measure μ_K supported by S_K with total mass $|\mu_K|=\mathrm{hcap}(K)$ such that,*

$$f_K(z)-z=\int\frac{-1}{z-x}d\mu_K(x),\quad z\in\mathbb{C}\setminus S_K. \tag{2.9}$$

Proof. This is [21, Formula (5.1)]. □

Lemma 2.5. *If a nonempty $\mathbb{H}$-hull K is contained in $\overline{D}^+_{x_0,r}$ for some $x_0\in\mathbb{R}$ and $r>0$, then $\mathrm{hcap}(K)\le r^2$, $S_K\subset[x_0-2r,x_0+2r]$, and*

$$|g_K(z)-z|\le 3r,\quad z\in\mathbb{C}\setminus K^{\mathrm{doub}}. \tag{2.10}$$

Proof. From the monotone property of hcap ([8]), we have $\mathrm{hcap}(K)\le\mathrm{hcap}(\overline{D}^+_{x_0,r})=r^2$. From [21, Lemma 5.3], we know that $S_K\subset S_{\overline{D}^+_{x_0,r}}=[x_0-2r,x_0+2r]$. Formula (2.10) follows from [8, Formula (3.12)] and that $g_{K-x_0}(z-x_0)=g_K(z)-x_0$. □

Lemma 2.6. *Let K be as in the above lemma. Then for any $z\in\mathbb{C}$ with $|z-x_0|\ge 5r$, we have*

$$|g_K(z)-z|\le 2|z-x_0|\Big(\frac{r}{|z-x_0|}\Big)^2; \tag{2.11}$$

$$\frac{|\,\mathrm{Im}\, g_K(z)-\mathrm{Im}\, z|}{|\,\mathrm{Im}\, z|}\le 4\Big(\frac{r}{|z-x_0|}\Big)^2; \tag{2.12}$$

$$|g_K'(z)-1|\le 5\Big(\frac{r}{|z-x_0|}\Big)^2. \tag{2.13}$$

Proof. Since $g_{K-x_0}(z-x_0)=g_K(z)-x_0$, we may assume that $x_0=0$. From the above two lemmas, we find that $|\mu_K|\le r^2$ and

$$f_K(w)-w=\int_{-2r}^{2r}\frac{-1}{z-w}d\mu_K(w),\quad w\in\mathbb{C}\setminus[-2r,2r]. \tag{2.14}$$

Thus, if $|w|>2r$, then $|f_K(w)-w|\le\frac{r^2}{|w|-2r}$. So f_K maps the circle $\{|z|=4r\}$ onto a Jordan curve that lies within the circles $\{|z|=3.5r\}$ and $\{|z|=4.5r\}$. Thus, if $|z|>5r$, then $|g_K(z)|>4r$, and $|z-g_K(z)|=|f(g_K(z))-g_K(z)|\le\frac{r^2}{|g_K(z)|-2r}\le r/2$, which implies $|z|\le|g_K(z)|+r/2$, and $|g_K(z)-z|\le\frac{r^2}{|g_K(z)|-2r}\le\frac{r^2}{|z|-2.5r}\le\frac{r^2}{|z|/2}$. So we get (2.11).

Taking the imaginary part of (2.14), we find that, if $w\in\mathbb{H}$ and $|w|>2r$, then $|\operatorname{Im} f_K(w)-\operatorname{Im} w|\le|\operatorname{Im} w|\frac{r^2}{(|w|-2r)^2}$. Letting $w=g_K(z)$ with $z\in\mathbb{H}$ and $|z|>5r$, we find that

$$|\operatorname{Im} z-\operatorname{Im} g_K(z)|\le|\operatorname{Im} g_K(z)|\frac{r^2}{(|g_K(z)|-2r)^2}\le|\operatorname{Im} z|\frac{r^2}{(|z|-2.5r)^2}\le|\operatorname{Im} z|\frac{r^2}{(|z|/2)^2},$$

which implies (2.12). Here we used that $|\operatorname{Im} g_K(z)|\le|\operatorname{Im} z|$ that can be seen from (2.14).

Differentiating (2.14) w.r.t. z, we find that, if $|w|>2r$, then $|f_K'(w)-1|\le\frac{r^2}{(|w|-2r)^2}$. Letting $w=g_K(z)$ with $z\in\mathbb{H}$ and $|z|>5r$, we find that

$$|1/g_K'(z)-1|\le\frac{r^2}{(|g_K(z)|-2r)^2}\le\frac{r^2}{(|z|-2.5r)^2}\le\frac{r^2}{(|z|/2)^2},$$

which then implies (2.13). □

Lemma 2.7. *Let K be a nonempty $\mathbb{H}$-hull. Suppose $z\in\mathbb{H}$ satisfies that $\operatorname{dist}(z,S_K)\ge 4\operatorname{diam}(S_K)$. Then $\operatorname{dist}(f_K(z),K)\ge 2\operatorname{diam}(K)$.*

Proof. Let $r=\operatorname{diam}(S_K)$. Since g_K maps $\mathbb{C}\setminus K^{\text{doub}}$ conformally onto $\mathbb{C}\setminus S_K$, fixes ∞, and satisfies that $g_K'(\infty)=1$, we see that K^{doub} and S_K have the same whole-plane capacity. Thus, $\operatorname{diam}(K)\le\operatorname{diam}(K^{\text{doub}})\le\operatorname{diam}(S_K)$. Take any $x_0\in\overline{K}\cap\mathbb{R}$. Then $K\subset\overline{D}^+_{x_0,r}$. So $|\mu_K|=\operatorname{hcap}(K)\le r^2$. Since $\operatorname{dist}(z,S_K)\ge 4r$, from (2.9) we get $|f_K(z)-z|\le r/4$. From [21, Lemma 5.2], we know $x_0\in[a_K,b_K]\subset[c_K,d_K]=S_K$. Thus, $\operatorname{dist}(f_K(z),K)\ge|f_K(z)-x_0|-r\ge|z-x_0|-|f_K(z)-z|-r\ge\operatorname{dist}(z,S_K)-2r>2r\ge 2\operatorname{diam}(K)$. □

Lemma 2.8. *Let K be an $\mathbb{H}$-hull, and w_0 be a prime end of $\mathbb{H}\setminus K$ that sits on ∂K. Let $z_0\in\mathbb{H}\setminus K$ and $R=\operatorname{dist}(z_0,K)>0$. Let g be any conformal map from $\mathbb{H}\setminus K$ onto $\mathbb{H}$ that fixes ∞ and sends w_0 to 0. Then for $z_1\in\mathbb{H}\setminus K$, we have*

$$\frac{|g(z_1)-g(z_0)|}{|g(z_0)|}=O\Big(\frac{|z_1-z_0|}{R}\Big); \tag{2.15}$$

$$\frac{|\operatorname{Im} g(z_1)-\operatorname{Im} g(z_0)|}{\operatorname{Im} g(z_0)}=O\Big(\frac{|\operatorname{Im} z_1-\operatorname{Im} z_0|}{\operatorname{Im} z_0}\Big)+O\Big(\frac{|z_1-z_0|}{R}\Big)^{1/2}. \tag{2.16}$$

Proof. By scaling invariance, we may assume that $g = g_K - x_0$, where $x_0 = g_K(w_0) \in [c_K, d_K]$. From Koebe's 1/4 theorem, we know that

$$|g(z_0)| = |g_K(z_0) - x_0| \geq \operatorname{dist}(g_K(z_0), [c_K, d_K]) \gtrsim |g'(z_0)|R.$$

Applying Koebe's distortion theorem, we find that, if $|z_1 - z_0| < R/5$, then

$$|g'(z_1) - g'(z_0)| \lesssim |g'(z_0)| \frac{|z_1 - z_0|}{R}. \tag{2.17}$$

$$|g'(z_1)| \asymp |g'(z_0)|, \quad |g(z_1) - g(z_0)| \lesssim |g'(z_0)||z_1 - z_0|. \tag{2.18}$$

Combining the second formula with the lower bound of $|g(z_0)|$, we get (2.15).

To derive (2.16), we assume $\frac{|\operatorname{Im} z_1 - \operatorname{Im} z_0|}{\operatorname{Im} z_0}$ and $\frac{|z_1 - z_0|}{R}$ are sufficiently small, and consider several cases. First, assume that $\operatorname{Im} z_0 \geq \frac{R}{C}$ for some big constant C. From Koebe's 1/4 theorem, we know that $\operatorname{Im} g(z_0) \gtrsim |g'(z_0)|R$. This together with the inequalities $|\operatorname{Im} g(z_1) - \operatorname{Im} g(z_0)| \leq |g(z_1) - g(z_0)|$ and (2.18) implies (2.16).

Now assume that $\operatorname{Im} z_0 \leq \frac{R}{C}$. Note that $z_0 - \overline{z_0} = 2i \operatorname{Im} z_0$ and $g(z_0) - g(\overline{z_0}) = 2i \operatorname{Im} g(z_0)$. From Koebe's distortion theorem, we see that when C is big enough,

$$|\operatorname{Im} g(z_0) - g'(z_0) \operatorname{Im} z_0| \lesssim |g'(z_0)| \operatorname{Im} z_0 \frac{\operatorname{Im} z_0}{R}, \tag{2.19}$$

which implies that

$$\operatorname{Im} g(z_0) \gtrsim |g'(z_0)| \operatorname{Im} z_0. \tag{2.20}$$

Now we assume that $\operatorname{Im} z_0 \geq \sqrt{R|z_1 - z_0|}$. Combining (2.20) with (2.18) and the inequalities $|\operatorname{Im} g(z_1) - \operatorname{Im} g(z_0)| \leq |g(z_1) - g(z_0)|$ and $\frac{|z_1 - z_0|}{\operatorname{Im} z_0} \leq (\frac{|z_1 - z_0|}{R})^{1/2}$, we get (2.16).

Finally, we assume that $\operatorname{Im} z_0 \leq \sqrt{R|z_1 - z_0|}$. Let $R_1 = R - |z_1 - z_0| \gtrsim R$. Then $\{|z - z_1| < R_1\} \subset \{|z - z_0| < R\}$. From Koebe's distortion theorem and (2.17), we get

$$|\operatorname{Im} g(z_1) - g'(z_1) \operatorname{Im} z_1| \lesssim |g'(z_1)| \operatorname{Im} z_1 \frac{\operatorname{Im} z_1}{R_1} \lesssim |g'(z_0)| \operatorname{Im} z_0 \frac{\operatorname{Im} z_0}{R}. \tag{2.21}$$

Now we have

$$\begin{aligned} |\operatorname{Im} g(z_1) - \operatorname{Im} g(z_0)| \leq & |\operatorname{Im} g(z_0) - g'(z_0) \operatorname{Im} z_0| + |\operatorname{Im} g(z_1) - g'(z_1) \operatorname{Im} z_1| \\ & + |g'(z_1) - g'(z_0)| \operatorname{Im} z_0 + |g'(z_1)||\operatorname{Im} z_1 - \operatorname{Im} z_0|. \end{aligned}$$

Combining the above inequality with the inequalities (2.17-2.21) and $\frac{\operatorname{Im} z_0}{R} \leq (\frac{|z_1 - z_0|}{R})^{1/2}$, we get (2.16) in the last case. $\square$

2.3 Lemma on extremal length

We use $d_\Omega(X, Y)$ to denote the extremal distance between X and Y in Ω.

Lemma 2.9. *Let S_1 and S_2 be a disjoint pair of connected bounded closed subsets of $\overline{\mathbb{H}}$ that intersect $\mathbb{R}$. Then*

$$\prod_{j=1}^{2}\Big(\frac{\operatorname{diam}(S_j)}{\operatorname{dist}(S_1,S_2)}\wedge 1\Big)\le 144e^{-\pi d_{\mathbb{H}}(S_1,S_2)}.$$

Proof. For $j=1,2$, let $S_j^{\rm doub}$ be the union of S_j and its reflection about $\mathbb{R}$. By reflection principle, $d_{\mathbb{H}}(S_1,S_2)=2d_{\mathbb{C}}(S_1^{\rm doub},S_2^{\rm doub})$. Choose $z_j\in S_j$, $j=1,2$, such that $|z_2-z_1|=d_S:=\operatorname{dist}(S_1,S_2)$. Let $r_j=\max_{z\in S_j^{\rm doub}}|z-z_j|$, $j=1,2$. From Teichmüller Theorem and conformal invariance of extremal distance ([1]), we find that

$$d_{\mathbb{C}}(S_1^{\rm doub},S_2^{\rm doub})\le d_{\mathbb{C}}([-r_1,0],[d_S,d_S+r_2])=d_{\mathbb{C}}([-1,0],[R,\infty))=\Lambda(R),$$

where $R>0$ satisfies that $\frac{1}{1+R}=\prod_{j=1}^2\frac{r_j}{d_S+r_j}$, and $\Lambda(R)$ is the modulus of the Teichmüller domain $\mathbb{C}\setminus([-1,0],[R,\infty))$. From [1, Formula (4-21)] and the above computation, we get

$$e^{-\pi d_{\mathbb{H}}(S_1,S_2)}=e^{-2\pi\Lambda(R)}\ge\frac{1}{16(R+1)}=\frac{1}{16}\prod_{j=1}^{2}\frac{r_j}{d_S+r_j}.$$

Since $\operatorname{diam}(S_j)\le 2r_j$ and $\frac{2r_j}{d_S}\wedge 1\le\frac{3r_j}{d_S+r_j}$, the proof is now complete. $\square$

2.4 Lemmas on two-sided radial SLE

For $z\in\mathbb{H}$, we use $\mathbb{P}_z^*$ to denote the law of a two-sided radial SLE$_\kappa$ curve through z. For $z\in\mathbb{R}\setminus\{0\}$, we use $\mathbb{P}_z^*$ to denote the law of a two-sided chordal SLE$_\kappa$ curve through z. Let $\mathbb{E}_z^*$ denote the corresponding expectation. In any case, we have $\mathbb{P}_z^*$-a.s., $T_z<\infty$. See [15, 16] for definitions and more details on these measures. For a random chordal Loewner curve γ, we use $(\mathcal{F}_t)$ to denote the filtration generated by γ.

Lemma 2.10. *Let $z\in\mathbb{H}$ and $R\in(0,|z|)$. Then $\mathbb{P}_z^*$ is absolutely continuous w.r.t. $\mathbb{P}[\cdot|\tau_R^z<\infty]$ on $\mathcal{F}_{\tau_R^z}\cap\{\tau_R^z<\infty\}$, and the Radon-Nikodym derivative is uniformly bounded.*

Proof. It is known ([15, 16]) that $\mathbb{P}_z^*$ is obtained by weighting $\mathbb{P}$ using $M_t^z/G(z)$, where $M_t^z=|g_t'(z)|^{2-d}G(Z_t(z))$ and $G(z)$ is given by (1.2). Since $\mathbb{P}[\cdot|\tau_R^z<\infty]$ is obtained by weighting the restriction of $\mathbb{P}$ to $\{\tau_R^z<\infty\}$ using $1/\mathbb{P}[\tau_R^z<\infty]$, it suffices to prove that $\frac{M_\tau^z}{G(z)}\cdot\mathbb{P}[\tau<\infty]$ is uniformly bounded, where $\tau=\tau_R^z$.

Let $y=\operatorname{Im} z$. From [18, Lemma 2.6] we have $\mathbb{P}[\tau<\infty]\lesssim\frac{P_y(R)}{P_y(|z|)}$. Let $\widetilde{z}=g_\tau(z)$ and $\widetilde{y}=\operatorname{Im}\widetilde{z}$. It suffices to show that

$$\frac{|\widetilde{z}|^{-\alpha}\widetilde{y}^{\alpha-(2-d)}}{|z|^{-\alpha}y^{\alpha-(2-d)}}\cdot|g_\tau'(z)|^{2-d}\cdot\frac{P_y(R)}{P_y(|z|)}\lesssim 1. \tag{2.22}$$

We consider two cases. First, suppose $y \geq R/10$. From Lemma 2.1, we get $\frac{P_y(R)}{P_y(|z|)} \lesssim (\frac{y}{|z|})^\alpha (\frac{R}{y})^{2-d}$. Applying Koebe's 1/4 theorem, we get $\widetilde{y} \gtrsim |g_\tau'(z)|R$. Thus,

$$\text{LHS of (2.22)} \lesssim \frac{(y/|\widetilde{z}|)^\alpha (|g_\tau'(z)|R)^{-(2-d)}}{|z|^{-\alpha} y^{\alpha-(2-d)}} \cdot |g_\tau'(z)|^{2-d} \cdot \Big(\frac{y}{|z|}\Big)^\alpha \Big(\frac{R}{y}\Big)^{2-d} = \Big(\frac{\widetilde{y}}{|\widetilde{z}|}\Big)^\alpha \leq 1.$$

So we get (2.22) in the first case. Second, assume that $y \leq R/10$. Then we have $\frac{P_y(R)}{P_y(|z|)} = (\frac{R}{|z|})^\alpha$. Applying Koebe's distortion theorem, we get $\widetilde{y} \asymp |g_\tau'(z)|y$. Applying Koebe's 1/4 theorem, we get $|\widetilde{z}| \gtrsim |g_\tau'(z)|R$. Thus,

$$\text{LHS of (2.22)} \lesssim \frac{(|g_\tau'(z)|R)^{-\alpha}(|g_\tau'(z)|y)^{\alpha-(2-d)}}{|z|^{-\alpha} y^{\alpha-(2-d)}} \cdot |g_\tau'(z)|^{2-d} \cdot \Big(\frac{R}{|z|}\Big)^\alpha.$$

So we get (2.22) in the second case. The proof is now complete. $\square$

Lemma 2.11. *Let $z \in \mathbb{H}$ and $R \in (0, |z|)$. Then for any $w \in \mathbb{H}$ such that $\frac{|w-z|}{R}$ is sufficiently small, $\mathbb{P}_z^*$ and $\mathbb{P}_w^*$ restricted to $\mathcal{F}_{\tau_R^z}$ are absolutely continuous w.r.t. each other, and*

$$\log\Big(\frac{d\mathbb{P}_w^*|_{\mathcal{F}_{\tau_R^z}}}{d\mathbb{P}_z^*|_{\mathcal{F}_{\tau_R^z}}}\Big) = O\Big(\frac{|z-w|}{R}\Big).$$

Proof. Let G and $M_t^\cdot$ be as in the above proof. Let $\tau = \tau_R^z$. It suffices to show that

$$\log\Big(\frac{M_\tau^z}{G(z)}\Big/\frac{M_\tau^w}{G(w)}\Big) = O\Big(\frac{|z-w|}{R}\Big).$$

Since $||z|-|w|| \leq |z-w|$ and $|z| \geq R$, we get $\log\frac{|w|}{|z|}| = O(\frac{|z-w|}{R})$. Let $\widetilde{z} = g_\tau(z) - U_\tau$ and $\widetilde{w} = g_\tau(w) - U_\tau$. From Koebe's 1/4 theorem and distortion theorem, we get $|\widetilde{z}| \gtrsim |g_\tau'(z)|R$ and $|\widetilde{z}-\widetilde{w}| \lesssim |g_\tau'(z)||z-w|$. So we get $\log\frac{|\widetilde{w}|}{|\widetilde{z}|} = O(\frac{|z-w|}{R})$. From Koebe's distortion theorem, we get $\log\frac{|g_\tau'(w)|}{|g_\tau'(z)|} = O(\frac{|z-w|}{R})$. So it suffices to show that

$$\log\Big(\frac{\operatorname{Im}\widetilde{w}}{\operatorname{Im} w}\Big/\frac{\operatorname{Im}\widetilde{z}}{\operatorname{Im} z}\Big) = O\Big(\frac{|z-w|}{R}\Big). \tag{2.23}$$

Now we consider two cases. First, suppose that $\operatorname{Im} z \geq R/8$. Since $|\operatorname{Im} w - \operatorname{Im} z| \leq |w-z|$ we get $\log\frac{\operatorname{Im} w}{\operatorname{Im} z} = O(\frac{|z-w|}{R})$. Applying Koebe's 1/4 theorem, we get $\operatorname{Im}\widetilde{z} \gtrsim |g_\tau'(z)|R$. Since $|\operatorname{Im}\widetilde{w} - \operatorname{Im}\widetilde{z}| \leq |\widetilde{w}-\widetilde{z}| \lesssim |g_\tau'(z)||z-w|$, from the above argument, we get $\log\frac{\operatorname{Im}\widetilde{w}}{\operatorname{Im}\widetilde{z}} = O(\frac{|z-w|}{R})$, which implies (2.23). Second, suppose that $\operatorname{Im} z \leq R/8$. Then $\operatorname{Im} w < R/4$ if $|z-w| < R/8$. Applying Koebe's distortion theorem, we get $\log(\frac{\operatorname{Im}\widetilde{z}}{|g_\tau'(z)|\operatorname{Im} z}), \log(\frac{\operatorname{Im}\widetilde{w}}{|g_\tau'(w)|\operatorname{Im} w}) = O(\frac{|z-w|}{R})$, which together with $\log\frac{|g_\tau'(w)|}{|g_\tau'(z)|} = O(\frac{|z-w|}{R})$ imply (2.23) in the second case. $\square$

Remark The above two lemmas still hold if z or w lies on $\mathbb{R}\setminus\{0\}$, and the two-sided radial measure is replaced by the two-sided chordal measure.

3 Main Estimates

Theorem 3.1. *Let $z_1,\dots,z_n$ be distinct points in $\overline{\mathbb{H}}\setminus\{0\}$, where $n\ge 2$. Let $r_j\in(0,d_j/8)$, $1\le j\le n$. Let $k_0\in\{2,\dots,n\}$ and $s_{k_0}\in(r_{k_0},|z_{k_0}-z_1|\wedge|z_{k_0}|)$. Then we have $\beta>0$ such that*

$$\mathbb{P}[\tau^{z_1}_{r_1}<\cdots<\tau^{z_n}_{r_n};\operatorname{inrad}_{H_{\tau^{z_1}_{r_1}}}(z_{k_0})\le s_{k_0}]\lesssim F(z_1,\dots,z_n;r_1,\dots,r_n)\Big(\frac{s_{k_0}}{|z_{k_0}-z_1|\wedge|z_{k_0}|}\Big)^{\beta}.$$

The proof of this theorem is long and similar to that of [18, Theorem 1.1], but is quite different from other proofs of this paper. So we postpone the proof to the Appendix.

Lemma 3.2. *Let $z_1\in\mathbb{H}$ and $0<r<\eta<R$. Further suppose $r<\operatorname{Im} z_1$. Let ξ be a connected component of $H_{\tau^{z_1}_\eta}\cap\{|z-z_1|=R\}$. Then*

(i) $\mathbb{P}[\gamma[\tau^{z_1}_\eta,\tau^{z_1}_r]\cap\xi\ne\emptyset|\tau^{z_1}_r<\infty]\lesssim(\frac{\eta}{R})^{\alpha/4}$.

(ii) $\mathbb{P}^*_{z_1}[\gamma[\tau^{z_1}_\eta,T_{z_1}]\cap\xi\ne\emptyset]\lesssim(\frac{\eta}{R})^{\alpha/4}$.

Proof. (i) From [12, Theorem 2.3], we know that there are constants $C,\delta>0$ such that, if $r<\delta\operatorname{Im} z_1$, then $\mathbb{P}[\tau^{z_1}_r<\infty]\ge CG(z_1)r^{2-d}$. Thus, for any $r<\operatorname{Im} z_1$,

$$\mathbb{P}[\tau^{z_1}_r<\infty]\ge C\delta^{2-d}G(z_1)r^{2-d}\gtrsim F(z_1)r^{2-d}=F(z_1;r). \tag{3.1}$$

We will show that

$$\mathbb{P}[\gamma[\tau^{z_1}_\eta,\tau^{z_1}_r]\cap\xi\ne\emptyset;\tau^{z_1}_r<\infty]\lesssim F(z_1;r)\Big(\frac{\eta}{R}\Big)^{\alpha/4}, \tag{3.2}$$

which together with (3.1) implies (i).

To prove (3.2), using Lemma 2.1, we may assume that $r=\eta e^{-n}$ for some $n\in\mathbb{N}$. Let $r_k=\eta e^{-k}$, $0\le k\le n$. Let E denote the event in (3.2). Then $E=\bigcup_{k=1}^n E_k$, where

$$E_k=\{\gamma[\tau^{z_1}_{r_0},\tau^{z_1}_{r_{k-1}}]\cap\xi=\emptyset;\gamma[\tau^{z_1}_{r_{k-1}},\tau^{z_1}_{r_k}]\cap\xi\ne\emptyset;\tau^{z_1}_{r_n}<\infty\}.$$

Let $y_1=\operatorname{Im} z_1$. From [18, Lemma 2.6] we know that

$$\mathbb{P}[\tau^{z_1}_{r_{k-1}}<\infty]\lesssim\frac{P_{y_1}(r_{k-1})}{P_{y_1}(|z_1|)};\quad \mathbb{P}[\tau^{z_1}_{r_n}<\infty|\mathcal{F}_{\tau^{z_1}_{r_k}},\tau^{z_1}_{r_k}<\infty]\lesssim\frac{P_{y_1}(r_n)}{P_{y_1}(r_k)}. \tag{3.3}$$

Suppose $\tau^{z_1}_{r_{k-1}}<\infty$ and $\gamma[\tau^{z_1}_{r_0},\tau^{z_1}_{r_{k-1}}]\cap\xi=\emptyset$. Let ρ be a connected component of $\{|z-z_1|=\sqrt{r_{k-1}R}\}\cap H_{\tau^{z_1}_{r_{k-1}}}$ that separates z_1 from ξ in $H_{\tau^{z_1}_{r_{k-1}}}$. Since ρ is a crosscut of $H_{\tau^{z_1}_{r_{k-1}}}$, it divides $H_{\tau^{z_1}_{r_{k-1}}}$ into a bounded domain and a unbounded domain. A crosscut in a domain D is an open simple curve in D whose two ends approach to two boundary points of D. Let E_b (resp.

E_u) denote the events that ξ lies in the bounded (resp. unbounded) domain. For the event E_b, we apply [18, Lemma 2.5] to the crosscuts ρ and ξ to get

$$\begin{aligned}&\mathbb{P}[\gamma[\tau^{z_1}_{r_{k-1}},\tau^{z_1}_{r_k}]\cap\xi\neq\emptyset;E_b|\mathcal{F}_{\tau^{z_1}_{r_{k-1}}},\tau^{z_1}_{r_{k-1}}<\infty,\gamma[\tau^{z_1}_{r_0},\tau^{z_1}_{r_{k-1}}]\cap\xi=\emptyset]\\ \lesssim& e^{-\alpha\pi d_{\mathbb{C}}(\rho,\xi)}\lesssim\Big(\frac{r_{k-1}}{R}\Big)^{\alpha/4}.\end{aligned}$$

Combining this estimate with (3.3) and Lemma 2.1, we get

$$\mathbb{P}[E_k\cap E_b]\lesssim F(z_1;r)\Big(\frac{r_{k-1}}{R}\Big)^{\alpha/4}\Big(\frac{r_k}{r_{k-1}}\Big)^{\alpha}. \tag{3.4}$$

If E_u happens, then ρ separates z_1 from ∞ in $H_{\tau^{z_1}_{r_{k-1}}}$. Let T_ρ denote the first time after $\tau^{z_1}_{r_{k-1}}$ that γ visits ρ, and let $\widetilde{\rho}$ (resp. J) be a connected component of $\rho\cap H_{T_\rho}$ (resp. $\{|z-z_1|=r_{k-1}\}\cap H_{T_\rho}$ that separates z_1 from ∞ in H_{T_ρ}. Applying [18, Lemma 2.5] to $\widetilde{\rho}$ and J, we get

$$\mathbb{P}[\tau^{z_1}_{r_k}<\infty;E_u|\mathcal{F}_{T_\rho},T_\rho<\infty]\lesssim e^{-\alpha\pi d_{\mathbb{C}}(\widetilde{\rho},J)}\lesssim\Big(\frac{r_{k-1}}{R}\Big)^{\alpha/4}.$$

Combining this estimate with (3.3) and Lemma 2.1, we get

$$\mathbb{P}[E_k\cap E_u]\lesssim F(z_1;r)\Big(\frac{r_{k-1}}{R}\Big)^{\alpha/4}\Big(\frac{r_k}{r_{k-1}}\Big)^{\alpha}. \tag{3.5}$$

Since $E=\bigcup_{k=1}^n E_k$, using (3.4) and (3.5), we get

$$\mathbb{P}[E]\lesssim F(z_1;r)\Big(\frac{r_k}{r_{k-1}}\Big)^{\alpha}\sum_{k=1}^n\Big(\frac{r_{k-1}}{R}\Big)^{\alpha/4}=F(z_1;r)\Big(\frac{\eta}{R}\Big)^{\alpha/4}\frac{e^\alpha}{1-e^{-\alpha/4}}.$$

From this we get (3.2) and finish the proof of (i).

(ii) From Lemma 2.10 and (i), we get $\mathbb{P}^*_{z_1}[\gamma[\tau^{z_1}_\eta,\tau^{z_1}_r]\cap\xi\neq\emptyset]\lesssim(\frac{\eta}{R})^{\alpha/4}$ for any $r>0$ smaller than η and $\operatorname{Im} z_1$. We then complete the proof by sending $r\to 0$. □

Corollary 3.3. *Let $z_1,z_0\in\mathbb{H}$ and $0<r<\eta<R$ be such that $R-\eta,\eta-r>2|z_1-z_0|$ and $r<\operatorname{Im} z_0$. Let ξ be a connected component of $H_{\tau^{z_1}_\eta}\cap\{|z-z_1|=R\}$. Then*

(i) $\mathbb{P}[\gamma[\tau^{z_1}_\eta,\tau^{z_0}_r]\cap\xi\neq\emptyset|\tau^{z_0}_r<\infty]\lesssim(\frac{\eta}{R})^{\alpha/4}$.

(ii) $\mathbb{P}^*_{z_0}[\gamma[\tau^{z_1}_\eta,T_{z_0}]\cap\xi\neq\emptyset]\lesssim(\frac{\eta}{R})^{\alpha/4}$.

Proof. (i) Let $\eta'=\eta+|z_1-z_0|$ and $R'=R-|z_1-z_0|$. Then $\tau^{z_0}_{\eta'}\le\tau^{z_1}_\eta$, and $\{|z-z_0|=R'\}$ disconnects z_1,z_0 from $\{|z-z_1|=R\}$. So there is a connected component ξ' of $\{|z-z_0|=R'\}\cap H_{\tau^{z_0}_{\eta'}}$ that disconnects z_1,z_0 from ξ in $H_{\tau^{z_0}_{\eta'}}$. Thus, by Lemma 3.2,

$$\mathbb{P}[\gamma[\tau^{z_1}_\eta,\tau^{z_0}_r]\cap\xi\neq\emptyset|\tau^{z_0}_r<\infty]\le\mathbb{P}[\gamma[\tau^{z_0}_{\eta'},\tau^{z_0}_r]\cap\xi'\neq\emptyset|\tau^{z_0}_r<\infty]\lesssim\Big(\frac{\eta'}{R'}\Big)^{\alpha/4}\lesssim\Big(\frac{\eta}{R}\Big)^{\alpha/4}.$$

(ii) This follows from Lemma 2.10 and (i) by sending $r\to 0$. □

The next lemma will be frequently used.

Lemma 3.4. *Let $z_1,\dots,z_n$ be distinct points in $\mathbb{H}$, where $n\ge 2$. Let K be an $\mathbb{H}$-hull such that $0\in\overline{K}$ and $\mathbb{H}\setminus K$ contains $z_1,\dots,z_n$. Let w_0 be a prime end of $\mathbb{H}\setminus K$ that sits on ∂K. Suppose that* $\operatorname{dist}(z_k,K)\ge s_k$, $2\le k\le n$, *where $s_k\in(0,|z_k|\wedge|z_k-z_1|)$. Then*

$$\begin{aligned}&F(z_1)F_{(\mathbb{H}\setminus K;w_0,\infty)}(z_2,\dots,z_n)\\ \lesssim& F(z_1,\dots,z_n)\prod_{k=2}^n\Big(\frac{|z_k|\wedge|z_k-z_1|}{s_k}\Big)^\alpha\min_{2\le k\le n}\Big(\frac{\operatorname{dist}(g_K(z_k),S_K)}{|g_K(z_k)-g_K(w_0)|}\Big)^\alpha\\ \lesssim& F(z_1,\dots,z_n)\prod_{k=2}^n\Big(\frac{|z_k|\wedge|z_k-z_1|}{s_k}\Big)^\alpha.\end{aligned}$$

Proof. Since $w_0\in\partial K$, we get $g_K(w_0)\in S_K$. So the first inequality immediately implies the second. Let y_k and l_k, $1\le k\le n$, be defined by (2.3). Let $g=g_K-g_K(w_0)$. Let $\widetilde{z}_k=g(z_k)$, $2\le k\le n$; and define $\widetilde{y}_k$ and $\widetilde{l}_k$ using (2.3) for the $n-1$ points: $\widetilde{z}_k$, $2\le k\le n$. In particular, $\widetilde{l}_2=|\widetilde{z}_2|$. Let $S=S_K-g_K(w_0)\ni 0$. Define for $2\le k\le n$,

$$\widetilde{l}_k^S=\operatorname{dist}(\widetilde{z}_k,S\cup\{\widetilde{z}_j:2\le j<k\}),\quad l_k^K=\operatorname{dist}(z_k,K\cup\{z_j:2\le j<k\}).$$

From Koebe's 1/4 theorem, we get $|g'(z_k)|l_k^K\asymp\widetilde{l}_k^S$. We claim that when ε is small,

$$\frac{P_{\widetilde{y}_k}(|g'(z_k)|\varepsilon)}{P_{\widetilde{y}_k}(\widetilde{l}_k^S)}\asymp\frac{P_{y_k}(\varepsilon)}{P_{y_k}(l_k^K)},\quad\text{if }\varepsilon\le\operatorname{dist}(z_k,K).\tag{3.6}$$

We consider two cases. If $y_k\le\operatorname{dist}(z_k,K)/10$, applying Koebe's distortion theorem, we get $\widetilde{y}_k\asymp|g'(z_k)|y_k$. Then we have (3.6) because $\frac{P_{ay}(ar)}{P_{ay}(aR)}=\frac{P_y(r)}{P_y(R)}$. If $y_k\ge\operatorname{dist}(z_k,K)/10$, then $y_k\gtrsim l_k^K$. Applying Koebe's 1/4 theorem, we get $\widetilde{y}_k\gtrsim|g'(z_k)|\operatorname{dist}(z_k,K)\gtrsim\widetilde{l}_k^K$. Thus, when $\varepsilon\le\operatorname{dist}(z_k,K)$, we have (3.6) because both sides of it are comparable to $(\frac{\varepsilon}{l_k^K})^{2-d}$.

Recall that

$$F(z_1)=\lim_{\varepsilon\to0^+}\varepsilon^{d-2}\frac{P_{y_1}(\varepsilon)}{P_{y_1}(l_1)};\quad F(z_1,\dots,z_n)=\lim_{\varepsilon\to0^+}\varepsilon^{n(d-2)}\prod_{k=1}^n\frac{P_{y_k}(\varepsilon)}{P_{y_k}(l_k)}.$$

Since g is a conformal map from D onto $\mathbb{H}$ that fixes ∞ and takes w_0 to 0, we have

$$F_{(D;w_0,\infty)}(z_2,\dots,z_n)=\prod_{k=2}^n|g'(z_k)|^{2-d}\lim_{\varepsilon\to0^+}\varepsilon^{(n-1)(d-2)}\prod_{k=2}^n\frac{P_{\widetilde{y}_k}(\varepsilon)}{P_{\widetilde{y}_k}(\widetilde{l}_k)}.$$

From (3.6), we get

$$F(z_1)F_{(D;w_0,\infty)}(z_2,\dots,z_n)\asymp\prod_{k=2}^n\Big(\frac{P_{y_k}(l_k)}{P_{y_k}(l_k^K)}\cdot\frac{P_{\widetilde{y}_k}(\widetilde{l}_k^S)}{P_{\widetilde{y}_k}(\widetilde{l}_k)}\Big)\cdot F(z_1,\dots,z_n).$$

Since $l_k^K = \text{dist}(z_k, K) \wedge \text{dist}(z_k : \{z_j : 2 \le j < k\}) \ge s_k \wedge \text{dist}(z_k : \{z_j : 2 \le j < k\})$, $l_k = |z_k| \wedge |z_k - z_1| \wedge \text{dist}(z_k : \{z_j : 2 \le j < k\})$, and $|z_k| \wedge |z_k - z_1| \ge s_k$, we get

$$\frac{P_{y_k}(l_k)}{P_{y_k}(l_k^K)} \le \Big(\frac{|z_k| \wedge |z_k - z_1| \wedge \text{dist}(z_k : \{z_j : 2 \le j < k\})}{s_k \wedge \text{dist}(z_k : \{z_j : 2 \le j < k\})}\Big)^\alpha \le \Big(\frac{|z_k| \wedge |z_k - z_1|}{s_k}\Big)^\alpha.$$

Note that $\frac{P_{\widetilde{y}_k}(\widetilde{l}_k^S)}{P_{\widetilde{y}_k}(\widetilde{l}_k)} \le 1$, $2 \le k \le n$, and $\frac{P_{\widetilde{y}_2}(\widetilde{l}_2^S)}{P_{\widetilde{y}_2}(\widetilde{l}_2)} = \frac{P_{\widetilde{y}_2}(\text{dist}(\widetilde{z}_2, S))}{P_{\widetilde{y}_2}(|\widetilde{z}_2|)} = (\frac{\text{dist}(\widetilde{z}_2, S)}{|\widetilde{z}_2|})^\alpha$. From Lemma 2.2, we get $\prod_{k=2}^n \frac{P_{\widetilde{y}_k}(\widetilde{l}_k^S)}{P_{\widetilde{y}_k}(\widetilde{l}_k)} \lesssim \min_{2 \le k \le n} \left(\frac{\text{dist}(\widetilde{z}_k, S)}{|\widetilde{z}_k|}\right)^\alpha$. Then the proof is completed. □

The next two lemmas are useful when we want to prove the lower bound.

Lemma 3.5. *Let $z_1, \dots, z_n$ be distinct points in $\overline{\mathbb{H}} \setminus \{0\}$. Let $r_j \in (0, d_j)$, $1 \le j \le n$, where d_j's are given by (2.3). Let K be an $\mathbb{H}$-hull such that $0 \in \overline{K}$, and let $U_0 \in S_K$. Suppose that $z_k \notin \overline{K}$ and*

$$\text{dist}(g_K(z_j), S_K) \asymp |\widetilde{z}_j| := |g_K(z_j) - U_0|, \quad 1 \le j \le n. \tag{3.7}$$

Suppose $I = \{1 = j_1 < \cdots < j_{|I|}\} \subset \{1, \dots, n\}$ satisfies that $r_j \lesssim \text{dist}(z_j, K)$. Then we have

$$\begin{aligned} &F(z_1; \text{dist}(z_1, K)) \cdot F(\widetilde{z}_{j_1}, \dots, \widetilde{z}_{j_{|I|}}; |g_K'(z_{j_1})| r_{j_1}, \dots, |g_K'(z_{j_{|I|}})| r_{j_{|I|}}) \\ \gtrsim &F(z_1, z_2, \dots, z_n; r_1, r_2, \dots, r_n). \end{aligned}$$

The implicit constant in the conclusion depends on the implicit constants in the assumption.

Proof. By reordering the points and using (2.7), we may assume that $I = \{1, \dots, m\}$. Let y_k and l_k, $1 \le k \le n$, be defined by (2.3). Also take $\widetilde{y}_k$ and $\widetilde{l}_k$ be the corresponding quantities for $\widetilde{z}_k$, $1 \le k \le m$. Let $S = S_K - U_0 \ni 0$. For $1 \le k \le m$ define.

$$\widetilde{l}_k^S = \text{dist}(\widetilde{z}_k, S \cup \{\widetilde{z}_j : 1 \le j < k\}), \quad l_k^K = \text{dist}(z_k, K \cup \{z_j : 1 \le j < k\}).$$

It is clear that $l_k^K \le l_k$. By Koebe's 1/4 theorem we have $|g_K'(z_k)| l_k^K \asymp \widetilde{l}_k^S$. From (3.7) we know that $\widetilde{l}_k^S \asymp \widetilde{l}_k$. Since $r_k \lesssim \text{dist}(z_k, K)$, $1 \le k \le m$, the argument of (3.6) gives us

$$\frac{P_{\widetilde{y}_k}(|g_K'(z_k)| r_k)}{P_{\widetilde{y}_k}(\widetilde{l}_k)} \asymp \frac{P_{y_k}(r_k)}{P_{y_k}(l_k^K)}, \quad 1 \le k \le m. \tag{3.8}$$

Since $l_k^K \le l_k$, we have

$$\frac{P_{\widetilde{y}_k}(|g_K'(z_k)| r_k)}{P_{\widetilde{y}_k}(\widetilde{l}_k)} \gtrsim \frac{P_{y_k}(r_k)}{P_{y_k}(l_k)}, \quad 1 \le k \le m. \tag{3.9}$$

Multiplying (3.8) for $k = 1$, (3.9) for $2 \le k \le m$, the equality $F(z_1; \text{dist}(z_1, K)) = \frac{P_{y_1}(l_1^K)}{P_{y_1}(l_1)}$, and the inequalities $1 \ge \frac{P_{y_k}(r_k)}{P_{y_k}(l_k)}$ for $m + 1 \le k \le n$, we get the desired inequality. □

Lemma 3.6. *Suppose we have set of distinct points $z_1, \dots, z_n$ in $\mathbb{H}$. Let l_j, $1 \le j \le n$, be defined by* (2.3)*. Let $m \in \{1, \dots, n-1\}$. Take $w_j = z_{m+j}$, $1 \le j \le n-m$. Let l_j^w, $1 \le j \le n-m$, be the corresponding quantity for w_j's. Suppose $l_{m+j} \asymp l_j^w$, $1 \le j \le n-m$. Then*

$$F(z_1, \dots, z_m; r_1, \dots, r_m) F(z_{m+1}, \dots, z_n; r_{m+1}, \dots, r_n) \asymp F(z_1, \dots, z_n; r_1, \dots, r_n).$$

The implicit constant in the result depends on the implicit constants in the assumption.

Proof. Just write the definition of F and note that $P_{\operatorname{Im} z_{m+j}}(l_{m+j}) \asymp P_{\operatorname{Im} w_j}(l_j^w)$. $\square$

4 Main Theorems

Following the approach in [15], we will prove the existence of ordered Green's function, i.e., the limit

$$\lim_{r_1, \dots, r_n \downarrow 0} \prod_{j=1}^n r_j^{d-2} \mathbb{P}[\tau_{r_1}^{z_1} < \cdots < \tau_{r_n}^{z_n} < \infty].$$

It is clear that if the ordered Green's function exists, then the (unordered) Green's function also exists.

For that purpose we define functions $\widehat{G}(z_1, \dots, z_n)$ by induction on n. For $n = 1$, let $\widehat{G}(z) = G(z)$ given by (1.2). Suppose $n \ge 2$ and $\widehat{G}$ has been defined for $n-1$ points. Now we define $\widehat{G}$ for distinct n points $z_1, \dots, z_n \in \mathbb{H}$. Given a chordal Loewner curve γ, for any $t \ge 0$, if $z_2, \dots, z_n \in H_t$, we define

$$\widehat{G}_t(z_2, \dots, z_n) = \prod_{j=1}^n |g_t'(z_j)|^{2-d} \widehat{G}(Z_t(z_2), \dots, Z_t(z_n));$$

otherwise define $\widehat{G}_t(z_2, \dots, z_n) = 0$. Recall that $Z_t = g_t - U_t$ is the centered Loewner map at time t. Now we define $\widehat{G}(z_1, \dots, z_n)$ by

$$\widehat{G}(z_1, \dots, z_n) = G(z_1) \mathbb{E}_{z_1}^* [\widehat{G}_{T_{z_1}}(z_2, \dots, z_n)].$$

Recall that $\mathbb{E}_{z_1}^*$ is the expectation w.r.t. the two-sided radial SLE$_\kappa$ curve through z_1.

The authors of [15] proved that the two-point (conformal radius version) Green's function exists and agrees with the $\widehat{G}(z_1, z_2)$ defined above (up to a constant). Their proof used the closed-form formula of one-point Green's function (1.2). We will show their result is also true for arbitrary number of points. The difficulty is that there is no closed-form formula known for two-point Green's function. We find a way to prove the above statement without knowing the exact formula of the Green's functions. Below is our first main theorem

Theorem 4.1. *There are finite constants $C_n, B_n > 0$ and $\beta_n, \delta_n \in (0,1)$ depending only on κ and n such that the following holds. Let $z_1, \dots, z_n$ be distinct points in $\mathbb{H}$. Let R_j, $1 \le j \le n$, Q and F be defined by (2.3,2.4). Then for any $r_1, \dots, r_n > 0$ that satisfy*

$$Q^{B_n} \frac{r_j}{R_j} < \delta_n, \quad 1 \le j \le n, \tag{4.1}$$

we have

$$\Big| \prod_{j=1}^{n} r_j^{d-2} \mathbb{P}[\tau_{r_1}^{z_1} < \cdots < \tau_{r_n}^{z_n} < \infty] - \widehat{G}(z_1, \dots, z_n) \Big| \le C_n F \sum_{j=1}^{n} \Big(Q^{B_n} \frac{r_j}{R_j} \Big)^{\beta_n}. \tag{4.2}$$

As an immediate consequence, the $G(z_1, \dots, z_n)$ defined by (1.1) exists and is equal to $\sum_\sigma \widehat{G}(z_{\sigma(1)}, \dots, z_{\sigma(n)})$, where the summation is over all permutations of $\{1, \dots, n\}$.

Proving the convergence of n-point Green's function requires certain modulus of continuity of $(n-1)$-point Green's functions, which is given by the following theorem.

Theorem 4.2. *There are finite constants $C_n, B_n > 0$ and $\beta_n, \delta_n \in (0,1)$ depending only on κ and n such that the following holds. Let $z_1, \dots, z_n$ be distinct points in $\mathbb{H}$. Let d_j, $1 \le j \le n$, Q and F be defined by (2.3,2.4). If $z'_1, \dots, z'_n \in \mathbb{H}$ satisfy that*

$$Q^{B_n} \frac{|z'_j - z_j|}{d_j} < \delta_n, \quad \frac{|\operatorname{Im} z'_j - \operatorname{Im} z_j|}{\operatorname{Im} z_j} < \delta_n, \quad 1 \le j \le n, \tag{4.3}$$

then

$$|\widehat{G}(z'_1, \dots, z'_n) - \widehat{G}(z_1, \dots, z_n)| \le C_n F \sum_{j=1}^{n} \Big(Q^{B_n} \frac{|z'_j - z_j|}{d_j} \Big)^{\beta_n} + \Big(\frac{|\operatorname{Im} z'_j - \operatorname{Im} z_j|}{\operatorname{Im} z_j} \Big)^{\beta_n}. \tag{4.4}$$

Moreover, the same inequality holds true (with bigger C_n) if $\widehat{G}$ is replaced by G.

The sharp lower bound for the Green's function is provided in the theorem below. The reader may compare it with Proposition 2.3.

Theorem 4.3. *Then there are finite constants $C_n, R_n > 0$ such that for any distinct points $z_1, \dots, z_n \in \overline{\mathbb{H}} \setminus \{0\}$ and any $r_j \in (0, d_j)$, $1 \le j \le n$, we have*

$$\mathbb{P}[\tau_{r_j}^{z_j} < \tau_{\{|z| = R_n(\sum_{i=1}^{n} |z_i|)\}}, 1 \le j \le n] \ge C_n F(z_1, \dots, z_n; r_1, \dots, r_n). \tag{4.5}$$

We have a local martingale related with the Green's function.

Corollary 4.4. *For fixed distinct $z_1, \dots, z_n \in \mathbb{H}$, $M_t := \widehat{G}_t(z_1, \dots, z_n)$ is a local martingale up to the first time any z_j, $1 \le j \le n$, is swallowed by γ.*

Proof. It suffices to prove the following. Let K be any $\mathbb{H}$-hull such that $0 \in K$ and $z_1, \dots, z_n \in \mathbb{H} \setminus K$. Let $\tau = \inf\{t > 0 : \gamma[0,t] \not\subset K\}$. Then $M_{t\wedge\tau}$ is a martingale. To prove this, we pick a small $r > 0$, and consider the martingale

$$M_t^{(r)} := r^{n(d-2)} \mathbb{P}[\tau_r^{z_1} < \cdots < \tau_r^{z_n} | \mathcal{F}_{t\wedge\tau}].$$

By the convergence theorem and Koebe's distortion theorem, we have $M_t^{(r)} \to M_{t\wedge\tau}$ as $r \to 0$. In order to have the desired result, we need uniform convergence. This can be done using the the convergence rate in Theorem 4.1 and a compactness result from [21]. Let $z_{j;t} = g_t(z_j) - U_t$; let Q_t and $R_{j;t}$ be the Q and R_j for $z_{1;t}, \dots, z_{n;t}$; let $F_t = \prod_{j=1}^n |g_t'(z_j)|^{2-d} F(z_{1;t}, \dots, z_{n;t})$. It suffices to show that $|g_t'(z_j)|, Q_t, R_{j;t}, F_t$, $1 \le j \le n$, $0 \le t \le \tau$, are all bounded from both above and below by a finite positive constant depending only on κ, K, and $z_1, \dots, z_n$. The existence of these bounds all follow directly or indirectly from [21, Lemma 5.4]. For example, to prove that F_t, $0 \le t \le \tau$, are bounded above, we need to prove that $|z_{j;t} - z_{k;t}|$, $j \ne k$, and $|z_{j,t}|$, $0 \le t \le \tau$, are all bounded below. It suffices to show that $|g_L(z_j) - g_L(z_k)$, $j \ne k$, and $\operatorname{dist}(g_L(z_j), S_L)$ for all L in $\mathcal{H}(K)$, the set of $\mathbb{H}$-hulls L with $L \subset K$, are bounded below. Suppose $|g_L(z_j) - g_L(z_k)|$, $j \ne k$, $L \in \mathcal{H}(K)$, are not bounded below by a constant. Then there are $z_j \ne z_k$ and a sequence $(L_n) \subset \mathcal{H}(K)$ such that $|g_{L_n}(z_j) - g_{L_n}(z_k)| \to 0$. Since $\mathcal{H}(K)$ is a compact metric space ([21, Lemma 5.4]), by passing to a subsequence, we may assume that $L_n \to L_0 \in \mathcal{H}(K)$. This then implies that $g_{L_0}(z_j) = \lim g_{L_n}(z_j) = \lim g_{L_n}(z_k) = g_{L_0}(z_k)$, which contradicts that g_{L_0} is injective on $\mathbb{H} \setminus K$. To prove that $\operatorname{dist}(g_L(z_j), S_L)$, $1 \le j \le n$, $L \in \mathcal{H}(K)$, are bounded from below, one may choose a pair of disjoint Jordan curve J_1, J_2 in $\mathbb{H} \setminus K$, both of which disconnects K from all of z_j's. Then $\operatorname{dist}(g_L(z_j), S_L) \ge \operatorname{dist}(g_L(J_1), g_L(J_2))$, and the same argument above shows that $\operatorname{dist}(g_L(J_1), g_L(J_2))$, $L \in \mathcal{H}(K)$, are bounded from below by a positive constant. □

Remark We may write $M_t = \prod_{j=1}^n |g_t'(z_j)|^{2-d} \widehat{G}(g_t(z_1) - U_t, \dots, g_t(z_n) - U_t)$. If we know that $\widehat{G}$ is smooth, then using Itô's formula and Loewner's equation (2.8), one can easily get a second order PDE for $\widehat{G}$. More specifically, if we view $\widehat{G}$ as a function on $2n$ real variables: $x_1, y_1, \dots, x_n, y_n$, then it satisfies

$$\frac{\kappa}{2}\Big(\sum_{j=1}^n \partial_{x_j}\Big)^2 \widehat{G} + \sum_{j=1}^n \partial_{x_j}\widehat{G} \cdot \frac{2x_j}{x_j^2 + y_j^2} + \sum_{j=1}^n \partial_{y_j}\widehat{G} \cdot \frac{-2y_j}{x_j^2 + y_j^2} + (2-d)\widehat{G} \cdot \sum_{j=1}^n \frac{-2(x_j^2 - y_j^2)}{(x_j^2 + y_j^2)^2} = 0.$$

Since the PDE does not depend on the order of points, it is also satisfied by the unordered Green's function G.

We expect that the smoothness of $\widehat{G}$ can be proved by Hörmander's theorem because the differential operator in the above displayed formula satisfies Hörmander's condition.

5 Proof of Theorems 4.1 and 4.2

At the beginning, we know that Theorems 4.1 and 4.2 hold for $n = 1$ with $\delta_1 = 1/2$ thanks to [12, Theorem 2.3] and the explicit formulas for $F(z)$ and $G(z)$. We will prove Theorems

4.1 and 4.2 together using induction. Let $n \geq 2$. Suppose that Theorems 4.1 and 4.2 hold for $n-1$ points. We now prove that they also hold for n points. We will frequently apply the Domain Markov Property (DMP) of SLE (c.f. [8]) without reference, i.e., if γ is a chordal SLE$_\kappa$ curve in $\mathbb{H}$ from 0 to ∞, and τ is a finite stopping time, then $Z_\tau(\gamma(\tau+\cdot))$ has the same law as γ, and is independent of $\mathcal{F}_\tau$.

Fix distinct points $z_1,\dots,z_n \in \mathbb{H}$. Let l_j, d_j, R_j, y_j, $1 \leq j \leq n$, Q, and F be as defined in (2.3,2.4). Throughout this section, a variable is a real number that depends on κ, n and $z_1,\dots,z_n$. From the induction hypothesis, Proposition 2.3, and (2.5), we see that $\widehat{G} \lesssim F$ holds for $(n-1)$ points. We write F_t for $F_{(H_t;\gamma(t),\infty)}$. Then Lemma 3.4 holds with $K = K_t$, $G(z_1)$ in place of $F(z_1)$, and $\widehat{G}_t$ in place of $F_{(\mathbb{H}\setminus K_t;w_0,\infty)}$. We will use the following lemma.

Lemma 5.1. *Let $k_0 \in \{2,\dots,n\}$ and $s_{k_0} \in (r_{k_0}, |z_{k_0}-z_1| \wedge |z_{k_0}|)$. Then there is some constant $\beta > 0$ such that*

$$G(z_1)E^*_{z_1}[\widehat{G}_{T_{z_1}}(z_2,\dots,z_n)\mathbf{1}\{\mathrm{inrad}_{H_{T_{z_1}}}(z_{k_0}) \leq s_{k_0}\}] \lesssim F \cdot \Big(\frac{s_{k_0}}{|z_{k_0}-z_1| \wedge |z_{k_0}|}\Big)^\beta.$$

Proof. This lemma essentially follows from the induction hypothesis, Theorem 3.1, and (2.5). Below are the details. Let $r_j \in (0, R_j/8)$, $1 \leq j \leq n$. From Theorem 3.1, there is a constant $\beta > 0$ such that

$$\mathbb{P}[\tau^{z_1}_{r_1} < \infty] \cdot \mathbb{E}[\mathbf{1}\{\mathrm{inrad}_{H_{\tau^{z_1}_{r_1}}}(z_{k_0}) \leq s_{k_0}\}\mathbb{P}[\tau^{z_1}_{r_1} < \cdots < \tau^{z_n}_{r_n} < \infty | \mathcal{F}_{\tau^{z_1}_{r_1}}, \tau^{z_1}_{r_1} < \infty]]$$

$$\lesssim F(z_1,\dots,z_n;r_1,\dots,r_n)\Big(\frac{s_{k_0}}{|z_{k_0}-z_1| \wedge |z_{k_0}|}\Big)^\beta.$$

By the convergence of $(n-1)$ point Green's function, we know that

$$\lim_{r_2,\dots,r_n \to 0} \prod_{k=2}^n r_k^{d-2}\mathbb{P}[\tau^{z_1}_{r_1} < \cdots < \tau^{z_n}_{r_n} | \mathcal{F}_{\tau^{z_1}_{r_1}}, \tau^{z_1}_{r_1} < \infty] = \widehat{G}_{\tau^{z_1}_{r_1}}(z_2,\dots,z_n).$$

Applying Fatou's lemma with $r_2,\dots,r_n \to 0$, we get

$$\mathbb{P}[\tau^{z_1}_{r_1} < \infty] \cdot \mathbb{E}[\mathbf{1}\{\mathrm{inrad}_{H_{\tau^{z_1}_{r_1}}}(z_{k_0}) \leq s_{k_0}\}\widehat{G}_{\tau^{z_1}_{r_1}}(z_2,\dots,z_n) | \tau^{z_1}_{r_1} < \infty]$$

$$\lesssim \lim_{r_2,\dots,r_n \to 0} \prod_{k=2}^n r_k^{d-2} F(z_1,\dots,z_n;r_1,\dots,r_n)\Big(\frac{s_{k_0}}{|z_{k_0}-z_1| \wedge |z_{k_0}|}\Big)^\beta,$$

which together with Lemma 2.10 implies that

$$\mathbb{P}[\tau^{z_1}_{r_1} < \infty] \cdot \mathbb{E}^*_{z_1}[\mathbf{1}\{\mathrm{inrad}_{H_{\tau^{z_1}_{r_1}}}(z_{k_0}) \leq s_{k_0}\}\widehat{G}_{\tau^{z_1}_{r_1}}(z_2,\dots,z_n)]$$

$$\lesssim \lim_{r_2,\dots,r_n \to 0} \prod_{k=2}^n r_k^{d-2} F(z_1,\dots,z_n;r_1,\dots,r_n)\Big(\frac{s_{k_0}}{|z_{k_0}-z_1| \wedge |z_{k_0}|}\Big)^\beta.$$

By the continuity two-sided radial SLE at its end point and the continuity of $(n-1)$ point Green's function, we see that, under the law $\mathbb{P}^*_{z_1}$, as $r_1 \to 0$, $\mathrm{inrad}_{H_{\tau^{z_1}_{r_1}}}(z_{k_0}) \to \mathrm{inrad}_{H_{T_{z_1}}}(z_{k_0})$ and $\widehat{G}_{\tau^{z_1}_{r_1}}(z_2, \ldots, z_n) \to \widehat{G}_{T_{z_1}}(z_2, \ldots, z_n)$. Since $\lim_{r_1\to 0} r_1^{d-2}\mathbb{P}[\tau^{z_1}_{r_1} < \infty] = G(z_1)$, applying Fatou's lemma with $r_1 \to 0$, we get the conclusion. $\square$

5.1 Convergence of Green's functions

In this subsection, we work on the inductive step for Theorem 4.1. Let $0 < r_j < R_j/8$, $1 \le j \le n$. Consider the event $\{\tau^{z_1}_{r_1} < \cdots < \tau^{z_n}_{r_n} < \infty\}$.

Fix $\vec{s} = (s_2, \ldots, s_n)$ with $s_j \in (r_j, |z_j - z_1| \wedge |z_j|)$ being variables to be determined later. We define events

$$E_{r;\vec{s}} = \bigcap_{j=2}^{n} \{\mathrm{dist}(z_j, K_{\tau^{z_1}_r}) \ge s_j\}, \quad r \ge 0. \tag{5.1}$$

Now we decompose the main event according to $E_{r_1;\vec{s}}$, and write

$$\mathbb{P}[\tau^{z_1}_{r_1} < \cdots < \tau^{z_n}_{r_n} < \infty] = \mathbb{P}[\tau^{z_1}_{r_1} < \cdots < \tau^{z_n}_{r_n} < \infty; E_{r_1;\vec{s}}] + e_1^*.$$

By Theorem 3.1 and (2.5), the term e_1^* satisfies that, for some $\beta > 0$,

$$0 \le e_1^* \lesssim \prod_{k=1}^{n} r_k^{2-d} F \sum_{j=2}^{n} \Big(\frac{s_j}{|z_j| \wedge |z_j - z_1|}\Big)^{\beta}.$$

We express

$$\begin{aligned}&\mathbb{P}[\tau^{z_1}_{r_1} < \cdots < \tau^{z_n}_{r_n} < \infty; E_{r_1;\vec{s}}]\\ =&\mathbb{P}[\tau^{z_1}_{r_1} < \infty] \cdot \mathbb{E}[1_{E_{r_1;\vec{s}}}\mathbb{P}[\tau^{z_2}_{r_2} < \cdots < \tau^{z_n}_{r_n} < \infty | \mathcal{F}_{\tau^{z_1}_{r_1}}; E_{r_1;\vec{s}}] | \tau^{z_1}_{r_1} < \infty].\end{aligned}$$

From Proposition 2.3 and Koebe's distortion theorem, we see that, if

$$\frac{r_k}{s_k \wedge R_k} < \frac{1}{6}, \quad 2 \le k \le n, \tag{5.2}$$

then

$$\mathbb{P}[\tau^{z_2}_{r_2} < \cdots < \tau^{z_n}_{r_n} < \infty | \mathcal{F}_{\tau^{z_1}_{r_1}}; E_{r_1;\vec{s}}] \lesssim \prod_{k=2}^{n} r_k^{2-d} F_{\tau^{z_1}_{r_1}}(z_2, \ldots, z_n). \tag{5.3}$$

Since Theorem 4.1 holds for $n = 1$, we see that, if

$$\frac{r_1}{R_1} < \delta_1, \tag{5.4}$$

then

$$|\mathbb{P}[\tau^{z_1}_{r_1} < \infty] - r_1^{2-d} G(z_1)| \lesssim r_1^{2-d} F(z_1) O(r_1/R_1)^{\beta_1}.$$

Now we express

$$\begin{aligned}&\mathbb{P}[\tau^{z_1}_{r_1}<\infty]\cdot\mathbb{E}[1_{E_{r_1;\vec{s}}}\mathbb{P}[\tau^{z_2}_{r_2}<\cdots<\tau^{z_n}_{r_n}<\infty|\mathcal{F}_{\tau^{z_1}_{r_1}};E_{r_1;\vec{s}}]|\tau^{z_1}_{r_1}<\infty]\\=&r_1^{2-d}G(z_1)\mathbb{E}[1_{E_{r_1;\vec{s}}}\mathbb{P}[\tau^{z_2}_{r_2}<\cdots<\tau^{z_n}_{r_n}<\infty|\mathcal{F}_{\tau^{z_1}_{r_1}};E_{r_1;\vec{s}}]|\tau^{z_1}_{r_1}<\infty]+e_2^*.\end{aligned}$$

From Lemma 3.4 and (5.3) we see that, if (5.2) and (5.4) hold, then

$$|e_2^*|\lesssim\prod_{k=1}^n r_k^{2-d}F\cdot\Big(\frac{r_1}{R_1}\Big)^{\beta_1}\prod_{k=2}^n\Big(\frac{|z_k|\wedge|z_k-z_1|}{s_k}\Big)^{\alpha}.$$

Define the events

$$E_{r;\theta}=\{\operatorname{dist}(g_{\tau^{z_1}_r}(z_j),S_{K_{\tau^{z_1}_r}})\ge\theta|g_{\tau^{z_1}_r}(z_j)-U_{\tau^{z_1}_r}|,2\le j\le n\},\quad r,\theta>0.\tag{5.5}$$

Fix a variable $\theta\in(0,1)$ to be determined later. According to the occurrence of $E_{r_1;\theta}$, we express

$$\begin{aligned}&r_1^{2-d}G(z_1)\mathbb{E}[1_{E_{r_1;\vec{s}}}\mathbb{P}[\tau^{z_2}_{r_2}<\cdots<\tau^{z_n}_{r_n}<\infty|\mathcal{F}_{\tau^{z_1}_{r_1}};E_{r_1;\vec{s}}]|\tau^{z_1}_{r_1}<\infty]\\=&r_1^{2-d}G(z_1)\mathbb{E}[1_{E_{r_1;\vec{s}}\cap E_{r_1;\theta}}\mathbb{P}[\tau^{z_2}_{r_2}<\cdots<\tau^{z_n}_{r_n}<\infty|\mathcal{F}_{\tau^{z_1}_{r_1}};E_{r_1;\vec{s}}\cap E_{r_1;\theta}]|\tau^{z_1}_{r_1}<\infty]+e_3^*.\end{aligned}$$

From Lemma 3.4 and and (5.3), we see that

$$0\le e_3^*\lesssim\prod_{k=1}^n r_k^{2-d}F\prod_{k=2}^n\Big(\frac{|z_k|\wedge|z_k-z_1|}{s_k}\Big)^{\alpha}\theta^{\alpha}.$$

Let $Z=Z_{\tau^{z_1}_{r_1}}$ and $\widehat{z}_k=Z(z_k)$, $2\le k\le n$. Define $\widehat{d}_k$, $2\le k\le n$, and $\widehat{Q}$, for the $(n-1)$ points $\widehat{z}_k$, $2\le k\le n$, using (2.3) and (2.4). Since Theorem 4.1 holds for $(n-1)$ points, using Koebe's distortion theorem, we conclude that, for some constants $B_{n-1}>0$ and $\beta_{n-1},\delta_{n-1}\in(0,1)$, if

$$\widehat{Q}^{B_{n-1}}\cdot\frac{r_j}{s_j\wedge R_j}<\frac{\delta_{n-1}}{8},\quad 2\le j\le n,$$

then

$$\begin{aligned}&\Big|\prod_{k=2}^n r_k^{d-2}\mathbb{P}[\tau^{z_2}_{r_2}<\cdots<\tau^{z_n}_{r_n}<\infty|\mathcal{F}_{\tau^{z_1}_{r_1}};E_{r_1;\vec{s}}]-\widehat{G}_{\tau^{z_1}_{r_1}}(z_2,\dots,z_n)\Big|\\\lesssim&F_{\tau^{z_1}_{r_1}}(z_2,\dots,z_n)\sum_{j=2}^n\Big(\widehat{Q}^{B_{n-1}}\frac{r_j}{s_j\wedge R_j}\Big)^{\beta_{n-1}}.\end{aligned}$$

Suppose $E_{r_1;\theta}$ happens. Let $S=S_{K_{\tau^{z_1}_{r_1}}}$. Since $U_{\tau^{z_1}_{r_1}}\in S$, from Koebe's 1/4 theorem, we get $\widehat{d}_k\gtrsim|g'(z_k)|(d_k\wedge\operatorname{dist}(z_k,\gamma[0,\tau^{z_1}_{r_1}])$ and

$$|\widehat{z}_k|\le\operatorname{dist}(g_{\tau^{z_1}_{r_1}}(z_k),S)/\theta\asymp|g'(z_k)|\operatorname{dist}(z_k,\gamma[0,\tau^{z_1}_{r_1}])/\theta,$$

which together imply that

$$\frac{|\widehat{z}_k|}{\widehat{d}_k} \le \frac{\operatorname{dist}(z_k,\gamma[0,\tau^{z_1}_{r_1}])/\theta}{d_k\wedge \operatorname{dist}(z_k,\gamma[0,\tau^{z_1}_{r_1}])} = \theta^{-1}\Big(\frac{\operatorname{dist}(z_k,\gamma[0,\tau^{z_1}_{r_1}])/\theta}{d_k}\vee 1\Big) \le \theta^{-1}\frac{|z_k|}{d_k},$$

where the last inequality holds because $d_k, \operatorname{dist}(z_k,\gamma[0,\tau^{z_1}_{r_1}]) \le |z_k|$. So for some constant $C>1$,

$$\widehat{Q} \le \frac{C}{\theta}Q. \tag{5.6}$$

Thus, if $E_{r_1;\theta}$ happens, and

$$Q^{B_{n-1}}\cdot\frac{r_j}{s_j\wedge R_j} < \frac{\theta^{B_{n-1}}\delta_{n-1}}{8C^{B_{n-1}}}, \quad 2\le j\le n, \tag{5.7}$$

then

$$\begin{aligned}&\Big|\prod_{k=2}^n r_k^{d-2}\mathbb{P}[\tau^{z_2}_{r_2}<\cdots<\tau^{z_n}_{r_n}<\infty|\mathcal{F}_{\tau^{z_1}_{r_1}};E_{r_1;\vec{s}}\cap E_{r_1;\theta}] - \widehat{G}_{\tau^{z_1}_{r_1}}(z_2,\dots,z_n)\Big|\\ \lesssim& F_{\tau^{z_1}_{r_1}}(z_2,\dots,z_n)\sum_{j=2}^n\Big(\theta^{-B_{n-1}}Q^{B_{n-1}}\frac{r_j}{s_j\wedge R_j}\Big)^{\beta_{n-1}}.\end{aligned}$$

Now we express

$$\begin{aligned}&r_1^{2-d}G(z_1)\mathbb{E}[1_{E_{r_1;\vec{s}}\cap E_{r_1;\theta}}\mathbb{P}[\tau^{z_2}_{r_2}<\cdots<\tau^{z_n}_{r_n}<\infty|\mathcal{F}_{\tau^{z_1}_{r_1}};E_{r_1;\vec{s}}\cap E_{r_1;\theta}]|\tau^{z_1}_{r_1}<\infty]\\ =&r_1^{2-d}G(z_1)\mathbb{E}[1_{E_{r_1;\vec{s}}\cap E_{r_1;\theta}}\prod_{k=2}^n r_k^{2-d}\widehat{G}_{\tau^{z_1}_{r_1}}(z_2,\dots,z_n)|\tau^{z_1}_{r_1}<\infty]+e_4^*.\end{aligned}$$

Using Lemma 3.4, we see that, when (5.7) holds,

$$|e_4^*| \lesssim \prod_{k=1}^n r_k^{2-d}F\prod_{k=2}^n\Big(\frac{|z_k|\wedge|z_k-z_1|}{s_k}\Big)^{\alpha}\sum_{j=2}^n\Big(\theta^{-B_{n-1}}Q^{B_{n-1}}\frac{r_j}{s_j\wedge R_j}\Big)^{\beta_{n-1}}.$$

Next, we express

$$\begin{aligned}&r_1^{2-d}G(z_1)\mathbb{E}[1_{E_{r_1;\vec{s}}\cap E_{r_1;\theta}}\prod_{k=2}^n r_k^{2-d}\widehat{G}_{\tau^{z_1}_{r_1}}(z_2,\dots,z_n)|\tau^{z_1}_{r_1}<\infty]\\ =&\prod_{k=1}^n r_k^{2-d}G(z_1)\mathbb{E}[1_{E_{r_1;\vec{s}}}\widehat{G}_{\tau^{z_1}_{r_1}}(z_2,\dots,z_n)|\tau^{z_1}_{r_1}<\infty]-e_5^*.\end{aligned}$$

The estimate on e_5^* is the same as that on e_3^* by Lemma 3.4.

To simplify the notation, we define for $r>0$ and $\vec{s}\in\mathbb{R}_+^{n-1}$,

$$\mathbb{E}^r_{z_1} = \mathbb{E}[\cdot|\tau^{z_1}_r<\infty]; \quad \widehat{G}_{r;\vec{s}} = 1_{E_{r;\vec{s}}}\widehat{G}_{\tau^{z_1}_r}.$$

So far we have

$$\mathbb{P}[\tau^{z_1}_{r_1} < \cdots < \tau^{z_n}_{r_n} < \infty] = \prod_{k=1}^{n} r_k^{2-d} G(z_1) \mathbb{E}^{r}_{z_1}[\widehat{G}_{r_1;\vec{s}}(z_2,\dots,z_n)] + e_1^* + e_2^* + e_3^* + e_4^* - e_5^*.$$

For $R > r > s \geq 0$, define $E_{r,s;R}$ to be the event

$$\begin{aligned} E_{r,s;R} = &\{\gamma[\tau^{z_1}_r, \tau^{z_1}_s] \text{ does not intersect any connected component of} \\ &\{|z-z_1| = R\} \cap H_{\tau^{z_1}_r} \text{ that separates } z_1 \text{ from any } z_k, 2 \leq k \leq n\}. \end{aligned} \tag{5.8}$$

Fix variables $\eta_1 < \eta_2 \in (r_1, d_1)$ to be determined later. According to whether $E_{\eta_1,r_1;\eta_2}$ occurs, we have the following decomposition:

$$G(z_1)\mathbb{E}^{r_1}_{z_1}[\widehat{G}_{r_1;\vec{s}}(z_2,\dots,z_n)] = G(z_1)\mathbb{E}^{r_1}_{z_1}[1_{E_{\eta_1,r_1;\eta_2}} \widehat{G}_{r_1;\vec{s}}(z_2,\dots,z_n)] + e_6.$$

From [18, Lemma 2.1] we know that, for each $2 \leq k \leq n$, there is a unique connected component, say ξ_k, of $\{|z - z_1| = \eta_2\} \cap H_{\tau^{z_1}_{\eta_1}}$, which separates z_1 from z_k in $H_{\tau^{z_1}_{\eta_1}}$, and if there is another ξ_k' with this property, then ξ_k also separates z_1 from ξ_k' in $H_{\tau^{z_1}_{\eta_1}}$. This means that, if $E_{\eta_1,r_1;\eta_2}$ does not occur, then $\gamma[\tau^{z_1}_{\eta_1}, \tau^{z_1}_{r_1}]$ must intersect $\cup_{k=1}^n \xi_k$. By Lemma 3.2 and Lemma 3.4, we have

$$0 \leq e_6 \lesssim F \prod_{j=2}^{n} \Big(\frac{|z_j| \wedge |z_j - z_1|}{s_j}\Big)^{\alpha} \Big(\frac{\eta_1}{\eta_2}\Big)^{\alpha/4}.$$

Changing the time from $\tau^{z_1}_{r_1}$ to $\tau^{z_1}_{\eta_1}$, we get another error term e_7:

$$G(z_1)\mathbb{E}^{r_1}_{z_1}[1_{E_{\eta_1,r_1;\eta_2}} \widehat{G}_{r_1;\vec{s}}(z_2,\dots,z_n)] = G(z_1)\mathbb{E}^{r_1}_{z_1}[1_{E_{\eta_1,r_1;r_2}} \widehat{G}_{\eta_1;\vec{s}}(z_2,\dots,z_n)] + e_7$$

To derive an estimate for e_7, we use the following lemma, whose proof is postponed to the end of this subsection.

Lemma 5.2. *There exist constants $B_* > 0$ and $\beta_*, \delta_* \in (0,1)$ such that the following holds. Let $0 \leq a < b$ be such that $z_1 \in H_a$, $\mathrm{dist}(z_1, K_a) < |z_j - z_1|$ and $\mathrm{dist}(z_j, K_b) \geq s_j$, $2 \leq j \leq n$. For $2 \leq j \leq n$, let ρ_j be the connected component of $\{|z - z_1| = |z_j - z_1|\} \cap H_a$ that contains z_j; and let ξ_j be a crosscuts of H_a, which is disjoint from ρ_j, and disconnects ρ_j from $K_b \setminus K_a$ in H_a. Let $d_* = \min_{2 \leq j \leq n} d_{H_a}(\rho_j, \xi_j)$. If*

$$Q^{B_*} \cdot e^{-2\pi d_*} < \delta_*, \tag{5.9}$$

then

$$G(z_1)|\widehat{G}_b(z_2,\dots,z_n) - \widehat{G}_a(z_2,\dots,z_n)| \lesssim F \prod_{k=2}^{n} \Big(\frac{|z_k| \wedge |z_k - z_1|}{s_k}\Big)^{\alpha} (Q^{B_*} e^{-2\pi d_*})^{\beta_*}.$$

We now apply Lemma 5.2 with $a = \tau^{z_1}_{\eta_1}$, $b = \tau^{z_1}_{r_1}$, and ξ_k being a connected component of $\{|z - z_1| = \eta_2\} \cap H_{\tau^{z_1}_{\eta_1}}$ that separates z_k from z_1. By comparison principle of extremal length, we have

$$d_{H_a}(\rho_k, \xi_k) \geq \log(|z_k - z_1|/\eta_2)/(2\pi) \geq \log(d_1/\eta_2)/(2\pi), \quad 2 \leq k \leq n.$$

Assume that

$$\eta_2 + s_k < |z_k - z_1|, \quad 2 \leq k \leq n. \tag{5.10}$$

Then $E_{\eta_1,r_1;\eta_2} \cap E_{r_1;\vec{s}} = E_{\eta_1,r_1;\eta_2} \cap E_{\eta_1;\vec{s}}$. Thus, for some constants $B_* > 0$ and $\beta_*, \delta_* \in (0,1)$, if

$$Q^{B_*} \cdot \frac{\eta_2}{d_1} < \delta_*, \tag{5.11}$$

and (5.10) holds, then

$$|e_7| \lesssim F \prod_{k=2}^{n} \Big(\frac{|z_k| \wedge |z_k - z_1|}{s_k}\Big)^{\alpha} \Big(Q^{B_*} \frac{\eta_2}{d_1}\Big)^{\beta_*}.$$

Removing the restriction of the event $E_{\eta_1,r_1;\eta_2}$, we get another error term e_8:

$$G(z_1)\mathbb{E}^{r_1}_{z_1}[1_{E_{\eta_1,r_1;\eta_2}} \widehat{G}_{\eta_1;\vec{s}}(z_2, \dots, z_n)] = G(z_1)\mathbb{E}^{r_1}_{z_1}[\widehat{G}_{\eta_1;\vec{s}}(z_2, \dots, z_n)] - e_8.$$

Here the estimate on e_8 is same as that on e_6 by Lemmas 3.2 and 3.4.

Changing the probability measure from the conditional chordal $\mathbb{E}_{r_1}$ to the two-sided radial $\mathbb{E}^*_{z_1}$, we get another error term e_9:

$$G(z_1)\mathbb{E}^{r_1}_{z_1}[\widehat{G}_{\eta_1;\vec{s}}(z_2, \dots, z_n)] = G(z_1)\mathbb{E}^*_{z_1}[\widehat{G}_{\eta_1;\vec{s}}(z_2, \dots, z_n)] + e_9.$$

From [15, Proposition 2.13] and Lemma 3.4, we find that for some constant $\beta_0 > 0$,

$$|e_9| \lesssim F \prod_{k=2}^{n} \Big(\frac{|z_k - z_1| \wedge |z_k|}{s_k}\Big)^{\alpha} \Big(\frac{r_1}{\eta_1}\Big)^{\beta_0}.$$

Let the event $E_{\eta_1,0;\eta_2}$ be defined by (5.8). We now express

$$G(z_1)\mathbb{E}^*_{z_1}[G_{\eta_1;\vec{s}}(z_2, \dots, z_n)] = G(z_1)\mathbb{E}^*_{z_1}[1_{E_{\eta_1,0;\eta_2}} \widehat{G}_{\eta_1;\vec{s}}(z_2, \dots, z_n)] + e_{10}$$

Here the estimate on e_{10} is same as that on e_6 by Lemmas 3.2 and 3.4.

Changing the time from $\tau^{z_1}_{\eta_1}$ to $\tau^{z_1}_0 = T_{z_1}$, we get another error term e_{11}:

$$G(z_1)\mathbb{E}^*_{z_1}[1_{E_{\eta_1,0;\eta_2}} \widehat{G}_{\eta_1;\vec{s}}(z_2, \dots, z_n)] = G(z_1)\mathbb{E}^*_{z_1}[1_{E_{\eta_1,0;\eta_2}} \widehat{G}_{0;\vec{s}}(z_2, \dots, z_n)] + e_{11}.$$

If (5.10) holds, then $E_{\eta_1,0;\eta_2} \cap E_{\eta_1;\vec{s}} = E_{\eta_1,0;\eta_2} \cap E_{0;\vec{s}}$. Apply Lemma 5.2 with $a = \tau^{z_1}_{\eta_1}$, $b = \tau^{z_1}_0 = T_{z_1}$, and ξ_k being a connected component of $\{|z - z_1| = \eta_2\} \cap H_{\tau^{z_1}_{\eta_1}}$ that separates z_k from z_1, we get an estimate on e_{11}, which is the same as that on e_7, provided that (5.11) holds. Note that the constants B_*, β_*, δ_* here may be different from those for e_7. But by taking the bigger B_* and smaller β_* and δ_*, we may make both estimates hold for the same set of constants.

Removing the restriction of the event $E_{\eta_1,0;\eta_2}$, we get another error term e_{12}:

$$G(z_1)\mathbb{E}^*_{z_1}[1_{E_{\eta_1,0;\eta_2}}\widehat{G}_{0;\vec{s}}(z_2,\dots,z_n)] = G(z_1)\mathbb{E}^*_{z_1}[\widehat{G}_{0;\vec{s}}(z_2,\dots,z_n)] - e_{12}.$$

Here the estimate on e_{12} is same as that e_6 by Lemmas 3.2 and 3.4.

Finally, note that $\widehat{G}_{0;\vec{s}} = 1_{E_{0;\vec{s}}}\widehat{G}_{T_{z_1}}$. Removing the restriction of the event $E_{0;\vec{s}}$, we get the last error term e_{13}:

$$G(z_1)\mathbb{E}^*_{z_1}[\widehat{G}_{0;\vec{s}}(z_2,\dots,z_n)] = G(z_1)\mathbb{E}^*_{z_1}[\widehat{G}_{T_{z_1}}(z_2,\dots,z_n)] - e_{13} = \widehat{G}(z_1,\dots,z_n) + e_{13}.$$

where by Lemma 5.1, the estimate on e_{13} is the same as that on $e_1^*/\prod_{k=1}^n r_k^{2-d}$.

At the end, we need to choose the variables $s_2,\dots,s_n$ and η_1,η_2,θ, and constants $C_n, B_n > 0$ and $\beta_n,\delta_n \in (0,1)$, such that if (4.1) holds, then (5.2,5.4,5.7,5.10,5.11) all hold, $r_j < R_j/8$, $1 \le j \le n$, and the upper bounds for $|e_s| := |e_s^*|/\prod_{k=1}^n r_k^{2-d}$, $1 \le s \le 5$, and $|e_s|$, $6 \le s \le 13$, are all bounded above by the RHS of (4.2).

We take $X \in (0,1)$ to be determined, and suppose that $\frac{s_j}{|z_j|\wedge|z_j-z_1|} = X$, $2 \le j \le n$. We have

$$\frac{r_j}{s_j \wedge R_j} = \left(1 \vee \frac{R_j}{s_j}\right)\cdot\frac{r_j}{R_j} \le X^{-1}\cdot\frac{r_j}{R_j}, \quad 2 \le j \le n. \tag{5.12}$$

In the argument below, we assume that (5.2,5.4,5.7,5.10,5.11) all hold so that we can freely use the estimates we have obtained.

From the estimate on $|e_4^*|$, we get

$$|e_4| \lesssim FQ^{B_{n-1}\beta_{n-1}}X^{-n\alpha-\beta_{n-1}}\theta^{-B_{n-1}\beta_{n-1}}\max_{2\le j\le n}\left(\frac{r_j}{R_j}\right)^{\beta_{n-1}}.$$

From the estimates on e_3^* and e_5^*, we get

$$|e_s| \lesssim FX^{-n\alpha}\theta^{\alpha}, \quad s \in \{3,5\}.$$

If we take θ such that $\theta^\alpha = \theta^{-B_{n-1}\beta_{n-1}}\max_{2\le j\le n}(\frac{r_j}{R_j})^{\beta_{n-1}}$, then we get

$$|e_s| \lesssim FQ^{B_{n-1}\beta_{n-1}}X^{-n\alpha-\beta_{n-1}}\max_{2\le j\le n}\left(\frac{r_j}{R_j}\right)^{\frac{\alpha\beta_{n-1}}{\alpha+B_{n-1}\beta_{n-1}}}, \quad 3 \le s \le 5.$$

Choose η_1 and η_2 such that $\frac{r_1}{\eta_1} = \frac{\eta_1}{\eta_2} = \frac{\eta_2}{d_1}$. Then we find that

$$|e_s| \lesssim FQ^{B_*\beta_*}X^{-n\alpha}\left(\frac{r_1}{d_1}\right)^{\frac{1}{3}(\frac{\alpha}{4}\wedge\beta_*\wedge\beta_0)}, \quad 6 \le s \le 12.$$

Since $R_1 \le d_1$, combining with the estimate on e_2^*, we get

$$|e_s| \lesssim FQ^{B_*\beta_*}X^{-n\alpha}\left(\frac{r_1}{R_1}\right)^{\frac{1}{3}(\frac{\alpha}{4}\wedge\beta_*\wedge\beta_0)\wedge\beta_1}, \quad s \in \{2,6,7,8,9,10,11,12\}.$$

Combining this with the estimates on $|e_s|$, $3 \le s \le 5$, we get

$$|e_s| \lesssim FQ^{B_{n-1}\beta_{n-1}+B_*\beta_*} X^{-n\alpha-\beta_{n-1}} \max_{1\le j\le n} \Big(\frac{r_j}{R_j}\Big)^{\beta_\#}, \quad 2 \le s \le 12,$$

where $\beta_\# := \frac{1}{3}(\frac{\alpha}{4} \wedge \beta_* \wedge \beta_0) \wedge \beta_1 \wedge \frac{\alpha\beta_{n-1}}{\alpha+B_{n-1}\beta_{n-1}}$. Since $|e_1|, |e_{13}| \lesssim FX^\beta$, if we choose X such that $X^\beta = X^{-n\alpha-\beta_{n-1}} \max_{1\le j\le n}(\frac{r_j}{R_j})^{\beta_\#}$, then with $\beta_n := \frac{\beta\beta_\#}{\beta+n\alpha+\beta_{n-1}}$, we get

$$|e_s| \lesssim FQ^{B_{n-1}\beta_{n-1}+B_*\beta_*} \max_{1\le j\le n} \Big(\frac{r_j}{R_j}\Big)^{\beta_n}, \quad 1 \le s \le 13. \tag{5.13}$$

Now we check Conditions (5.2,5.4,5.7,5.10,5.11) and $r_j < R_j/8$, $1 \le j \le n$. Clearly, (5.7) implies (5.2). The LHS of (5.11) equals to $Q^{B_*}(\frac{r_1}{d_1})^{1/3} \le Q^{B_*}(\frac{r_1}{R_1})^{1/3}$, and so it holds if $Q^{3B_*}\frac{r_1}{R_1} < \delta_*^3$. Thus, (5.4) and (5.11) both hold if $Q^{3B_*}\frac{r_1}{R_1} < \delta_*^3 \wedge \delta_1$. Condition (5.10) holds if $\eta_2 < \frac{d_1}{2}$ and $s_k < \frac{1}{2}|z_k - z_1| \wedge |z_k|$, which are equivalent to $\frac{r_1}{d_1} < \frac{1}{8}$ and $X < \frac{1}{2}$, respectively, which further follow from

$$\max_{1\le j\le n} \frac{r_j}{R_j} < \Big(\frac{1}{2}\Big)^{3+\frac{\beta+n\alpha+\beta_{n-1}}{\beta_\#}}.$$

From (5.12) and the choices of X and θ, we see that (5.7) follows from

$$Q^{B_{n-1}} \max_{1\le j\le n} \frac{r_j}{R_j} < \frac{X\theta^{B_{n-1}}\delta_{n-1}}{8C^{B_{n-1}}} = \frac{\delta_{n-1}}{8C^{B_{n-1}}} \max_{1\le j\le n} \Big(\frac{r_j}{R_j}\Big)^{\frac{\beta_\#}{\beta+n\alpha+\beta_{n-1}}+\frac{B_{n-1}\beta_{n-1}}{\alpha+B_{n-1}\beta_{n-1}}}.$$

Let $\beta_\& = 1 - \frac{\beta_\#}{\beta+n\alpha+\beta_{n-1}} - \frac{B_{n-1}\beta_{n-1}}{\alpha+B_{n-1}\beta_{n-1}}$. Since $\beta_\# \le \frac{\alpha\beta_{n-1}}{\alpha+B_{n-1}\beta_{n-1}}$, we get $\beta_\& > 0$. So (5.2) and (5.7) hold if $Q^{B_{n-1}/\beta_\&} \max_{1\le j\le n}\frac{r_j}{R_j} < (\frac{\delta_{n-1}}{8C^{B_{n-1}}})^{1/\beta_\&}$. Thus, (5.2,5.4,5.7,5.10,5.11) all hold if

$$Q^{3B_*+\frac{B_{n-1}}{\beta_\&}} \max_{1\le j\le n} \frac{r_j}{R_j} < \delta_n,$$

where $\delta_n := \delta_*^3 \wedge \delta_1 \wedge (\frac{1}{2})^{3+\frac{\beta+n\alpha+\beta_{n-1}}{\beta_\#}} \wedge (\frac{\delta_{n-1}}{8C^{B_{n-1}}})^{\frac{1}{\beta_\&}}$. Combining this with (5.13), we see that, if we set $B_n = 3B_* + \frac{B_{n-1}}{\beta_\&} + \frac{B_{n-1}\beta_{n-1}+B_*\beta_*}{\beta_n}$, then whenever (4.1) holds, (5.2,5.4,5.7,5.10,5.11) and $r_j < R_j/8$, $1 \le j \le n$, all hold, and the upper bounds for $|e_s|$, $1 \le s \le 13$, are all bounded above by the RHS of (4.2). It remains to prove Lemma 5.2 to finish this subsection.

5.1.1 Proof of Lemma 5.2

Since $K_a \subset K_b$ we also have $\operatorname{dist}(z_j, K_a) \ge s_j$, $2 \le j \le n$. Let $K = g_a(K_b \setminus K_a)$. Then K is an $\mathbb{H}$-hull, and $g_b = g_K \circ g_a$. Since $g_a(\gamma(a)) = U_a$, we have $U_a \in \overline{K} \cap \mathbb{R}$. Since $g_b(\gamma(b)) = U_b$, we have $U_b \in S_K$. Let $r_K = \sup\{|z - U_a| : z \in K\}$. From Lemma 2.5, we get $S_K \subset [U_a - 2r_K, U_a + 2r_K]$. Thus, $|U_b - U_a| \le 2r_K$.

Define $z_j^a = g_a(z_j)$, $\rho_j^a = g_a(\rho_j)$, $\xi_j^a = g_a(\xi_j)$, $z_j^b = g_b(z_j)$, $\rho_j^b = g_b(\rho_j)$, $2 \le j \le n$. Then $\rho_j^a, \rho_j^b, \xi_j^a$ are crosscuts of $\mathbb{H}$, $z_j^a \in \rho_j^a$, $z_j^b \in \rho_j^b$, and ξ_j^a disconnects K from ρ_j^a. By conformal invariance of extremal distance, we get

$$d_{\mathbb{H}}(\rho_j^b, S_K) = d_{\mathbb{H}}(\rho_j^a, K) = d_{H_a}(\rho_j, K_b \setminus K_a) \ge d_{H_a}(\rho_j, \xi_j) \ge d_*.$$

Applying Lemma 2.9 to $\overline{\rho_j^a}$ and $\overline{K}$, and to ρ_j^b and S_K, respectively, we get

$$\Big(\frac{\operatorname{diam}(\rho_j^a)}{\operatorname{dist}(\rho_j^a, K)} \wedge 1\Big) \cdot \Big(\frac{\operatorname{diam}(K)}{\operatorname{dist}(\rho_j^a, K)} \wedge 1\Big) \le 144 e^{-\pi d_*}, \quad 2 \le j \le n; \tag{5.14}$$

$$\Big(\frac{\operatorname{diam}(\rho_j^b)}{\operatorname{dist}(\rho_j^b, S_K)} \wedge 1\Big) \cdot \Big(\frac{\operatorname{diam}(S_K)}{\operatorname{dist}(\rho_j^b, S_K)} \wedge 1\Big) \le 144 e^{-\pi d_*}, \quad 2 \le j \le n. \tag{5.15}$$

Fix a variable $\phi \in (0,1)$ to be determined later. Define the event $E_{a;\phi}$ using (5.5) but with $\tau_r^{z_1}$ replaced by a (instead of $\tau_a^{z_1}$). First, suppose $E_{a;\phi}$ does not occur. Since $\operatorname{dist}(z_j, K_a) \ge s_j$, $2 \le j \le n$, from Lemma 3.4 we get

$$G(z_1)\widehat{G}_a(z_2, \dots, z_n) \lesssim F \prod_{k=2}^n \Big(\frac{|z_k| \wedge |z_k - z_1|}{s_k}\Big)^\alpha \phi^\alpha. \tag{5.16}$$

Fix some $j \in \{2, \dots, n\}$ for a while. Applying Koebe's 1/4 theorem, we get

$$\begin{aligned} \operatorname{dist}(z_j^b, S_{K_b}) &\asymp |g_b'(z_j)| \operatorname{dist}(z_j, K_b) \le |g_b'(z_j)| \operatorname{dist}(z_j, K_a) \\ &= |g_K'(z_j^a)||g_a'(z_j)| \operatorname{dist}(z_j, K_a) \asymp |g_K'(z_j^a)| \operatorname{dist}(z_j^a, S_{K_a}) \end{aligned}$$

and

$$|z_j^b - U_b| \ge \operatorname{dist}(z_j^b, S_K) \asymp |g_K'(z_j^a)| \operatorname{dist}(z_j^a, K).$$

Now we consider two cases.

Case 1. $\operatorname{diam}(S_K) \le \operatorname{dist}(z_j^b, S_K)/4$. In this case, since $z_j^a = f_K(z_j^b)$, applying Lemma 2.7, we get $\operatorname{dist}(z_j^a, K) \ge 2 \operatorname{diam}(K)$, which implies that $\operatorname{dist}(z_j^a, K) \asymp |z_j^a - U_a|$ since $U_a \in \overline{K}$. From the above two displayed formulas, we get $\frac{\operatorname{dist}(z_j^b, S_{K_b})}{|z_j^b - U_b|} \lesssim \frac{\operatorname{dist}(z_j^a, S_{K_a})}{|z_j^a - U_a|}$.

Case 2. $\operatorname{diam}(S_K) \ge \operatorname{dist}(z_j^b, S_K)/4$. From (5.15), we have

$$\frac{\operatorname{diam}(\rho_j^b)}{\operatorname{dist}(\rho_j^b, S_K)} \le 576 e^{-\pi d_*}, \tag{5.17}$$

if

$$144 e^{-\pi d_*} < 1/4. \tag{5.18}$$

Since $\operatorname{dist}(z_1, K_a) < |z_j - z_1|$, and $\rho_j \subset \{|z - z_1| = |z_j - z_1|\}$, we see that either ρ_j disconnects K_b from ∞, or ρ_j touches K_b. The former case implies that $\operatorname{diam}(\rho_j^b) \ge \operatorname{dist}(\rho_j^b, S_K)$ because ρ_j^b disconnects K from ∞, which is impossible by (5.17) if (5.18) holds. In the latter case,

$\rho_j^b := g_b(\rho_j)$ touches S_{K_b}, and so $\operatorname{dist}(z_j^b, S_{K_b}) \le \operatorname{diam}(\rho_j^b)$. On the other hand, since $U_b \in S_K$ and $z_j^b \in \rho_j^b$, we get $|z_j^b - U_b| \ge \operatorname{dist}(\rho_j^b, S_K)$. Thus by (5.17), we have $\operatorname{dist}(z_j^b, S_{K_b}) \le 576 e^{-\pi d_*}|z_j^b - U_b|$ if (5.18) holds.

Combining Case 1 with Case 2, we see that, if (5.18) holds and $E_{a;\phi}$ does not occur, then for some $2 \le j \le n$, $\operatorname{dist}(z_j^b, S_{K_b}) \lesssim (\phi + e^{-\pi d_*})|z_j^b - U_b|$. This together with Lemmas 3.4 and that $\operatorname{dist}(z_j, K_b) \ge s_j$, $2 \le j \le n$, implies that

$$G(z_1)\widehat{G}_b(z_2, \dots, z_n) \lesssim F \prod_{k=2}^{n} \Big(\frac{|z_k| \wedge |z_k - z_1|}{s_k}\Big)^{\alpha} (\phi^\alpha + e^{-\alpha\pi d_*}). \tag{5.19}$$

Now suppose that $E_{a;\phi}$ occurs. Since $z_j^a \in \rho_j^a$ and $U_a \in \overline{K}$, we have $|z_j^a - U_a| \ge \operatorname{dist}(\rho_j^a, K)$. We claim that $\operatorname{diam}(\rho_j^a) \ge \operatorname{dist}(z_j^a, S_{K_a})$. If this is not true, then the region bounded by ρ_j^a in $\mathbb{H}$ is disjoint from S_{K_a}, which implies that $\rho_j = g_a^{-1}(\rho_j^a)$ is also a crosscut of $\mathbb{H}$, and the region bounded by ρ_j in $\mathbb{H}$ is disjoint from K_a. Since ρ_j is an arc on the circle $\{|z - z_1| = |z_j - z_1|\}$, this would imply that $\operatorname{dist}(z_1, K_a) \ge |z_j - z_1|$, which is a contradiction. So the claim is proved. Thus, we have

$$\frac{\operatorname{diam}(\rho_j^a)}{\operatorname{dist}(\rho_j^a, K)} \ge \frac{\operatorname{dist}(z_j^a, S_{K_a})}{|z_j^a - U_a|} \ge \phi. \tag{5.20}$$

From (5.14), (5.20), $r_K \le \operatorname{diam}(K)$ and $z_j^a \in \rho_j^a$, we see that

$$\frac{r_K}{\operatorname{dist}(z_j^a, K)} \le \frac{144}{\phi} e^{-\pi d_*}, \quad 2 \le j \le n, \tag{5.21}$$

as long as the RHS is less than 1. Applying Lemma 2.6 with $x_0 = U_a$, $r = r_K$, and $z = z_j^a$, from $z_j^b = g_K(z_j^a)$, we see that, if

$$\frac{144}{\phi} e^{-\pi d_*} < \frac{1}{5}, \tag{5.22}$$

then

$$|z_j^b - z_j^a| \le r_K, \quad \frac{|\operatorname{Im} z_j^b - \operatorname{Im} z_j^a|}{\operatorname{Im} z_j^a} \le 4\Big(\frac{r_K}{\operatorname{dist}(z_j^a, K)}\Big)^2; \tag{5.23}$$

$$|g_K'(z_j^a) - 1| \le 5\Big(\frac{r_K}{\operatorname{dist}(z_j^a, K)}\Big)^2. \tag{5.24}$$

Let $\widehat{z}_j^a = z_j^a - U_a$ and $\widehat{z}_j^b = z_j^b - U_b$, $2 \le j \le n$. Since $|U_b - U_a| \le 2r_K$, from (5.23), we find that, if (5.22) holds, then

$$\frac{|\widehat{z}_j^b - \widehat{z}_j^a|}{|\widehat{z}_j^a|} \le 3\frac{r_K}{\operatorname{dist}(z_j^a, K)}, \quad \frac{|\operatorname{Im} \widehat{z}_j^b - \operatorname{Im} \widehat{z}_j^a|}{\operatorname{Im} \widehat{z}_j^a} \le 4\Big(\frac{r_K}{\operatorname{dist}(z_j^a, K)}\Big)^2. \tag{5.25}$$

By definition, we have

$$\widehat{G}_a(z_2,\dots,z_n) = \prod_{j=2}^n |g_a'(z_j)|^{2-d}\widehat{G}(\widehat{z}_2^a,\dots,\widehat{z}_n^a);$$

$$\begin{aligned}\widehat{G}_b(z_2,\dots,z_n) &= \prod_{j=2}^n |g_b'(z_j)|^{2-d}\widehat{G}(\widehat{z}_2^b,\dots,\widehat{z}_n^b)\\ &= \prod_{j=2}^n |g_K'(z_j^a)|^{2-d}\prod_{j=2}^n |g_a'(z_j)|^{2-d}\widehat{G}(\widehat{z}_2^b,\dots,\widehat{z}_n^b).\end{aligned}$$

Define $\widehat{G}_{a,b}(z_2,\dots,z_n) = \prod_{j=2}^n |g_a'(z_j)|^{2-d}\widehat{G}(\widehat{z}_2^b,\dots,\widehat{z}_n^b)$. From (5.24) we see that there is a constant $\delta \in (0,1)$ (depending on n) such that, if

$$\frac{r_K}{\mathrm{dist}(z_j^a,K)} < \delta, \tag{5.26}$$

then

$$|\widehat{G}_b(z_2,\dots,z_n) - \widehat{G}_{a,b}(z_2,\dots,z_n)| \lesssim \Big(\frac{r_K}{\mathrm{dist}(z_j^a,K)}\Big)^2 \widehat{G}_{a,b}(z_2,\dots,z_n). \tag{5.27}$$

Define $\widehat{d}_k$, $2 \le k \le n$, and $\widehat{Q}$ using (2.3) and (2.4) for the $(n-1)$ points $\widehat{z}_2^a,\dots,\widehat{z}_n^a$. Since Theorem 4.2 holds for $(n-1)$ points, from (5.25) we see that, for some constants $B_{n-1} > 0$ and $\beta_{n-1},\delta_{n-1} \in (0,1)$, if

$$\widehat{Q}^{B_{n-1}} \cdot \frac{|\widehat{z}_j^b - \widehat{z}_j^a|}{\widehat{d}_j} < \delta_{n-1}, \qquad \frac{|\operatorname{Im}\widehat{z}_j^b - \operatorname{Im}\widehat{z}_j^a|}{\operatorname{Im}\widehat{z}_j^a} < \delta_{n-1},$$

then

$$\begin{aligned}&|\widehat{G}_{a,b}(z_2,\dots,z_n) - \widehat{G}_a(z_2,\dots,z_n)|/F_a(z_2,\dots,z_n)\\ \lesssim &\sum_{j=2}^n \Big(\Big(\widehat{Q}^{B_{n-1}}\frac{|\widehat{z}_j^b - \widehat{z}_j^a|}{\widehat{d}_j}\Big)^{\beta_{n-1}} + \Big(\frac{|\operatorname{Im}\widehat{z}_j^b - \operatorname{Im}\widehat{z}_j^a|}{\operatorname{Im}\widehat{z}_j^a}\Big)^{\beta_{n-1}}\Big).\end{aligned}$$

Since $E_{a;\phi}$ occurs, (5.6) holds here with ϕ in place of θ by the same argument. Let $B_0 = B_{n-1}+1$. Then, for some constant $C > 1$, if

$$Q^{B_0} \cdot \frac{|\widehat{z}_j^b - \widehat{z}_j^a|}{|\widehat{z}_j^a|} < \frac{\phi^{B_0}\delta_{n-1}}{C^{B_0}}, \qquad \frac{|\operatorname{Im}\widehat{z}_j^b - \operatorname{Im}\widehat{z}_j^a|}{\operatorname{Im}\widehat{z}_j^a} < \delta_{n-1}, \tag{5.28}$$

then

$$\begin{aligned}&|\widehat{G}_{a,b}(z_2,\dots,z_n) - \widehat{G}_a(z_2,\dots,z_n)|/F_a(z_2,\dots,z_n)\\ \lesssim &\sum_{j=2}^n \Big(\Big(\phi^{-B_0}Q^{B_0}\frac{|\widehat{z}_j^b - \widehat{z}_j^a|}{|\widehat{z}_j^a|}\Big)^{\beta_{n-1}} + \Big(\frac{|\operatorname{Im}\widehat{z}_j^b - \operatorname{Im}\widehat{z}_j^a|}{\operatorname{Im}\widehat{z}_j^a}\Big)^{\beta_{n-1}}\Big).\end{aligned} \tag{5.29}$$

From (5.28) we see that the RHS of (5.29) is bounded above by a constant. Since $\widehat{G}_a \lesssim F_a$ by induction hypothesis, we get $\widehat{G}_{a,b} \lesssim F_a$ as well. From (5.27) and (5.29), we see that if (5.26) and (5.28) both hold, then

$$\begin{aligned}&|\widehat{G}_b(z_2,\dots,z_n) - \widehat{G}_a(z_2,\dots,z_n)|/F_a(z_2,\dots,z_n)\\ \lesssim&\Big(\frac{r_K}{\operatorname{dist}(z_j^a,K)}\Big)^2 + \sum_{j=2}^n \Big(\Big(\phi^{-B_0}Q^{B_0}\frac{|\widehat{z}_j^b - \widehat{z}_j^a|}{|\widehat{z}_j^a|}\Big)^{\beta_{n-1}} + \Big(\frac{|\operatorname{Im}\widehat{z}_j^b - \operatorname{Im}\widehat{z}_j^a|}{\operatorname{Im}\widehat{z}_j^a}\Big)^{\beta_{n-1}}\Big)\\ \lesssim&\phi^{-2}e^{-2\pi d_*} + (\phi^{-B_0-1}Q^{B_0}e^{-\pi d_*})^{\beta_{n-1}} + (\phi^{-2}e^{-2\pi d_*})^{\beta_{n-1}} \lesssim (\phi^{-B_0-1}Q^{B_0}e^{-\pi d_*})^{\beta_{n-1}}.\end{aligned}$$

where the second last inequality follows from (5.21), (5.25), and that $|z_j - z_1| \ge d_1$, and the last inequality holds provided that

$$\phi^{-2}e^{-2\pi d_*} < 1. \tag{5.30}$$

Since $\operatorname{dist}(z_j, K_a) \ge s_j$, $2 \le j \le n$, from Lemma 3.4, we get

$$G(z_1)|\widehat{G}_b(z_2,\dots,z_n) - \widehat{G}_a(z_2,\dots,z_n)| \lesssim F\prod_{k=2}^n \Big(\frac{|z_k| \wedge |z_k - z_1|}{s_k}\Big)^\alpha (\phi^{-B_0-1}Q^{B_0}e^{-\pi d_*})^{\beta_{n-1}}.$$

Combining the above with (5.16,5.19), which holds when $E_{a;\phi}$ does not occur, we find that, as long as Conditions (5.18,5.22,5.26,5.28,5.30) all hold, no matter whether $E_{a;\phi}$ happens, we have

$$\begin{aligned}&G(z_1)|\widehat{G}_b(z_2,\dots,z_n) - \widehat{G}_a(z_2,\dots,z_n)|\\ \lesssim&F\prod_{k=2}^n \Big(\frac{|z_k| \wedge |z_k - z_1|}{s_k}\Big)^\alpha [e^{-\alpha\pi d_*} + \phi^\alpha + (\phi^{-B_0-1}Q^{B_0}e^{-\pi d_*})^{\beta_{n-1}}].\end{aligned}$$

Finally, we may find constants $b, B_* > 0$ and $\beta_*, \delta_* \in (0,1)$, such that, with $\phi = e^{-b2\pi d_*}$, if (5.9) holds, then (5.18,5.22,5.26,5.28,5.30) all hold, and the quantity in the square bracket of the above displayed formula is bounded above by a constant times $(Q^{B_*}e^{-\pi d_*})^{\beta_*}$. This is analogous to the argument after the estimate on e_{13} and before this proof.

5.2 Continuity of Green's functions

We work on the inductive step for Theorem 4.2 in this subsection. Suppose $z_1', \dots, \widetilde{z}_n'$ are distinct points in $\mathbb{H}$ such that z_j' is close to z_j, $1 \le j \le n$. Let $T = T_{z_1} = \tau_0^{z_1}$ and $T' = T_{z_1'} = \tau_0^{z_1'}$. The main part of this subsection is composed of two lemmas.

Lemma 5.3. *With the induction hypothesis, Theorem 4.2 holds if* $z_1' = z_1$.

Lemma 5.4. *With the induction hypothesis, Theorem 4.2 holds if* $z_k' = z_k$, $2 \le k \le n$.

Before proving these lemmas, we first show how they can be used to prove the inductive step for Theorem 4.2 from $n-1$ to n. We have

$$\begin{aligned}&|\widehat{G}(z_1',z_2',\dots,z_n')-\widehat{G}(z_1,z_2,\dots,z_n)|\\ \le&|\widehat{G}(z_1',z_2',\dots,z_n')-\widehat{G}(z_1',z_2,\dots,z_n)|+|\widehat{G}(z_1',z_2,\dots,z_n)-\widehat{G}(z_1,z_2,\dots,z_n)|=:I_1+I_2.\end{aligned}$$

By Lemma 5.4, for some constants $B_n^{(2)}>0$ and $\beta_n^{(2)},\delta_n^{(2)}\in(0,1)$, I_2 is bounded by the RHS of (4.4) when (4.3) holds for $j=1$. We need to use Lemma 5.3 to estimate I_1 with the assumption that z_1' is close to z_1 but may not equal to z_1. Define d_k' and l_k', $1\le k\le n$, Q' and F' using (2.3) and (2.4) for the n points $z_1',z_1,\dots,z_n$. From Lemma 5.3, we know that, for some constants $B_n'>0$ and $\beta_n',\delta_n'\in(0,1)$, I_1 is bounded by the RHS of (4.4) when (4.3) holds for $2\le j\le n$, with d_j', Q' and F' in place of d_j, Q and F, respectively. Suppose

$$|z_1'-z_1|<d_1/2,\quad \operatorname{Im} z_1'\asymp \operatorname{Im} z_1. \tag{5.31}$$

Then we have $|z_1'|\asymp|z_1$ and $|z_k-z_1'|\asymp|z_k-z_1|$, $2\le k\le n$, which imply that $d_k'\asymp d_k$ and $l_k'\asymp l_k$, $1\le k\le n$, which in turn imply that $Q'\asymp Q$ and $F'\asymp F$.

Thus, there are constants $B_n^{(1)}>0$ and $\beta_n^{(1)},\delta_n^{(1)}\in(0,1)$, such that I_1 is bounded by the RHS of (4.4) when (4.3) holds for $2\le j\le n$. Finally, taking $B_n=B_n^{(1)}\vee B_n^{(2)}$, $\beta_n=\beta_n^{(1)}\wedge\beta_n^{(2)}$ and $\delta_n=\delta_n^{(1)}\wedge\delta_n^{(2)}\wedge 1/8$, we then finish the inductive step for Theorem 4.2 from $n-1$ to n.

Proof of Lemma 5.3. Define $E_{0;\vec{s}}$ and $E_{0;\theta}$ using (5.1) and (5.5) for $z_1,z_2,\dots,z_n$; and define $E'_{0;\vec{s}}$ and $E'_{0;\theta}$ using (5.1) and (5.5) for $z_1,z_2',\dots,z_n'$.

Fix $\vec{s}=(s_2,\dots,s_n)$ with $s_j\in(|z_j'-z_j|,|z_j-z_1|\wedge|z_j|)$ and $\theta\in(0,1)$ being variables to be determined later. From Koebe's 1/4 theorem and distortion theorem, we see that there is a constant $\delta\in(0,1/10)$ such that, if

$$\frac{|z_j'-z_j|}{s_j}<\delta,\quad 2\le j\le n, \tag{5.32}$$

and $E_{0;\vec{s}}$ occurs, then

$$4|g_T(z_j')-g_T(z_j)|<\operatorname{dist}(g_T(z_j),S_{K_T})\le|g_T(z_j)-U_T|,\quad 2\le j\le n,$$

which implies that

$$E_{0;\vec{s}}\cap E'_{0;2\theta}\subset E_{0;\vec{s}}\cap E_{0;\theta}\subset E_{0;\vec{s}}\cap E'_{0;\theta/2}. \tag{5.33}$$

Since $\delta<1/2$, (5.32) clearly implies that

$$E'_{0;2\vec{s}}\subset E_{0;\vec{s}}\subset E'_{0;\vec{s}/2}. \tag{5.34}$$

Suppose (5.32) holds. First, we express

$$\widehat{G}(z_1,z_2,\dots,z_n)=G(z_1)\mathbb{E}^*_{z_1}[\widehat{G}_T(z_2,\dots,z_n)]=G(z_1)\mathbb{E}^*_{z_1}[\mathbf{1}_{E_{0;\vec{s}}}\widehat{G}_T(z_2,\dots,z_n)]+e_1;$$

$$\widehat{G}(z_1,z_2'\dots,z_n')=G(z_1)\mathbb{E}^*_{z_1}[\widehat{G}_T(z_2',\dots,z_n')]=G(z_1)\mathbb{E}^*_{z_1}[\mathbf{1}_{E_{0;\bar{s}}}\widehat{G}_T(z_2',\dots,z_n')]+e_1'.$$

Using Lemma 5.1 and (5.34), we find that there is a constant $\beta>0$ such that

$$0\le e_1,e_1'\lesssim F\sum_{j=2}^n\Big(\frac{s_j}{|z_j|\wedge|z_j-z_1|}\Big)^\beta.$$

Second, we express

$$G(z_1)\mathbb{E}^*_{z_1}[\mathbf{1}_{E_{0;\bar{s}}}\widehat{G}_T(z_2,\dots,z_n)]=G(z_1)\mathbb{E}^*_{z_1}[\mathbf{1}_{E_{0;\bar{s}}\cap E_{0;\theta}}\widehat{G}_T(z_2,\dots,z_n)]+e_2;$$

$$G(z_1)\mathbb{E}^*_{z_1}[\mathbf{1}_{E_{0;\bar{s}}}\widehat{G}_T(z_2',\dots,z_n')]=G(z_1)\mathbb{E}^*_{z_1}[\mathbf{1}_{E_{0;\bar{s}}\cap E_{0;\theta}}\widehat{G}_T(z_2',\dots,z_n')]+e_2'.$$

From Lemma 3.4, (5.33,5.34), and that $\widehat{G}\lesssim F$ holds for $(n-1)$ points, we get

$$0\le e_2,e_2'\lesssim F\prod_{j=2}^n\Big(\frac{|z_j|\wedge|z_j-z_1|}{s_j}\Big)^\alpha\theta^\alpha.$$

Now suppose $E_{0;\bar{s}}$ and $E_{0;\theta}$ both occur. Let $Z=Z_T$, $\widehat{z}_j=Z(z_j)$ and $\widehat{z}_j'=Z(z_j')$, $2\le j\le n$. By definition, we have

$$\widehat{G}_T(z_2,\dots,z_n)=\prod_{j=2}^n|g_T'(z_j)|^{2-d}\widehat{G}(\widehat{z}_2,\dots,\widehat{z}_n);$$
$$\widehat{G}_T(z_2',\dots,z_n')=\prod_{j=2}^n|g_T'(z_j')|^{2-d}\widehat{G}(\widehat{z}_2',\dots,\widehat{z}_n').$$

Define $\widehat{G}_T'(z_2',\dots,z_n')=\prod_{j=2}^n|g_T'(z_j)|^{2-d}\widehat{G}(\widehat{z}_2',\dots,\widehat{z}_n')$. From Koebe's distortion theorem, there is a constant $\delta'\in(0,1)$ such that, if

$$\frac{|z_j'-z_j|}{s_j}<\delta',\quad 2\le j\le n, \tag{5.35}$$

then

$$|\widehat{G}_T(z_2',\dots,z_n')-\widehat{G}_T'(z_2',\dots,z_n')|\lesssim\sum_{j=2}^n\frac{|z_j'-z_j|}{s_j}\cdot\widehat{G}_T'(z_2',\dots,z_n'). \tag{5.36}$$

Define $\widehat{d}_k$, $2\le k\le n$, and $\widehat{Q}$ using (2.3) and (2.4) for the $(n-1)$ points $\widehat{z}_2,\dots,\widehat{z}_n$. Since Theorem 4.2 holds for $(n-1)$ points, we see that, for some constants $B_{n-1}>0$ and $\beta_{n-1},\delta_{n-1}\in(0,1)$, if

$$\widehat{Q}^{B_{n-1}}\cdot\frac{|\widehat{z}_j'-\widehat{z}_j|}{\widehat{d}_j}<\delta_{n-1},\quad\frac{|\operatorname{Im}\widehat{z}_j'-\operatorname{Im}\widehat{z}_j|}{\operatorname{Im}\widehat{z}_j}<\delta_{n-1},$$

then

$$|\widehat{G}(\widehat{z}_2',\dots,\widehat{z}_n')-\widehat{G}(\widehat{z}_2,\dots,\widehat{z}_n)|/F(\widehat{z}_2,\dots,\widehat{z}_n)$$
$$\lesssim\sum_{j=2}^n\Big(\Big(\widehat{Q}^{B_{n-1}}\frac{|\widehat{z}_j'-\widehat{z}_j|}{\widehat{d}_j}\Big)^{\beta_{n-1}}+\Big(\frac{|\operatorname{Im}\widehat{z}_j'-\operatorname{Im}\widehat{z}_j|}{\operatorname{Im}\widehat{z}_j}\Big)^{\beta_{n-1}}\Big).$$

If $E_{0;\theta}$ occurs,(5.6) holds here by the same argument. Let $B_0=B_{n-1}+1$. Then, for some constant $C>1$, if

$$Q^{B_0}\cdot\frac{|\widehat{z}_j'-\widehat{z}_j|}{|\widehat{z}_j|}<\frac{\theta^{B_0}\delta_{n-1}}{C^{B_0}},\quad \frac{|\operatorname{Im}\widehat{z}_j'-\operatorname{Im}\widehat{z}_j|}{\operatorname{Im}\widehat{z}_j}<\delta_{n-1}, \tag{5.37}$$

then

$$|\widehat{G}_T'(z_2',\dots,z_n')-\widehat{G}_T(z_2,\dots,z_n)|/F_T(z_2,\dots,z_n)$$
$$\lesssim\sum_{j=2}^n\Big(\Big(\theta^{-B_0}Q^{B_0}\frac{|\widehat{z}_j'-\widehat{z}_j|}{|\widehat{z}_j|}\Big)^{\beta_{n-1}}+\Big(\frac{|\operatorname{Im}\widehat{z}_j'-\operatorname{Im}\widehat{z}_j|}{\operatorname{Im}\widehat{z}_j}\Big)^{\beta_{n-1}}\Big). \tag{5.38}$$

From (5.37) we see that the RHS of (5.38) is bounded above by a constant. Since $\widehat{G}_T\lesssim F_T$, we get $\widehat{G}_T'(z_2',\dots,z_n')\lesssim F_T(z_2,\dots,z_n)$. From (5.36) and (5.38), we see that, if (5.35) and (5.37) both hold, then

$$|\widehat{G}_T(z_2',\dots,z_n')-\widehat{G}_T(z_2,\dots,z_n)|/F_T(z_2,\dots,z_n)$$
$$\lesssim\sum_{j=2}^n\Big(\frac{|z_j'-z_j|}{s_j}+\Big(\theta^{-B_0}Q^{B_0}\frac{|\widehat{z}_j'-\widehat{z}_j|}{|\widehat{z}_j|}\Big)^{\beta_{n-1}}+\Big(\frac{|\operatorname{Im}\widehat{z}_j'-\operatorname{Im}\widehat{z}_j|}{\operatorname{Im}\widehat{z}_j}\Big)^{\beta_{n-1}}\Big). \tag{5.39}$$

Applying Lemma 2.8 to $K=K_T$ and using $Z=g_T-U_T$ and $U_T\in S_{K_T}$, we find that, if (5.32) holds, then for $2\le j\le n$,

$$\frac{|\widehat{z}_j'-\widehat{z}_j|}{|\widehat{z}_j|}\lesssim\frac{|z_j'-z_j|}{s_j},\quad \frac{|\operatorname{Im}\widehat{z}_j'-\operatorname{Im}\widehat{z}_j|}{\operatorname{Im}\widehat{z}_j}\lesssim\frac{|\operatorname{Im}z_j'-\operatorname{Im}z_j|}{\operatorname{Im}z_j}+\Big(\frac{|z_j'-z_j|}{s_j}\Big)^{1/2}. \tag{5.40}$$

Thus, there is a constant $C_0>0$, such that if

$$Q^{B_0}\cdot\frac{|z_j'-z_j|}{s_j}<\frac{\theta^{B_0}\delta_{n-1}^2}{C_0},\quad \frac{|\operatorname{Im}z_j'-\operatorname{Im}z_j|}{\operatorname{Im}z_j}<\frac{\delta_{n-1}}{C_0}, \tag{5.41}$$

then (5.37) holds.

Now we express

$$G(z_1)\mathbb{E}_{z_1}^*[\mathbf{1}_{E_{0;\bar{s}}\cap E_{0;\theta}}\widehat{G}_T(z_2',\dots,z_n')]=G(z_1)\mathbb{E}_{z_1}^*[\mathbf{1}_{E_{0;\bar{s}}\cap E_{0;\theta}}\widehat{G}_T(z_2,\dots,z_n)]+e_3.$$

From (5.39,5.40) and Lemma 3.4, we find that, if (5.32,5.35,5.41) all hold, then

$$|e_3|\lesssim F\prod_{j=2}^n\Big(\frac{|z_j|\wedge|z_j-z_1|}{s_j}\Big)^\alpha\sum_{j=2}^n\Big(\Big(\theta^{-B_0}Q^{B_0}\frac{|z_j'-z_j|}{s_j}\Big)^{\beta_{n-1}/2}+\Big(\frac{|\operatorname{Im}z_j'-\operatorname{Im}z_j|}{\operatorname{Im}z_j}\Big)^{\beta_{n-1}}\Big).$$

At the end, we follow the argument after the estimate on e_{13} in Section 5.1. First suppose that $\frac{s_j}{|z_j|\wedge|z_j-z_1|} = X$, $2 \le j \le n$, for some $X \in (0,1)$ to be determined. Then we have $\frac{|z_j'-z_j|}{s_j} \le X^{-1} \cdot \frac{|z_j'-z_j|}{d_j}$, $2 \le j \le n$. Then we may set

$$\theta = \max_{2\le j\le n} \Big(\frac{|z_j'-z_j|}{d_j}\Big)^a, \quad X = \max_{2\le j\le n} \Big(\frac{|z_j'-z_j|}{d_j}\Big)^b \bigvee \max_{2\le j\le n} \Big(\frac{|\operatorname{Im} z_j' - \operatorname{Im} z_j|}{\operatorname{Im} z_j}\Big)^c$$

for some suitable constants $a,b,c>0$. It is easy to find those a,b,c and some constants $B_n > 0$ and $\beta_n, \delta_n \in (0,1)$ such that the upper bounds for $|e_1|, |e_1'|, |e_2|, |e_2'|, |e_3|$ are all bounded by the RHS of (4.4) with $z_1' = z_1$, and if (4.3) holds, then (5.32,5.35,5.41) all hold. The proof is now complete. $\square$

Proof of Lemma 5.4. Fix $s_j \in (|z_1'-z_1|, |z_j-z_1|\wedge|z_j|)$, $2\le j\le n$, and $\eta_2 > \eta_1 > |z_1'-z_1|$ depending on $\kappa, n, z_1, z_1', z_2, \dots, z_n$ to be determined later. Define $E_{0;\vec{s}}$, $E_{\eta_1;\vec{s}}$, and $E_{\eta_1,0;\eta_2}$ using (5.1), (5.1), and (5.8), respectively, for $z_1, z_2, \dots, z_n$. Define $E'_{0;\vec{s}}$ using (5.1) for $z_1', z_2, \dots, z_n$, let $E'_{\eta_1,\vec{s}} = E_{\eta_1;\vec{s}}$, and define

$$\begin{aligned} E'_{\eta_1,0;\eta_2} = &\{\gamma[\tau^{z_1}_{\eta_1}, T_{z_1'}] \text{ does not intersect any connected component of} \\ &\{|z-z_1| = \eta_2\} \cap H_{\tau^{z_1}_{\eta_1}} \text{ that separates } z_1' \text{ from any } z_k, 2\le k\le n\}. \end{aligned}$$

First, we express

$$\widehat{G}(z_1, z_2, \dots, z_n) = G(z_1)\mathbb{E}^*_{z_1}[\mathbf{1}_{E_{0;\vec{s}}} \widehat{G}_{T_{z_1}}(z_2, \dots, z_n)] + e_1;$$

$$\widehat{G}(z_1', z_2, \dots, z_n) = G(z_1')\mathbb{E}^*_{z_1'}[\mathbf{1}_{E'_{0;\vec{s}}} \widehat{G}_{T_{z_1'}}(z_2, \dots, z_n)] + e_1'.$$

Now suppose (5.31) holds. Recall that we have $|z_j - z_1'| \asymp |z_j - z_1|$, $2\le j\le n$, $Q' \asymp Q$ and $F' \asymp F$. By Lemma 5.1, we see that there is a constant $\beta > 0$ such that

$$0 \le e_1, e_1' \lesssim F \sum_{j=2}^n \Big(\frac{s_j}{|z_j|\wedge|z_j-z_1|}\Big)^\beta.$$

Second, we express

$$G(z_1)\mathbb{E}^*_{z_1}[\mathbf{1}_{E_{0;\vec{s}}} \widehat{G}_{T_{z_1}}(z_2, \dots, z_n)] = G(z_1)\mathbb{E}^*_{z_1}[\mathbf{1}_{E_{0;\vec{s}} \cap E_{\eta_1,0;\vec{s}}} \widehat{G}_{T_{z_1}}(z_2, \dots, z_n)] + e_2;$$

$$G(z_1')\mathbb{E}^*_{z_1'}[\mathbf{1}_{E'_{0;\vec{s}}} \widehat{G}_{T_{z_1'}}(z_2, \dots, z_n)] = G(z_1')\mathbb{E}^*_{z_1'}[\mathbf{1}_{E'_{0;\vec{s}} \cap E'_{\eta_1,0;\vec{s}}} \widehat{G}_{T_{z_1'}}(z_2, \dots, z_n)] + e_2'.$$

From Lemma 3.2, Corollary 3.3, Lemma 3.4, and that $|z_j - z_1'| \asymp |z_j - z_1|$ and $F' \asymp F$, we get

$$0 \le e_2, e_2' \lesssim F \prod_{j=2}^n \Big(\frac{|z_j|\wedge|z_j-z_1|}{s_j}\Big)^\alpha \Big(\frac{\eta_1}{\eta_2}\Big)^{\alpha/4}.$$

Third, we change the times in the two expressions from T_{z_1} and $T_{z_1'}$, respectively, to the same time $\tau^{z_1}_{\eta_1}$, and express

$$G(z_1)\mathbb{E}^*_{z_1}[\mathbf{1}_{E_{0;\vec{s}}\cap E_{\eta_1,0;\vec{s}}}\widehat{G}_{T_{z_1}}(z_2,\dots,z_n)] = G(z_1)\mathbb{E}^*_{z_1}[\mathbf{1}_{E_{\eta_1;\vec{s}}\cap E_{\eta_1,0;\vec{s}}}\widehat{G}_{\tau^{z_1}_{\eta_1}}(z_2,\dots,z_n)] + e_3;$$

$$G(z_1')\mathbb{E}^*_{z_1'}[\mathbf{1}_{E'_{0;\vec{s}}\cap E'_{\eta_1,0;\vec{s}}}\widehat{G}_{T_{z_1'}}(z_2,\dots,z_n)] = G(z_1')\mathbb{E}^*_{z_1'}[\mathbf{1}_{E'_{\eta_1;\vec{s}}\cap E'_{\eta_1,0;\vec{s}}}\widehat{G}_{\tau^{z_1}_{\eta_1}}(z_2,\dots,z_n)] + e_3'.$$

Now suppose (5.10) holds. Then $E_{\eta_1,0;\eta_2}\cap E_{\eta_1;\vec{s}} = E_{\eta_1,0;\eta_2}\cap E_{0;\vec{s}}$ and $E'_{\eta_1,0;\eta_2}\cap E'_{\eta_1;\vec{s}} = E'_{\eta_1,0;\eta_2}\cap E'_{0;\vec{s}}$. Applying Lemma 5.2 with $a=\tau^{z_1}_{\eta_1}$, $b=T_{z_1}$ or $b=T_{z_1'}$, and using $Q'\asymp Q$, $F'\asymp F$ and $|z_j-z_1'|\asymp|z_j-z_1|$, we find that, for some constants $B_*>0$ and $\beta_*,\delta_*\in(0,1)$, if (5.11) holds, then

$$|e_3|,|e_3'|\lesssim F\prod_{j=2}^n\Big(\frac{|z_j|\wedge|z_j-z_1|}{s_k}\Big)^\alpha\Big(Q^{B_*}\frac{\eta_2}{d_1}\Big)^{\beta_*}.$$

Note that the proof of Lemma 5.2 uses Theorem 4.2 for $n-1$ points so we can use it here by induction hypothesis. Removing the restriction of the events $E_{\eta_1,0;\eta_2}$ and $E'_{\eta_1,0;\eta_2}$, we express

$$G(z_1)\mathbb{E}^*_{z_1}[\mathbf{1}_{E_{\eta_1;\vec{s}}\cap E_{\eta_1,0;\vec{s}}}\widehat{G}_{\tau^{z_1}_{\eta_1}}(z_2,\dots,z_n)] = G(z_1)\mathbb{E}^*_{z_1}[\mathbf{1}_{E_{\eta_1;\vec{s}}}\widehat{G}_{\tau^{z_1}_{\eta_1}}(z_2,\dots,z_n)] - e_4;$$

$$G(z_1')\mathbb{E}^*_{z_1'}[\mathbf{1}_{E'_{\eta_1;\vec{s}}\cap E'_{\eta_1,0;\vec{s}}}\widehat{G}_{\tau^{z_1}_{\eta_1}}(z_2,\dots,z_n)] = G(z_1')\mathbb{E}^*_{z_1'}[\mathbf{1}_{E'_{\eta_1;\vec{s}}}\widehat{G}_{\tau^{z_1}_{\eta_1}}(z_2,\dots,z_n)] - e_4'.$$

The estimates on e_4,e_4' are the same as that on e_2,e_2' by Lemma 3.2, Corollary 3.3, Lemma 3.4, and that $F'\asymp F$ and $|z_j-z_1'|\asymp|z_j-z_1|$.

Changing $G(z_1')$ to $G(z_1)$ on the RHS of the second displayed formula, we express

$$G(z_1')\mathbb{E}^*_{z_1'}[\mathbf{1}_{E'_{\eta_1;\vec{s}}}\widehat{G}_{\tau^{z_1}_{\eta_1}}(z_2,\dots,z_n)] = G(z_1)\mathbb{E}^*_{z_1'}[\mathbf{1}_{E'_{\eta_1;\vec{s}}}\widehat{G}_{\tau^{z_1}_{\eta_1}}(z_2,\dots,z_n)] + e_5.$$

From (1.2) and Lemma 3.4 we see that there is a constant $\delta>0$ such that, if

$$\frac{|z_1'-z_1|}{|z_1|}<\delta,\quad \frac{|\operatorname{Im} z_1'-\operatorname{Im} z_1|}{\operatorname{Im} z_1}<\delta, \tag{5.42}$$

then

$$|e_5|\lesssim F\prod_{k=2}^n\Big(\frac{|z_k|\wedge|z_k-z_1|}{s_k}\Big)^\alpha\Big(\frac{|z_1'-z_1|}{|z_1|}+\frac{|\operatorname{Im} z_1'-\operatorname{Im} z_1|}{\operatorname{Im} z_1}\Big).$$

Finally, we express

$$G(z_1)\mathbb{E}^*_{z_1'}[\mathbf{1}_{E'_{\eta_1;\vec{s}}}\widehat{G}_{\tau^{z_1}_{\eta_1}}(z_2,\dots,z_n)] = G(z_1)\mathbb{E}^*_{z_1}[\mathbf{1}_{E_{\eta_1;\vec{s}}}\widehat{G}_{\tau^{z_1}_{\eta_1}}(z_2,\dots,z_n)] + e_6.$$

Since $E'_{\eta_1,\vec{s}}=E_{\eta_1;\vec{s}}$, the random variables in the two square brackets are the same, which is $\mathcal{F}_{\tau^{z_1}_{\eta_1}}$-measurable. By Lemmas 2.11 and 3.4, we see that there is a constant δ such that, if

$$\frac{|z_1'-z_1|}{\eta_1}<\delta, \tag{5.43}$$

then

$$|e_6| \lesssim F \prod_{k=2}^{n} \Big(\frac{|z_k| \wedge |z_k - z_1|}{s_k}\Big)^{\alpha} \Big(\frac{|z_1' - z_1|}{\eta_1}\Big).$$

At the end, we follow the argument after the estimate on e_{13} in Section 5.1. Suppose that $\frac{s_j}{|z_j| \wedge |z_j - z_1|} = X$, $2 \leq j \leq n$, for some $X \in (0,1)$ to be determined. Pick η_1, η_2 such that $|z_1' - z_1|/\eta_1 = \eta_1/\eta_2 = \eta_2/d_1$. It is easy to find constants $a, B_n > 0$ and $\beta_n, \delta_n \in (0,1)$ such that with $X = (\frac{|z_1' - z_1|}{d_1})^a$, if (4.3) holds for $j = 1$, then Conditions (5.31,5.10,5.11,5.42,5.43) all hold, and the upper bounds for $|e_j|$, $1 \leq j \leq 6$, and $|e_j'|$, $1 \leq j \leq 4$, are all bounded by the RHS of (4.4). The proof is now complete. □

6 Proof of Theorem 4.3

In this section we want to show the desired lower bound for the multi-point Green's function. The method of the proof is based on the generalization of the method used in [16] and [13] to show the lower bound. We find the best point (almost means the nearest point but we make it precise) to go near first and we consider the event to go near that point before going near other points (as much as possible). This can be done by staying in a L-shape as defined in [16]. It is possible that we can not go all the way to a specific given point since couple of points are very near each other. In this case we can stop in an earlier time and separate points by a conformal map. We will go through the details about this general strategy in this section. Following Lawler and Zhou in [16], we define for $z \in \overline{\mathbb{H}}$ and $\rho \in (0,1)$,

$$L_z = [0, \operatorname{Re} z] \cup [\operatorname{Re} z, z],$$

and

$$L_{z,\rho} = \{z' \in \overline{\mathbb{H}} |\operatorname{dist}(z', L_z) \leq \rho|z|\}.$$

A simple geometry argument shows that, for any $z_0 \in \overline{\mathbb{H}} \setminus \{0\}$ and $\rho \in (0,1)$,

$$L_{z_0,\rho} \cap \{z \in \overline{\mathbb{H}} : |z| \geq |z_0|\} \subset \{|z - z_0| \leq \sqrt{2\rho}|z_0|\}. \tag{6.1}$$

Now we state a lemma which shows what happens to points which are not in the L-shape when we flatten the domain.

Lemma 6.1. *Suppose $0 < \rho \leq \frac{1}{4}$. Then there exists $c < \infty$ such that the following holds. Suppose $z \in \mathbb{H}$, $z_1, z_2 \in \mathbb{H} \setminus L_{z,2\rho}$, and $\gamma(t)$, $0 \leq t \leq T$, is a chordal Loewner curve such that $\gamma(0) = 0$, $\gamma(T) = z$, and $\gamma[0,T] \subset L_{z,\rho}$. Let $Z = Z_T$ be the centered Loewner map at time T. Then we have the followings.*

$$|Z'(z_1)| \asymp 1.$$

$$\operatorname{Im}(Z(z_1)) \asymp \operatorname{Im}(z_1).$$

$$|Z(z_1)| \asymp |z_1|.$$

$$|Z(z_1) - Z(z_2)| \lesssim |z_1 - z_2|.$$

Finally if $z_1, z_2, \ldots, z_n$ are distinct points in $\mathbb{H} \setminus L_{z,2\rho}$ and $r_1, \ldots, r_n > 0$ we have

$$F(Z(z_1), \ldots, Z(z_n); |Z'(z_1)|r_1, \ldots, |Z'(z_n)|r_n) \gtrsim F(z_1, \ldots, z_n; r_1, \ldots, r_n).$$

The implicit constants depend only on κ and ρ.

Proof. The proofs for first 3 equations above are in [16, Proposition 3.2]. For the second to last one, suppose η is a curve in $\mathbb{H} \setminus L_{z,2\rho}$ which connects z_1 and z_2 and has length at most $c_1|z_1 - z_2|$. If the closed line l passing through z_1 and z_2 does not pass through $L_{z,2\rho}$ then it works otherwise we go on the l until we hit $L_{z,2\rho}$ then we go up on $L_{z,2\rho}$ to modify pass such that it does not pass through $L_{z,2\rho}$. Then the length of the image of η under Z is at most $c_2|z_1 - z_2|$ by derivative estimate. The last statement is a result of the definition of F and the previous equations. $\square$

Remark We expect that $|Z(z_1) - Z(z_2)| \asymp |z_1 - z_2|$ holds in the statement of the lemma. We do not try to prove it since it is not needed.

The same proof gives us the following modification of Lemma 6.1. Suppose the chordal Loewner curve γ satisfies that $\gamma[0, T] \subset \{|z| \le R\}$. Suppose $z_1, \ldots, z_n \notin \{|z| \le 2R\}$. Then all the results of the Lemma 6.1 holds for $z_1, \ldots, z_n$ as well. These results also follow from [21, Lemma 5.4]. See, e.g., the proof of Corollary 4.4.

Now we strengthen [16, Proposition 3.1]. We quantify the chance that we stay in the L-shape and at the same time the tip of the curve behaves nicely.

Proposition 6.2. *There are uniform constants $C_0, C_1 > 0, N > 2, b_2 > 1 > b_1 > 0$ such that for every $0 < \delta < 1$, there is $C_\delta > 0$ such that for every $z_0 \in \overline{\mathbb{H}} \setminus \{0\}$ and $0 < r \le \frac{\delta|z_0|}{N}$ there exists stopping time $\tau_0 = \tau_0^\delta(z_0, r)$ such that the event E_{τ_0} defined by $\tau_0 < \infty$ and*

(i) $\operatorname{dist}(z_0, \gamma[0, \tau_0]) \in (b_1 r, b_2 r)$,

(ii) $\gamma[0, \tau_0] \subset L_{z_0,\delta}$,

(iii) $\operatorname{dist}(g_{\tau_0}(z_0), S_{K_{\tau_0}}) \ge C_0|g_{\tau_0}(z_0) - U_{\tau_0}| = C_0|Z_{\tau_0}(z_0)|$,

(iv) $|Z_{\tau_0}(z_0)| \le C_1\sqrt{r|z_0|}$,

satisfies that

$$\mathbb{P}^*_{z_0}[E_{\tau_0}] \ge C_\delta; \tag{6.2}$$

$$\mathbb{P}[E_{\tau_0}] \ge C_\delta F(z_0; r). \tag{6.3}$$

Proof. By scaling we may assume $\max\{|x_0|, y_0\} = 1$, where $x_0 = \operatorname{Re} z_0$ and $y_0 = \operatorname{Im} z_0$. Then $|z_0| \asymp 1$. We first prove (6.2), and consider two different cases to prove this. First we consider the interior case when r is smaller or comparable to y_0, and then we consider the boundary case when r is bigger or comparable to y_0. Also throughout the proof we consider N as a fixed number (greater than 2) which we will determine at the end.

Interior Case: Suppose for this case that $r < 10y_0$. Define the stopping time τ by

$$\tau = \inf\{t : \mathrm{dist}(\gamma(t), z_0) = \frac{y_0}{10} \wedge r\}.$$

By [16, Proposition 3.1], we know that there is $u > 0$ depending only on κ and $\frac{\delta}{N}$ such that for every $z_0 \in \mathbb{H}$, $\mathbb{P}^*_{z_0}[\gamma[0, T_{z_0}] \subset L_{z_0, \frac{\delta}{N}}] \geq u$. By this we know that

$$\mathbb{P}^*_{z_0}[\gamma[0, \tau] \subset L_{z_0, \frac{\delta}{N}}] \geq u.$$

Let $\widetilde{E}$ denote the event $\gamma[0, \tau] \subset L_{z_0, \frac{\delta}{N}}$. Now define τ_0 by

$$\tau_0 = \inf\{t : \Upsilon_t(z_0) = \frac{y_0}{100} \wedge \frac{r}{10}\},$$

where $\Upsilon_t(z_0)$ is the conformal radius of z_0 in H_t.

Now we want to show $\mathbb{P}^*_{z_0}[E_{\tau_0} | \widetilde{E}] \geq u_0$ for some constant $u_0 > 0$. Since $\mathbb{P}^*_{z_0}$-a.s. $T_{z_0} < \infty$, we have $\mathbb{P}^*_{z_0}[\tau_0 < \infty] = 1$. By Koebe's 1/4 theorem, we immediately have Property (i).

For Property (ii) let $E^1_{\tau_0}$ denote the event that after time τ, γ stays in $L_{z_0,\delta}$ till T_z. Since $\mathrm{dist}(\gamma(\tau), z_0) \leq r \leq \frac{\delta}{N}$, there exists at least one connected component of $\{|z - z_0| = \frac{\delta}{2})\} \cap H_\tau$ that disconnects $\gamma(\tau)$ from ∞ in H_τ. After τ, in order for γ to reach $\partial L_{z_0,\delta}$, it must intersect that arc. By Lemma 3.2 we have $\mathbb{P}^*_{z_0}[E^1_{\tau_0} | \widetilde{E}] \geq 1 - CN^{-c}$ for some constants $C, c > 0$.

For Property (iii) we use [16, Lemma 2.2]. By Koebe's 1/4 theorem we know that $\log(\Upsilon_{\tau_0}) - \log(\Upsilon_\tau) \leq -1$. By [16, Lemma 2.2], for any $\rho < 1$ we have $\theta_0 > 0$ such that

$$\mathbb{P}^*_{z_0}[\mathrm{Im}\, Z_{\tau_0}(z_0)/|Z_{\tau_0}(z_0)| \geq \theta_0 | \mathcal{F}_\tau] \geq \rho.$$

Call the event $\mathrm{Im}\, Z_{\tau_0}(z_0)/|Z_{\tau_0}(z_0)| \geq \theta_0$ as $E^2_{\tau_0}$. If $E^2_{\tau_0}$ occurs then Property (iii) is satisfied (with the constant depending on θ_0) because $\mathrm{dist}(g_{\tau_0}(z_0), S_{K_{\tau_0}}) \geq \mathrm{Im}\, Z_{\tau_0}(z_0)$.

If we choose $\rho \in (0, 1)$ and $N > 2$ such that $u_0 = \rho - CN^{-c} > 0$ then we have

$$\mathbb{P}^*_{z_0}[E^1_{\tau_0} \cap E^2_{\tau_0} | \widetilde{E}] \geq \mathbb{P}^*_{z_0}[E^1_{\tau_0} | \widetilde{E}] + \mathbb{P}^*_{z_0}[E^2_{\tau_0} | \widetilde{E}] - 1 \geq \rho - CN^{-c} = u_0 > 0.$$

So $\mathbb{P}^*_{z_0}[E^1_{\tau_0} \cap E^2_{\tau_0}] \geq uu_0 > 0$. We have seen that Properties (i)-(iii) are satisfied on the event $E^1_{\tau_0} \cap E^2_{\tau_0}$. For Property (iv), set $Z = Z_{\tau_0}$, and let $\Pi = \{z \in \mathbb{H} : \mathrm{Im}(z) = 10\}$. Then $\mathrm{Im}\, Z(z) \leq \mathrm{Im}\, z = 10$ for $z \in \Pi$. Consider the event that Brownian motion starting at z_0 hits Π before hitting $\gamma[0, \tau_0] \cup \mathbb{R}$. By Property (i) and Beurling estimate it has chance less than $c\sqrt{r}$ for some fixed constant c. After map Z, the chance that Brownian motion starting at $Z(z_0)$ hits $Z(\Pi)$ before hitting $\mathbb{R}$ is at least $\mathrm{Im}(Z(z_0))/10$ by gambler's ruin estimate which has the same order as $|Z(z_0)|$ when $E^2_{\tau_0}$ happens. So we have Property (iv) on the event $E^1_{\tau_0} \cap E^2_{\tau_0}$. Thus, $E^1_{\tau_0} \cap E^2_{\tau_0} \subset E_{\tau_0}$. This finishes the proof of (6.2) in the interior case.

Boundary Case: For this case assume that $1 > r \geq 10y_0$. Without loss of generality we assume $x_0 = 1$. Then $z_0 = 1 + iy_0$. We follow the steps as in the interior case just we have to modify some definitions for the boundary case. First, following [11] we consider

$$x_t = \inf\{x > 0 : T_x > t\}, \quad D_t = H_t \cup \{\bar{z} : z \in H_t\} \cup (x_t, \infty),$$

$$X_t = Z_t(1) = g_t(1) - U_t, \quad O_t = g_t(x_t) - U_t,$$
$$J_t = \frac{X_t - O_t}{X_t}, \quad \Upsilon_t(1) = \frac{X_t - O_t}{X_t} g_t'(1).$$

Note that Υ_t is 1/4 times the conformal radius of 1 in D_t. So we have

$$\frac{1}{4}\operatorname{dist}(1, \partial D_t) \le \Upsilon_t(1) \le \operatorname{dist}(1, \partial D_t). \tag{6.4}$$

Take

$$\tau = \inf\{t : \operatorname{dist}(\gamma(t), 1) = 100r\}.$$

By [16, Proposition 3.1], we know that there is $u > 0$ depending on κ and $\frac{\delta}{N}$ such that $\mathbb{P}_1^*[\gamma[0, T_1] \subset L_{1,\frac{\delta}{N}}] \ge u$. Let $\widetilde{E}$ denote the event that $\gamma[0, \tau] \subset L_{1,\frac{\delta}{N}}$. Then $\mathbb{P}_1^*[\widetilde{E}] \ge u$. Now take τ_0 as

$$\tau_0 = \inf\{t : \Upsilon_t(1) = 8r\}.$$

Since $\mathbb{P}_1^*$-a.s. $T_1 < \infty$, we have $\mathbb{P}_1^*[\tau_0 < \infty] = 1$. By (6.4), we immediately have Property (i). Let $E_1^{\tau_0}$ denote the event that after τ, the curve stays in $L_{1,\delta}$ till T_1. Using Lemma 3.2 as in the interior case, we get $\mathbb{P}_1^*[E_{\tau_0}^1 | \widetilde{E}] \ge 1 - CN^{-c}$ for some constants $C, c > 0$. If $E_{\tau_0}^1$ happens, since $L_{1,\delta} \subset L_{z_0,\delta}$, we have Property (ii).

By Koebe's 1/4 theorem we know that $\log(\Upsilon_{\tau_0}) - \log(\Upsilon_\tau) \le -1$. By [11, Section 4] we have that for any $\rho < 1$ there is $\theta_0 > 0$ such that

$$\mathbb{P}_1^*[J_{\tau_0} \ge \theta_0 | \mathcal{F}_\tau] \ge \rho.$$

Call the event $J_{\tau_0} \ge \theta_0$ as $E_{\tau_0}^2$. Since $|z_0 - 1| = y_0$ and $\operatorname{dist}(z_0, K_{\tau_0}) \ge 2r \ge 20y_0$, by Koebe's 1/4 theorem and distortion theorem, we get $|g_{\tau_0}(z_0) - g_{\tau_0}(1)| \le \frac{2}{9}\operatorname{dist}(g_{\tau_0}(z_0), S_{K_{\tau_0}})$. Thus, by triangle inequality, $\operatorname{dist}(g_{\tau_0}(z_0), S_{K_{\tau_0}}) \asymp \operatorname{dist}(g_{\tau_0}(1), S_{K_{\tau_0}})$. Since $U_{\tau_0} \in S_{K_{\tau_0}}$, we have $|g_{\tau_0}(z_0) - g_{\tau_0}(1)| \le \frac{2}{9}|g_{\tau_0}(z_0) - U_{\tau_0}|$. So we also get $|g_{\tau_0}(z_0) - U_{\tau_0}| \asymp |g_{\tau_0}(1) - U_{\tau_0}|$. If $E_{\tau_0}^2$ happens then the Property (iii) is satisfied at the point 1 with $C_0 = \theta_0$, and so is also satisfied at the point z_0 with a bigger constant by the above estimates.

If we choose $\rho \in (0, 1)$ and $N > 2$ such that $u_0 = \rho - CN^{-c} > 0$ then we have $\mathbb{P}_1^*[E_{\tau_0}^1 \cap E_{\tau_0}^2 | \widetilde{E}] \ge u_0$. So $\mathbb{P}_1^*[E_{\tau_0}^1 \cap E_{\tau_0}^2] \ge uu_0 > 0$. Since $\operatorname{dist}(z_0, \gamma[0, \tau_0]) \ge 2r$, until time τ_0 the two probability measures $\mathbb{P}_{z_0}^*$ and $\mathbb{P}_1^*$ are comparable by a universal constant c by [16, Proposition 2.9]. So we get $\mathbb{P}_{z_0}^*[E_{\tau_0}^1 \cap E_{\tau_0}^2] \ge uu_0/c > 0$.

We have seen that Properties (i)-(iii) are satisfied on the event $E_{\tau_0}^1 \cap E_{\tau_0}^2$. For Property (iv), similar to the interior case, we use Beurling estimate. Take $D = D_{\tau_0}$. Brownian motion starting at 1 has chance less than $c\sqrt{r}$ to hit $\Pi = \{\operatorname{Im} z = 10\}$ before exiting D. By conformal invariance of Brownian motion, this implies that distance between $(-\infty, O_{\tau_0})$ and $Z_{\tau_0}(1)$ which is $X_{\tau_0} - O_{\tau_0}$ is not more than $c\sqrt{r}$, which then implies $g_{\tau_0}'(1) \lesssim \frac{1}{\sqrt{r}}$ because $\Upsilon_{\tau_0} \asymp r$. Since $J_{\tau_0} \ge \theta_0$, we have $|Z_{\tau_0}(1)| \lesssim \sqrt{r}$. By Koebe's distortion theorem we get $|Z_{\tau_0}(z_0) - Z_{\tau_0}(1)| \lesssim g_{\tau_0}'(1)|z_0 - 1| \lesssim \sqrt{r}$. So we get $|Z_{\tau_0}(z_0)| \lesssim \sqrt{r}$, as desired. So we get $E_{\tau_0}^1 \cap E_{\tau_0}^2 \subset E_{\tau_0}$. This finishes the proof of (6.2) in the boundary case.

Finally, we prove (6.3). From [15, 16] we know that $\mathbb{P}$ is absolutely continuous with respect to $\mathbb{P}^*_{z_0}$ on $\mathcal{F}_{\tau_0} \cap \{\tau_0 < \infty\}$, and the Radon-Nikodym derivative is

$$R = \begin{cases} \frac{|Z_{\tau_0}(z_0)|^\alpha \operatorname{Im}(Z_{\tau_0}(z_0))^{(2-d)-\alpha}}{|g'_{\tau_0}(z_0)|^{2-d}|z_0|^\alpha y_0^{(2-d)-\alpha}}, & z_0 \in \mathbb{H}; \\ \\ \frac{|Z_{\tau_0}(z_0)|^\alpha}{|g'_{\tau_0}(z_0)|^\alpha |z_0|^\alpha}, & z_0 \in \mathbb{R} \setminus \{0\}. \end{cases}$$

Recall that in both of the above two cases, we defined events $E^1_{\tau_0}$ and $E^2_{\tau_0}$ such that $E^1_{\tau_0} \cap E^2_{\tau_0} \subset E_{\tau_0}$ and $\mathbb{P}^*_{z_0}[E^1_{\tau_0} \cap E^2_{\tau_0}] \gtrsim 1$. So it suffices to show that $R \asymp F(z_0; r)$ on $E^2_{\tau_0}$.

In the interior case, suppose $E^2_{\tau_0}$ happens. Then $\operatorname{Im} Z_{\tau_0}(z_0) \asymp |Z_{\tau_0}(z_0)|$. They are also comparable to $\operatorname{dist}(g_{\tau_0}(z_0), S_{K_{\tau_0}})$ because $\operatorname{Im} Z_{\tau_0}(z_0) \le \operatorname{dist}(g_{\tau_0}(z_0), S_{K_{\tau_0}}) \le |Z_{\tau_0}(z_0)|$. By Koebe's 1/4 theorem we get

$$R \asymp \frac{\operatorname{dist}(g_{\tau_0}(z_0), S_{K_{\tau_0}})^{2-d}}{|g'_{\tau_0}(z_0)|^{2-d}|z_0|^\alpha y_0^{(2-d)-\alpha}} \asymp \frac{\operatorname{dist}(z_0, K_{\tau_0})^{2-d}}{|z_0|^\alpha y_0^{(2-d)-\alpha}} \asymp \frac{r^{2-d}}{|z_0|^\alpha y_0^{(2-d)-\alpha}} = F(z_0; r).$$

In the boundary case, by Koebe's distortion theorem, we get $R \asymp \frac{|Z_{\tau_0}(z_0)|^\alpha}{|g'_{\tau_0}(z_0)|^\alpha |z_0|^\alpha}$. Suppose $E^2_{\tau_0}$ happens. Then $|Z_{\tau_0}(z_0)| \asymp \operatorname{dist}(g_{\tau_0}(z_0), S_{K_{\tau_0}})$. By Koebe's 1/4 theorem we get

$$R \asymp \frac{\operatorname{dist}(g_{\tau_0}(z_0), S_{K_{\tau_0}})^\alpha}{|g'_{\tau_0}(z_0)|^\alpha |z_0|^\alpha} \asymp \frac{\operatorname{dist}(z_0, K_{\tau_0})^\alpha}{|z_0|^\alpha} \asymp \frac{r^\alpha}{|z_0|^\alpha} = F(z_0; r).$$

So we get $R \asymp F(z_0; r)$ on $E^2_{\tau_0}$ in both cases. The proof is now complete. □

Remark. Note that we expect that $F(z_0; r)$ is comparable to the probability that SLE goes to distance r of z_0. So We showed that there is a good chance to go to distance r of z_0 in a "good way ". Once we have this we can prove Theorem 4.3.

Proof of Theorem 4.3. We prove the theorem by induction on n. For $n = 1$ it is a corollary of Proposition 6.2. Suppose that $n \ge 2$ and the theorem is true for $1, \dots, n-1$ and we want to prove it for n. We consider different cases.

Case A: There exist $R \ge 2(\max_{1 \le j \le n-1} R_j) r > 0$ and $1 \le m \le n-1$ such that $|z_j| < r$, $1 \le j \le m$, and $|z_j| > R$, $m+1 \le j \le n$. Let $\tau_0 = \vee_{j=1}^m \tau^{z_j}_{r_j}$ and $r' = R/2$. From the induction hypothesis, we have $\mathbb{P}[\tau_0 < \tau_{\{|z|=r'\}}] \gtrsim F(z_1, \dots, z_m; r_1, \dots, r_m)$. Let E_{τ_0} denote the event $\tau_0 < \tau_{\{|z|=r'\}}$. Let $\widetilde{\gamma}(t) = Z_{\tau_0}(\gamma(\tau_0 + t))$, $\widetilde{z}_j = Z_{\tau_0}(z_j)$, and $\widetilde{r}_j = |Z'_{\tau_0}(z_j)| r_j / 4$, $m+1 \le j \le n$. By DMP of SLE, conditionally on $\mathcal{F}_{\tau_0}$, $\widetilde{\gamma}$ has the same law as γ. Let $\widetilde{\tau}_S$ and $\widetilde{\tau}^z_r$ be the stopping times that correspond to $\widetilde{\gamma}$. By induction hypothesis, we have

$$\mathbb{P}[\widetilde{\tau}^{\widetilde{z}_j}_{\widetilde{r}_j} < \widetilde{\tau}_{\{|z|=R_{n-m}\sum_{j=m+1}^n |\widetilde{z}_j|\}}, m+1 \le j \le n | \mathcal{F}_{\tau_0}, E_{\tau_0}] \gtrsim F(\widetilde{z}_{m+1}, \dots, \widetilde{z}_n; \widetilde{r}_{m+1}, \dots, \widetilde{r}_n).$$

Suppose E_{τ_0} happens. Then $K_{\tau_0} \subset \{|z| \le r'\}$. By Lemma 2.5 and that $U_{\tau_0} \in S_{K_{\tau_0}}$ we have $|Z_{\tau_0}(z) - z| \le 5r'$ for any $z \notin \overline{K_{\tau_0}}$. Let $\widetilde{E}$ denote the event on the LHS of the above displayed formula. By Koebe's 1/4 theorem, we see that $E_{\tau_0}\cap\widetilde{E}\subset\bigcap_{j=1}^n\{\tau^{z_j}_{r_j}<\tau_{\{|z|=r''\}}\}$, where $r''=6r'+R_{n-m}\sum_{j=m+1}^n(|z_j|+5r')$. Since $r'\le R\le|z_n|$, we can find a constant R_n such that $r''\le R_n\sum_{j=1}^n|z_j|$. Thus,

$$\begin{aligned}
&\mathbb{P}[\tau^{z_j}_{r_j}<\tau_{\{|z|=R_n\sum_{j=1}^n|z_j|\}}]\ge\mathbb{P}[E_{\tau_0}\cap\widetilde{E}]=\mathbb{E}[E_{\tau_0}]\cdot\mathbb{E}[\mathbb{P}[\widetilde{E}|\mathcal{F}_{\tau_0},E_{\tau_0}]]\\
\gtrsim&F(z_1,\dots,z_m;r_1,\dots,r_m)\cdot\mathbb{E}[F(\widetilde{z}_{m+1},\dots,\widetilde{z}_n;\widetilde{r}_{m+1},\dots,\widetilde{r}_n)|\mathcal{F}_{\tau_0},E_{\tau_0}].\\
\gtrsim&F(z_1,\dots,z_m;r_1,\dots,r_m)\cdot F(z_{m+1},\dots,z_n;r_{m+1},\dots,r_n)\\
\asymp&F(z_1,\dots,z_n;r_1,\dots,r_n).
\end{aligned}$$

where the second last estimate follows from the remark after Lemma 6.1, and the last estimate follows from Lemma 3.6 because $\operatorname{dist}(z_j,\{z_1,\dots,z_m\})\asymp|z_j|$, $m+1\le j\le n$. The proof of Case A is now complete.

We will reduce other cases to Case A or the case of fewer points. By (2.7) we may assume that z_1 has the smallest norm among z_j, $1\le j\le n$. Fix constants $\rho_1>\cdots>\rho_n\in(0,1/2)$ to be determined later.

Case B: $\{z_1,\dots,z_n\}\setminus L_{z_1,\rho_1}\ne\emptyset$. By pigeonhole principle, Case B is a union of subcases: Case B.k, $1\le k\le n-1$, where Case B.k denotes the case that Case B happens and $\{z_1,\dots,z_n\}\cap(L_{z_1,\rho_k}\setminus L_{z_1,\rho_{k+1}})=\emptyset$.

Case B.k: In this case we have $\{z_1,\dots,z_n\}\setminus L_{z_1,\rho_k}\ne\emptyset$, $\{z_1,\dots,z_n\}\cap(L_{z_1,\rho_k}\setminus L_{z_1,\rho_{k+1}})=\emptyset$, and $\{z_1,\dots,z_n\}\cap L_{z_1,\rho_{k+1}}\ne\emptyset$ because $z_1\in L_{z_1,\rho_{k+1}}$. By (2.7) we may assume that $z_1,\dots,z_m\in L_{z_1,\rho_{k+1}}$ and $z_{m+1},\dots,z_n\notin L_{z_1,\rho_k}$, where $1\le m\le n-1$.

We will apply Proposition 6.2. Let N,b_1,C_1 be the constants there. Let $\delta=\frac{2N}{b_1}\sqrt{2\rho_{k+1}}$, and $r=\frac{\delta|z_1|}{N}$. Let $\tau_0=\tau_0^\delta(z_1,r)$ and E_{τ_0} be given by Proposition 6.2. For $1\le j\le m$, since $z_j\in L_{z_1,\rho_{k+1}}$ and $|z_j|\ge|z_1|$, by (6.1), we have $|z_j-z_1|\le\sqrt{2\rho_{k+1}}|z_1|\le\frac{b_1r}{2}$. Suppose E_{τ_0} happens. By Koebe's 1/4 theorem, we have

$$|g'_{\tau_0}(z_1)|b_1r\le|g'_{\tau_0}(z_1)|\operatorname{dist}(z_1,K_{\tau_0})\le4\operatorname{dist}(g_{\tau_0}(z_1),S_{K_{\tau_0}})\le4|Z_{\tau_0}(z_1)|\le4C_1\sqrt{r|z_1|}.$$

For $1\le j\le m$, since $\operatorname{dist}(z_1,K_{\tau_0})\ge b_1r\ge2|z_j-z_1|$, by Koebe's distortion theorem, we have

$$|Z_{\tau_0}(z_j)-Z_{\tau_0}(z_1)|\le2|g'_{\tau_0}(z_1)||z_j-z_1|\le|g'_{\tau_0}(z_1)|b_1r\le4C_1\sqrt{r|z_1|}.$$

Since $|Z_{\tau_0}(z_1)|\le C_1\sqrt{r|z_1|}$, we get

$$|Z_{\tau_0}(z_j)|\le5C_1\sqrt{r|z_1|},\quad 1\le j\le m.$$

Suppose that

$$\delta\le\rho_k/2. \tag{6.5}$$

Since $K_{\tau_0}\subset L_{z_1,\delta}$, and $z_j\notin L_{z_1,\rho_k}$, $m+1\le j\le n$, by Lemma 6.1, we see that $|g'_{\tau_0}(z_j)|\ge C_{\rho_k}$, where $C_{\rho_k}>0$ depends only on κ and ρ_k. By Koebe's 1/4 theorem, we get

$$|Z_{\tau_0}(z_j)|\ge\operatorname{dist}(g_{\tau_0}(z_j),S_{K_{\tau_0}})\ge|g'_{\tau_0}(z_j)|\operatorname{dist}(z_j,K_{\tau_0})/4\ge C_{\rho_k}\rho_k|z_1|/8,\quad m+1\le j\le n.$$

Suppose now that

$$C_{\rho_k}\rho_k|z_1|/8 \geq 2(\max_{1\leq j\leq n-1} R_j)5C_1\sqrt{r|z_1|}. \tag{6.6}$$

Then we see that $Z_{\tau_0}(z_1),\dots,Z_{\tau_0}(z_n)$ satisfy the condition in Case A.

We will apply Lemma 3.5 with $K = K_{\tau_0}$ and $U_0 = U_{\tau_0}$. Let $I = \{1\} \cup \{1 \leq j \leq n : r_j \leq \operatorname{dist}(z_j, K_{\tau_0})\}$. We check the conditions of that lemma when E_{τ_0} happens. By the definition of I, we have $r_j \leq \operatorname{dist}(z_j, K_{\tau_0})$ for $j \in I \setminus \{1\}$. For $j = 1$, since $\operatorname{dist}(z_1, K_{\tau_0}) \geq b_1 r \gtrsim |z_1|$ and $r_1 \leq d_1 \leq |z_1|$, we have $r_1 \lesssim \operatorname{dist}(z_1, K_{\tau_0})$. We have to check Condition (3.7). First, (3.7) holds for $j = 1$ by Property (iii) of E_{τ_0}. Second, for $2 \leq j \leq m$, since $|z_j - z_1| \leq \frac{1}{2}\operatorname{dist}(z_1, K_{\tau_0})$, by Koebe's 1/4 theorem and distortion theorem, (3.7) also holds for these j. Third, for $m+1 \leq j \leq n$, by Lemma 6.1 and Koebe's 1/4 theorem, we have $\operatorname{dist}(g_{\tau_0}(z_j), S_{K_{\tau_0}}) \gtrsim \operatorname{dist}(z_j, L_{z_1,\delta})$. On the other hand, since $K_{\tau_0} \subset L_{z_1,\delta} \subset \{|z| \leq r'\}$, where $r' := 2|z_1|$, we have $|Z_{\tau_0}(z) - z| \leq 5r' = 10|z_1|$ for any $z \in \overline{\mathbb{H}} \setminus \overline{K_{\tau_0}}$ by Lemma 2.5. Thus, $|Z_{\tau_0}(z_j)| \lesssim |z_j|$. Since $\rho_k \geq 2\delta$, it is clear that $|z| \lesssim \operatorname{dist}(z, L_{z_1,\delta})$ for any $z \in \overline{\mathbb{H}} \setminus L_{z,\rho_k}$. So we see that (3.7) also holds for $m+1 \leq j \leq n$.

Let $\widetilde{\gamma}$, $\widetilde{z}_j$, $\widetilde{r}_j$, $\widetilde{\tau}_S$ and $\widetilde{\tau}^z_r$ be as defined in Case A. Then $\widetilde{z}_j = Z_{\tau_0}(z_j)$, $1 \leq j \leq n$, satisfy the condition in Case A. By the result of Case A (if $|I| = n$) or the induction hypothesis (if $|I| < n$), we see that

$$\mathbb{P}[\widetilde{\tau}^{\widetilde{z}_j}_{\widetilde{r}_j} < \widetilde{\tau}_{\{|z|=R\sum_{j\in I}|\widetilde{z}_j|\}}, j \in I|\mathcal{F}_{\tau_0}, E_{\tau_0}] \gtrsim F(\widetilde{z}_{j_1},\dots,\widetilde{z}_{j_{|I|}};\widetilde{r}_{j_1},\dots,\widetilde{r}_{j_{|I|}}),$$

where R is the maximum of R_j, $1 \leq j \leq n-1$, and the R_n in Case A. Let $\widetilde{E}$ denote the event on the LHS of the above displayed formula. Since $|\widetilde{z}_j - z_j| \leq 5r'$, by Koebe's 1/4 theorem, we see that $E_{\tau_0} \cap \widetilde{E} \subset \bigcap_{j=1}^n \{\tau^{z_j}_{r_j} < \tau_{\{|z|=r''\}}\}$, where $r'' = 6r' + R\sum_{j\in I}(|z_j| + 5r') \leq R_n \sum_{j=1}^n |z_j|$ for some constant $R_n > 0$. Thus,

$$\begin{aligned}
&\mathbb{P}[\tau^{z_j}_{r_j} < \tau_{\{|z|=R_n\sum_{j=1}^n|z_j|\}}] \geq \mathbb{P}[E_{\tau_0} \cap \widetilde{E}] = \mathbb{E}[E_{\tau_0}] \cdot \mathbb{E}[\mathbb{P}[\widetilde{E}|\mathcal{F}_{\tau_0}, E_{\tau_0}]] \\
\gtrsim &F(z_1; r) \cdot \mathbb{E}[F(\widetilde{z}_{j_1},\dots,\widetilde{z}_{j_{|I|}};\widetilde{r}_{j_1},\dots,\widetilde{r}_{j_{|I|}})|\mathcal{F}_{\tau_0}, E_{\tau_0}] \gtrsim F(z_1,\dots,z_n; r_1,\dots,r_n),
\end{aligned}$$

where the last inequality follows from Lemma 3.5 and that $\operatorname{dist}(z_1, K_{\tau_0}) \leq b_2 r$. We remark that the implicit constant in the above estimate depends on ρ_k and ρ_{k+1}. This does not matter because ρ_k and ρ_{k+1} are constants once they are determined. Now we have finished the proof of Case B.k assuming Conditions (6.5,6.6).

Case C: $z_1,\dots,z_n \in L_{z_1,\rho_1}$. This case is the complement of Case B, and we will reduce it to Case B. Let

$$e_n = \max_{1\leq j\leq n} |z_j - z_1|.$$

From (6.1) we know that $e_n \leq \sqrt{2\rho_1}|z_1|$.

We apply Proposition 6.2 with $z_0 = z_1$, $\delta = \frac{4N}{b_1}\sqrt{\rho_1}$ and $r = \frac{2e_n}{b_1}$. Let $\tau = \tau^\delta_0(z_1, r)$ and E_{τ_0} given by that proposition. Suppose E_{τ_0} happens. By Properties (i,iii) and Koebe's 1/4 theorem, we have

$$|Z_{\tau_0}(z_1)| \leq \operatorname{dist}(g_{\tau_0}(z_1), S_{K_{\tau_0}})/C_0 \leq 4|g'_{\tau_0}(z_1)| \operatorname{dist}(z_1, K_{\tau_0})/C_0 \leq \frac{8b_2}{b_1 C_n}|g'_{\tau_0}(z_1)|e_n.$$

By Koebe's distortion theorem, we have

$$\max_{1\le j\le n} |Z_{\tau_0}(z_j) - Z_{\tau_0}(z_1)| \ge \frac{2}{9}|g'_{\tau_0}(z_1)|e_n.$$

Thus, if $Z_{\tau_0}(z_s)$ has the smallest norm among $Z_{\tau_0}(z_j)$, $1 \le j \le n$, then

$$\max_{1\le j\le n} |Z_{\tau_0}(z_j) - Z_{\tau_0}(z_s)| \ge \frac{b_1 C_n}{72 b_2}|Z_{\tau_0}(z_s)|.$$

If ρ_1 satisfies that

$$\sqrt{2\rho_1} < \frac{b_1 C_n}{72 b_2}, \tag{6.7}$$

then from (6.1) we see that not all $Z_{\tau_0}(z_j)$, $1 \le j \le n$, are contained in $L_{Z_{\tau_0}(z_s),\rho_1}$. After reordering the points, we see that $Z_{\tau_0}(z_j)$, $1 \le j \le n$, satisfy the condition in Case B.

We will apply Lemma 3.5 with $K = K_{\tau_0}$ and $U_0 = U_{\tau_0}$. Let $I = \{1, \dots, n\}$. We check the conditions of that lemma when E_{τ_0} happens. Since $r_1 \le |z_1 - z_1| \le e_n$ and $\operatorname{dist}(z_1, K_{\tau_0}) \ge 2e_1$, we have $r_1 < \operatorname{dist}(z_1, K_{\tau_0})$. For $2 \le j \le n$, since $r_j \le d_j \le |z_j - z_1| \le e_n$ and $\operatorname{dist}(z_1, K_{\tau_0}) \ge 2e_n$, we see that $r_j \le \operatorname{dist}(z_j, K_{\tau_0})$. So I satisfies the property there. We have to check Condition (3.7). First, (3.7) holds for $j = 1$ by Property (iii) of E_{τ_0}. Second, for $2 \le j \le n$, since $|z_j - z_1| \le \frac{1}{2}\operatorname{dist}(z_1, K_{\tau_0})$, by Koebe's 1/4 theorem and distortion theorem, (3.7) also holds for these j.

Let $\widetilde\gamma$, $\widetilde z_j$, $\widetilde r_j$, $\widetilde\tau_S$ and $\widetilde\tau^z_r$ be as defined in Case A. By the result of Case B we see that

$$\mathbb{P}[\widetilde\tau^{\widetilde z_j}_{\widetilde r_j} < \widetilde\tau_{\{|z|=R\sum_{1\le j\le n}|\widetilde z_j|\}}, 1 \le j \le n | \mathcal{F}_{\tau_0}, E_{\tau_0}] \gtrsim F(\widetilde z_1, \dots, \widetilde z_n; \widetilde r_1, \dots, \widetilde r_n),$$

where R is the R_n in Case B. Let $r' = 2|z_1|$. Then $K_{\tau_0} \subset \{|z| \le r'\}$. So $|Z_{\tau_0}(z) - z| \le 5r'$ for $z \in \overline{\mathbb{H}} \setminus \overline{K_{\tau_0}}$. Let $\widetilde E$ denote the event on the LHS of the above displayed formula. By Koebe's 1/4 theorem, we see that $E_{\tau_0} \cap \widetilde E \subset \bigcap_{j=1}^n \{\tau^{z_j}_{r_j} < \tau_{\{|z|=r''\}}\}$, where $r'' = 6r' + R\sum_{j=1}^n (|z_j| + 5r') \le R_n \sum_{j=1}^n |z_j|$ for some constant $R_n > 0$. Thus,

$$\begin{aligned}\mathbb{P}[\tau^{z_j}_{r_j} < \tau_{\{|z|=R_n\sum_{j=1}^n |z_j|\}}] &\ge \mathbb{P}[E_{\tau_0} \cap \widetilde E] = \mathbb{E}[E_{\tau_0}] \cdot \mathbb{E}[\mathbb{P}[\widetilde E | \mathcal{F}_{\tau_0}, E_{\tau_0}]] \\ &\gtrsim F(z_1; r) \cdot \mathbb{E}[F(\widetilde z_1, \dots, \widetilde z_n; \widetilde r_1, \dots, \widetilde r_n) | \mathcal{F}_{\tau_0}, E_{\tau_0}] \gtrsim F(z_1, \dots, z_n; r_1, \dots, r_n),\end{aligned}$$

where the last inequality follows from Lemma 3.5 and that $\operatorname{dist}(z_1, K_{\tau_0}) \le b_2 r$. Now we have finished the proof of Case B.k assuming Condition (6.7).

In the end, we need to find $\rho_1, \dots, \rho_n$ such that Conditions (6.5,6.6,6.7) all hold. To do this, we may first use (6.7) to choose ρ_1. Once ρ_k is chosen, we may use (6.5,6.6) to choose ρ_{k+1} because these two inequalities are satisfied when ρ_{k+1} is sufficiently small given ρ_k. $\square$

Appendices

A Proof of Theorem 3.1

In order to prove Theorem 3.1, we need some lemmas. The proof of Theorem 3.1 will be given after the proof of Lemma A.4. We still let γ be a chordal SLE$_\kappa$ curve in $\mathbb{H}$ from 0 to ∞. Throughout the appendix, we use C (without subscript) to denote a positive constant depending only on κ, and use C_x to denote a positive constant depending only on κ and some variable x. The value of a constant may vary between occurrences.

First, let's recall the one-point estimate and the boundary estimate for chordal SLE$_\kappa$. (see Lemma 2.6 and Lemma 2.5 in [18, Lemma 2.6, Lemma 2.5]).

Lemma A.1 (One-point Estimate)**.** *Let T be a stopping time for γ. Let $z_0 \in \overline{\mathbb{H}}$, $y_0 = \operatorname{Im} z_0 \geq 0$, and $R \geq r > 0$. Then*

$$\mathbb{P}[\tau_r^{z_0} < \infty | \mathcal{F}_T, \operatorname{dist}(z_0, K_T) \geq R] \leq C \frac{P_{y_0}(r)}{P_{y_0}(R)}.$$

Lemma A.2 (Boundary Estimate)**.** *Let T be a stopping time. Let ξ_1 and ξ_2 be a disjoint pair of crosscuts of H_T such that*

1. *either ξ_1 disconnects $\gamma(T)$ from ξ_2 in H_T, or $\gamma(T)$ is an end point of ξ_1;*
2. *among the three bounded components of $H_T \setminus (\xi_1 \cup \xi_2)$, the boundary of the unbounded component does not contain ξ_2.*

Then

$$\mathbb{P}[\tau_{\xi_2} < \infty | \mathcal{F}_T] \leq C e^{-\alpha \pi d_{H_T}(\xi_1, \xi_2)}.$$

The lemma below is similar to and stronger than [18, Theorem 3.1]. The symbols z_j, R_j, r_j in this lemma are not related with the symbols with the same names in Theorem 3.1 or main theorems of this paper.

Lemma A.3. *Let $m \in \mathbb{N}$, $z_j \in \overline{\mathbb{H}}$, $y_j = \operatorname{Im} z_j$, and $|z_j| > R_j \geq r_j > 0$, $1 \leq j \leq m$. Let $D_j = \{|z - z_j| < r_j\}$ and $\widehat{D}_j = \{|z - z_j| < R_j\}$, $1 \leq j \leq m$. Let $\widehat{J}_0, J_0, J_0'$ be three mutually disjoint Jordan curves in $\mathbb{C}$, which bound Jordan domains $\widehat{D}_0, D_0, D_0'$, respectively. Suppose that $D_0' \subset D_0 \subset \widehat{D}_0$, and $0 \notin \overline{D_0}$. Let $A = \widehat{D}_0 \setminus \overline{D_0}$ be the doubly connected domain bounded by J_0 and $\widehat{J}_0$. Suppose that $A \cap \widehat{D}_j = \emptyset$, $1 \leq j \leq m$, and there is some $n_0 \in \{1, \ldots, m\}$ such that $\widehat{D}_0 \cap \widehat{D}_{n_0} = \emptyset$. Let $\xi_j = \partial D_j \cap \mathbb{H}$, $\widehat{\xi}_j = \partial \widehat{D}_j \cap \mathbb{H}$, $0 \leq j \leq m$, and $\xi_0' = \partial D_0' \cap \mathbb{H}$. Let*

$$E = \{\tau_{\xi_0} < \tau_{\widehat{\xi}_1} \leq \tau_{\xi_1} < \cdots < \tau_{\widehat{\xi}_m} \leq \tau_{\xi_m} < \tau_{\xi_0'} < \infty\}.$$

Then

$$\mathbb{P}[E | \mathcal{F}_{\tau_{\xi_0}}] \leq C^m e^{-\alpha \pi d_{\mathbb{C}}(J_0, \widehat{J}_0)/2} \prod_{j=1}^m \frac{P_{y_j}(r_j)}{P_{y_j}(R_j)}.$$

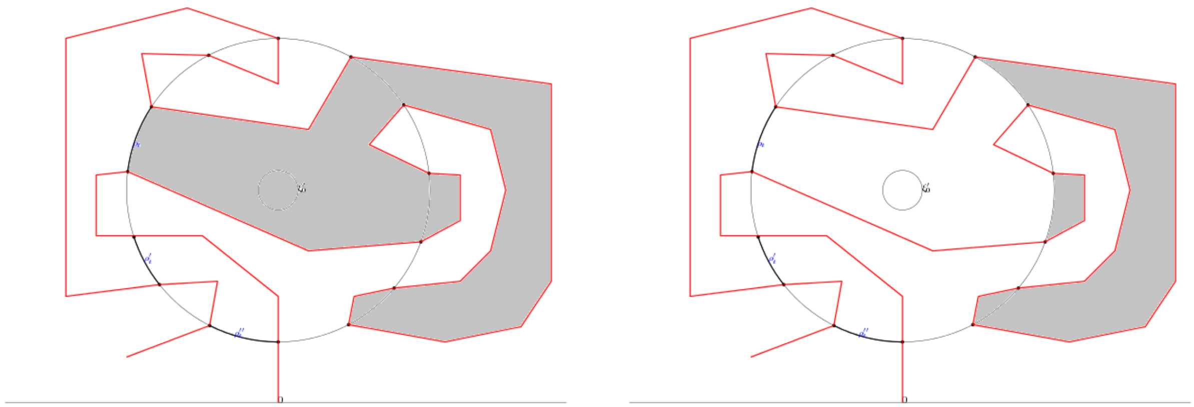

Figure 1: The two pictures above illustrate $\widehat{U}_t^\rho$ and U_t^ρ, respectively. The red curve is γ up to time t, the big circle is ρ, and the small circle is ξ_0'. The connected components of $\rho\cap H_t$ that disconnects ξ_0' from ∞ are labeled as ρ_t,ρ_t',ρ_t'', where ρ_t is closest to ξ_0' in H_t. The grey region on the left picture is $\widehat{U}_t^\rho$; and the grey region on the right picture is U_t^ρ.

Proof. We write $\tau_0=\tau_{\xi_0}$, $\widehat{\tau}_j=\tau_{\widehat{\xi}_j}$ and $\tau_j=\tau_{\xi_j}$, $1\le j\le m$, and $\tau_{m+1}=\tau_{\xi_0'}$.

From the one-point estimate, we have

$$\mathbb{P}[\tau_j<\infty|\mathcal{F}_{\widehat{\tau}_j}]\le C\frac{P_{y_j}(r_j)}{P_{y_j}(R_j)},\quad 1\le j\le m. \tag{A.1}$$

Thus, $\mathbb{P}[E|\mathcal{F}_{\tau_0}]\le C^m\prod_{j=1}^m\frac{P_{y_j}(r_j)}{P_{y_j}(R_j)}$. Now we need to derive the factor $e^{-\alpha\pi d_{\mathbb{C}}(J_0,\widehat{J}_0)/2}$.

By mapping A conformally onto an annulus, we see that there is a Jordan curve ρ in A that disconnects J_0 from $\widehat{J}_0$, such that

$$d_{\mathbb{C}}(\rho,J_0)=d_{\mathbb{C}}(\rho,\widehat{J}_0)=d_{\mathbb{C}}(J,\widehat{J}_0)/2. \tag{A.2}$$

Let $T=\inf\{t\ge 0:\xi_0'\not\subset H_t\}$. Let $t\in[\tau_0,T)$. Each connected component η of $\rho\cap H_t$ is a crosscut of H_t, and $H_t\setminus\eta$ is the disjoint union of a bounded domain and an unbounded domain. We use $H_t^*(\eta)$ to denote the bounded domain. First, consider the connected components η of $\rho\cap H_t$ such that $\xi_0'\subset H_t^*(\eta)$. If such η is unique, we denote it by ρ_t. Otherwise, applying [18, Lemma 2.1], we may find the unique component η_0, such that $H_t^*(\eta_0)$ is the smallest among all of these $H_t^*(\eta)$. Again we use ρ_t to denote this η_0. Let $\widehat{U}_t^\rho=H_t^*(\rho_t)$. Then $\xi_0'\subset\widehat{U}_t^\rho$. Next, consider the connected components η of $\rho\cap H_t$ such that $H_t^*(\eta)\subset\widehat{U}_t^\rho\setminus\xi_0'$. Let the union of $H_t^*(\eta)$ for these η be denoted by U_t^ρ. Then we have $U_t^\rho\subset\widehat{U}_t^\rho$ and $U_t^\rho\cap\xi_0'=\emptyset$.

Now we define a family of events.

- Let $A_{(0,1)}$ be the event that $\tau_0<\widehat{\tau}_1\wedge T$ and $D_1\cap\mathbb{H}\subset U_{\tau_0}^\rho$.
- For $1\le j\le n_0-1$, let $A_{(j,j)}$ be the event that $\tau_{j-1}<\tau_j<T$, and $D_j\cap\mathbb{H}\not\subset U_{\tau_{j-1}}^\rho$, but $D_j\cap\mathbb{H}\subset U_{\tau_j}^\rho$.

- For $1 \le j \le n_0 - 1$, let $A_{(j,j+1)}$ be the event that $\tau_j < \widehat{\tau}_{j+1} \wedge T$, and $D_j \cap \mathbb{H} \not\subset U^\rho_{\tau_j}$, but $D_{j+1} \cap \mathbb{H} \subset U^\rho_{\tau_j}$.

- For $n_0 \le j \le m$, let $A_{(j,j)}$ be the event that $\tau_{j-1} < \tau_j < T$, and $D_j \cap \mathbb{H} \not\subset \widehat{U}^\rho_{\tau_{j-1}}$, but $D_j \cap \mathbb{H} \subset \widehat{U}^\rho_{\tau_j}$.

- For $n_0 \le j \le m-1$, let $A_{(j,j+1)}$ be the event that $\tau_j < \widehat{\tau}_{j+1} \wedge T$, and $D_j \cap \mathbb{H} \not\subset \widehat{U}^\rho_{\tau_j}$, but $D_{j+1} \cap \mathbb{H} \subset \widehat{U}^\rho_{\tau_j}$.

- Let $A_{(m,m+1)}$ be the event that $\tau_m < \tau_{m+1} \wedge T$ and $D_m \cap \mathbb{H} \not\subset \widehat{U}^\rho_{\tau_m}$.

Let $I = \{(j,j+1) : 0 \le j \le m\} \cup \{(j,j) : 1 \le j \le m\}$. We claim that $E \subset \bigcup_{\iota \in I} A_\iota$. To see this, note that, if none of the events $A_{(j,j+1)}$, $0 \le j \le n_0 - 1$, and $A_{(j,j)}$, $1 \le j \le n_0 - 1$, happens, then $D_{n_0} \cap \mathbb{H} \not\subset U^\rho_{\tau_{n_0}}$. Since D_{n_0} is disjoint from $\widehat{D}_0$, we can conclude that $D_{n_0} \cap \mathbb{H} \not\subset \widehat{U}^\rho_{\tau_{n_0}}$. In fact, if $D_{n_0} \cap \mathbb{H} \subset \widehat{U}^\rho_{\tau_{n_0}}$, then from $D_{n_0} \cap \widehat{D}_0 = \emptyset$, $\rho \subset \widehat{D}_0$, and ρ surrounds ξ_0', we may find a connected component η of $\rho \cap H_{\tau_{n_0}}$ that disconnects $D_{n_0} \cap \mathbb{H}$ from ξ_0' in $H_{\tau_{n_0}}$. Since $D_{n_0} \cap \mathbb{H}, \xi_0' \subset \widehat{U}^\rho_{\tau_{n_0}}$, we have $\eta \subset \widehat{U}^\rho_{\tau_{n_0}}$. From the definitions of ρ_{n_0} and $\widehat{U}^\rho_{n_0}$, we see that η does not disconnect ξ_0' from ∞ in $H_{\tau_{n_0}}$. Thus, $D_{n_0} \cap \mathbb{H} \subset H^*_{\tau_{n_0}}(\eta) \subset \widehat{U}^\rho_{\tau_{n_0}}$, and $\xi_0' \cap H^*_{\tau_{n_0}}(\eta) = \emptyset$. This shows that $D_{n_0} \cap \mathbb{H} \subset U^\rho_{\tau_{n_0}}$, which is a contradiction. Since $D_{n_0} \cap \mathbb{H} \not\subset \widehat{U}^\rho_{\tau_{n_0}}$, one of the events $A_{(j,j)}$ and $A_{(j,j+1)}$, $n_0 \le j \le m$, must happen. So the claim is proved. We will finish the proof by showing that

$$\mathbb{P}[E \cap A_\iota | \mathcal{F}_{\tau_0}] \le C^m e^{-\alpha\pi d_{\mathbb{C}}(J_0, \widehat{J}_0)/2} \prod_{j=1}^m \frac{P_{y_j}(r_j)}{P_{y_j}(R_j)}, \quad \iota \in I. \tag{A.3}$$

Case 1. Suppose $A_{(0,1)}$ occurs. Then at time τ_0, there is a connected component, denoted by $\widetilde{\rho}_{\tau_0}$, of $\rho \cap H_{\tau_0}$, that disconnects $\widehat{\xi}_1$ from both ξ_0' and ∞ in H_{τ_0}. Since $\xi_0' \subset D_0 \cap \mathbb{H} \subset H_{\tau_0}$ and $\gamma(\tau_0) \in \partial D_0$, we see that $\widetilde{\rho}_{\tau_0}$ disconnects $\widehat{\xi}_1$ also from $\gamma(\tau_0)$ in H_{τ_0}. Since $\widehat{\xi}_1$ is disjoint from A, it is contained in either D_0 or $\mathbb{C} \setminus \widehat{D}_0$. If $\widehat{\xi}_1$ is contained in D_0 (resp. $\mathbb{C} \setminus \widehat{D}_0$), then $J_0 \cap H_{\tau_0}$ (resp. $\widehat{J}_0 \cap H_{\tau_0}$) contains a connected component, denoted by η_{τ_0}, which disconnects $\widehat{\xi}_1$ from $\widetilde{\rho}_{\tau_0}$ and ∞ in H_{τ_0}. Using the boundary estimate and (A.2), we get

$$\mathbb{P}[\widehat{\tau}_1 < \infty | \mathcal{F}_{\tau_0}, A_{(0,1)}] \le C e^{-\alpha\pi d_{H_{\tau_0}}(\widetilde{\rho}_{\tau_0}, \eta_{\tau_0})} \le C e^{-\alpha\pi d_{\mathbb{C}}(J_0, \widehat{J}_0)/2},$$

which together with (A.1) implies that (A.3) holds for $\iota = (0,1)$.

Case 2. Suppose for some $1 \le j \le n_0 - 1$, $A_{(j,j+1)}$ occurs. Then at time τ_j, there is a connected component, denoted by $\widetilde{\rho}_{\tau_j}$, of $\rho \cap H_{\tau_j}$, that disconnects $\widehat{\xi}_{j+1}$ from both ξ_j and ∞ in H_{τ_j}. Since $\gamma(\tau_j) \in \xi_j$, we see that $\widetilde{\rho}_{\tau_j}$ disconnects $\widehat{\xi}_{j+1}$ also from $\gamma(\tau_j)$ in H_{τ_j}. According to whether ξ_{j+1} belongs to D_0 or $\mathbb{C} \setminus \widehat{D}_0$, we may find a connected component η_{τ_j} of $J_0 \cap H_{\tau_0}$

or $\widehat{J}_0 \cap H_{\tau_0}$ that disconnects $\widehat{\xi}_{j+1}$ from $\widetilde{\rho}_{\tau_j}$ and ∞ in H_{τ_j}. Using the boundary estimate and (A.2), we get

$$\mathbb{P}[\widehat{\tau}_{j+1} < \infty | \mathcal{F}_{\tau_j}, A_{(j,j+1)}, \tau_j < \widehat{\tau}_{j+1}] \le C e^{-\alpha\pi d_{H_{\tau_j}}(\widetilde{\rho}_{\tau_j}, \eta_{\tau_j})} \le C e^{-\alpha\pi d_{\mathbb{C}}(J_0, \widehat{J}_0)/2},$$

which together with (A.1) implies that (A.3) holds for $\iota = (j, j+1)$, $1 \le j \le n_0 - 1$.
Case 3. Suppose for some $n_0 \le j \le m$, $A_{(j,j+1)}$ occurs. We write $\xi_{m+1} = \xi_0'$. Then ρ_{τ_j} disconnects $\widehat{\xi}_{j+1}$ from $\gamma(\tau_j)$ and ∞ in H_{τ_j}. According to whether ξ_{j+1} belongs to D_0 or $\mathbb{C} \setminus \widehat{D}_0$, we may find a connected component η_{τ_j} of $J_0 \cap H_{\tau_0}$ or $\widehat{J}_0 \cap H_{\tau_0}$ that disconnects $\widehat{\xi}_{j+1}$ from ρ_{τ_j} and ∞ in H_{τ_j}. Using the boundary estimate and (A.2), we get

$$\mathbb{P}[\widehat{\tau}_{j+1} < \infty | \mathcal{F}_{\tau_j}, A_{(j,j+1)}, \tau_j < \widehat{\tau}_{j+1}] \le C e^{-\alpha\pi d_{H_{\tau_j}}(\rho_{\tau_j}, \eta_{\tau_j})} \le C e^{-\alpha\pi d_{\mathbb{C}}(J_0, \widehat{J}_0)/2},$$

which together with (A.1) implies that (A.3) holds for $\iota = (j, j+1)$, $n_0 \le j \le m$.
Case 4. Suppose for some $n_0 \le j \le m-1$, $A_{(j,j)}$ occurs. Define a stopping time

$$\sigma_j = \inf\{t \ge \tau_{j-1} : D_j \cap \mathbb{H} \subset \widehat{U}_t^\rho\}.$$

Then $\tau_{j-1} \le \sigma_j \le \tau_j$. From [18, Lemma 2.2], we know that

- $\gamma(\sigma_j)$ is an endpoint of ρ_{σ_j};
- $D_j \cap \mathbb{H} \subset \widehat{U}_{\sigma_j}^\rho$.

The second property implies that $\tau_{j-1} < \sigma_j < \tau_j$. Now we define two events. Let $F_< = \{\sigma_j < \widehat{\tau}_j\}$ and $F_\ge = \{\widehat{\tau}_j \le \sigma_j < \tau_j\}$. Then $A_{(j,j)} \subset F_< \cup F_\ge$.
Case 4.1. Suppose $F_\ge$ occurs. Let $N = \lceil \log(R_j/r_j) \rceil \in \mathbb{N}$. Let $\zeta_k = \{z \in \mathbb{H} : |z - z_j| = (R_j^{N-k} r_j^k)^{1/N}\}$, $0 \le k \le N$. Note that $\zeta_0 = \widehat{\xi}_j$ and $\zeta_N = \xi_j$. Then $F_\ge \subset \bigcup_{k=1}^N F_k$, where

$$F_k := \{\tau_{\zeta_{k-1}} \le \sigma_j < \tau_{\zeta_k} < \infty\}, \quad 1 \le k \le N.$$

If F_k occurs, then $\zeta_k \subset \widehat{U}_{\sigma_j}^\rho$. Since $\zeta_{k-1} \cap H_{\sigma_j}$ has a connected component $\zeta_{k-1}^{\sigma_j}$, which disconnects ζ_k from ρ_{σ_j} in H_{σ_j}, by the boundary estimate, we get

$$\mathbb{P}[\tau_{\zeta_k} < \infty | \mathcal{F}_{\sigma_j}, F_k] \le C e^{-\alpha\pi d_{H_{\sigma_j}}(\rho_{\sigma_j}, \zeta_{k-1}^{\sigma_j})}.$$

According to whether ζ_k belongs to D_0 or $\widehat{D}_0$, we may find a connected component η_{σ_j} of $J_0 \cap H_{\sigma_j}$ or $\widehat{J}_0 \cap H_{\sigma_j}$ that disconnects $\zeta_{k-1}^{\sigma_j}$ from ρ_{σ_j} and ∞ in H_{σ_j}. Moreover, we may find a connected component $\zeta_0^{\sigma_j}$ of $\zeta_0 \cap H_{\sigma_j}$ that disconnects η_{σ_j} from $\zeta_{k-1}^{\sigma_j}$. From (A.2) we get

$$d_{H_{\sigma_j}}(\rho_{\sigma_j}, \zeta_{k-1}^{\sigma_j}) \ge d_{H_{\sigma_j}}(\rho_{\sigma_j}, \eta_{\sigma_j}) + d_{H_{\sigma_j}}(\zeta_0^{\sigma_j}, \zeta_{k-1}^{\sigma_j}) \ge \frac{1}{2} d_{\mathbb{C}}(J_0, \widehat{J}_0) + \frac{k-1}{2\pi N} \log\Big(\frac{R_j}{r_j}\Big)$$

Thus, we get

$$\mathbb{P}[\tau_{\zeta_k} < \infty | \mathcal{F}_{\sigma_j}, F_k] \le C e^{-\alpha\pi d_{\mathbb{C}}(J_0, \widehat{J}_0)/2} \Big(\frac{r_j}{R_j}\Big)^{\frac{\alpha}{2}\frac{k-1}{N}}.$$

From the one-point estimate, we get

$$\mathbb{P}[F_k|\mathcal{F}_{\tau_{j-1}},\tau_{j-1}<\widehat{\tau}_j]\le C\frac{P_{y_j}((R_j^{N-k+1}r_j^{k-1})^{1/N})}{P_{y_j}(R_j)};$$

$$\mathbb{P}[\tau_j<\infty|\mathcal{F}_{\tau_{\zeta_k}},F_k]\le C\frac{P_{y_j}(r_j)}{P_{y_j}((R_j^{N-k}r_j^{k})^{1/N})}.$$

The above three displayed formulas together imply that

$$\mathbb{P}[\tau_j<\infty,F_k|\mathcal{F}_{\tau_{j-1}},\tau_{j-1}<\widehat{\tau}_j]\le Ce^{-\alpha\pi d_{\mathbb{C}}(J_0,\widehat{J}_0)/2}\Big(\frac{r_j}{R_j}\Big)^{\frac{\alpha}{2}\frac{k-1}{N}}\Big(\frac{r_j}{R_j}\Big)^{-\alpha/N}\frac{P_{y_j}(r_j)}{P_{y_j}(R_j)}.$$

Since $F_\ge\subset\bigcup_{k=1}^N F_k$, by summing up the above inequality over k, we get

$$\begin{aligned}\mathbb{P}[\tau_j<\infty,F_\ge|\mathcal{F}_{\tau_{j-1}},\tau_{j-1}<\widehat{\tau}_j]&\le Ce^{-\alpha\pi d_{\mathbb{C}}(J_0,\widehat{J}_0)/2}\frac{P_{y_j}(r_j)}{P_{y_j}(R_j)}\left[\Big(\frac{r_j}{R_j}\Big)^{-\alpha/N}\frac{1-(\frac{r_j}{R_j})^{\alpha/2}}{1-(\frac{r_j}{R_j})^{\alpha/(2N)}}\right]\\&\le Ce^{-\alpha\pi d_{\mathbb{C}}(J_0,\widehat{J}_0)/2}\frac{P_{y_j}(r_j)}{P_{y_j}(R_j)},\end{aligned}\tag{A.4}$$

where the second inequality holds because the quantity inside the square bracket is bounded above by $\frac{e^\alpha}{1-e^{-\alpha/4}}$. To see this, consider the cases $R_j/r_j\le e$ and $R_j/r_j>e$ separately.
Case 4.2. Suppose $F_<$ occurs. Then $\widehat{\xi}_j\subset\widehat{U}^\rho_{\sigma_j}$. According to whether $\widehat{\xi}_j$ belongs to D_0 or $\widehat{D}_0$, we may find a connected component η_{σ_j} of $J_0\cap H_{\sigma_j}$ or $\widehat{J}_0\cap H_{\sigma_j}$ that disconnects $\widehat{\xi}_j$ from ρ_{σ_j} and ∞ in H_{σ_j}. By the boundary estimate, we get

$$\mathbb{P}[\widehat{\tau}_j<\infty|\mathcal{F}_{\sigma_j},F_<]\le Ce^{-\alpha\pi d_{H_{\sigma_j}}(\rho_{\sigma_j},\eta_{\sigma_j})}\le Ce^{-\alpha\pi d_{\mathbb{C}}(J_0,\widehat{J}_0)/2},$$

which together with (A.1) implies that

$$\mathbb{P}[\tau_j<\infty,F_<|\mathcal{F}_{\tau_{j-1}}]\le Ce^{-\alpha\pi d_{\mathbb{C}}(J_0,\widehat{J}_0)/2}\frac{P_{y_j}(r_j)}{P_{y_j}(R_j)}.\tag{A.5}$$

Combining (A.4) and (A.5), we get

$$\mathbb{P}[\tau_j<\infty,A_{(j,j)}|\mathcal{F}_{\tau_{j-1}},\tau_{j-1}<\widehat{\tau}_j]\le Ce^{-\alpha\pi d_{\mathbb{C}}(J_0,\widehat{J}_0)/2}\frac{P_{y_j}(r_j)}{P_{y_j}(R_j)},$$

which together with (A.1) implies that (A.3) holds for $\iota=(j,j)$, $n_0\le j\le m$.
Case 5. Suppose for some $1\le j\le n_0-1$, $A_{(j,j)}$ occurs. Define a stopping time

$$\sigma_j=\inf\{t\ge\tau_{j-1}:D_j\cap\mathbb{H}\subset U^\rho_t\}.$$

To derive properties of σ_j, we claim that the following are true.

(i) If $D_j \cap \mathbb{H} \subset H_{t_0} \setminus U^\rho_{t_0}$, then there is $\varepsilon > 0$ such that $D_j \cap \mathbb{H} \subset H_t \setminus U^\rho_t$ for $t_0 \le t < t_0 + \varepsilon$;

(ii) If $D_j \cap \mathbb{H} \subset U^\rho_{t_0}$, and if $\gamma(t_0)$ is not an endpoint of a connected component of $\rho \cap H_{t_0}$ that disconnects $D_j \cap \mathbb{H}$ from ∞ in H_{t_0}, then there is $\varepsilon > 0$ such that $D_j \cap \mathbb{H} \subset U^\rho_t$ for $t_0 - \varepsilon < t \le t_0$.

To see that (i) holds, we consider two cases. Case 1. $D_j \cap \mathbb{H} \subset H_{t_0} \setminus \widehat{U}^\rho_{t_0}$. From [18, Lemma 2.2], there is $\varepsilon > 0$ such that for $t_0 \le t < t_0 + \varepsilon$, $D_j \cap \mathbb{H} \subset H_t \setminus \widehat{U}^\rho_t$, which implies that $D_j \cap \mathbb{H} \subset H_t \setminus U^\rho_t$. Case 2. $D_j \cap \mathbb{H} \subset \widehat{U}^\rho_{t_0} \setminus U^\rho_{t_0}$. Then there is a curve ζ in H_{t_0}, which connects ξ'_0 with D_j, and does not intersect ρ. In this case, there is $\varepsilon > 0$ such that for $t_0 \le t < t_0 + \varepsilon$, $\zeta \subset H_t$ and $D_j \cap \mathbb{H} \subset H_t$, which imply that $D_j \cap \mathbb{H} \subset H_t \setminus U^\rho_t$.

Now we consider (ii). Since $D_j \cap \mathbb{H} \subset U^\rho_{t_0}$, there is a connected component ζ of $\rho \cap H_{t_0}$, which is contained in $\widehat{U}^\rho_{t_0}$, and disconnects $D_j \cap \mathbb{H}$ from ξ'_0 and ∞ in H_{t_0}. From the assumption, $\gamma(t_0)$ is not an end point of ζ. By the continuity of γ, there is $\varepsilon_1 > 0$ such that $\gamma[t_0 - \varepsilon_1, t_0] \cap \overline{\zeta} = \emptyset$. This implies that, for $t_0 - \varepsilon_1 < t \le t_0$, ζ is also a crosscut of H_t. Since H_t is simply connected, ζ also disconnects $D_j \cap \mathbb{H}$ from ξ'_0 and ∞ in H_t. Since ρ_{t_0} is a connected component of $\rho \cap H_{t_0}$ that disconnects $\widehat{U}^\rho_{t_0} \supset U^\rho_{t_0} \supset D_j \cap \mathbb{H}$ from ∞, $\gamma(t_0)$ is also not an endpoint of ρ_{t_0}. Since $\zeta \subset \widehat{U}^\rho_{t_0}$, from [18, Lemma 2.2], there is $\varepsilon \in (0, \varepsilon_1)$ such that for $t_0 - \varepsilon < t \le t_0$, $\zeta \subset \widehat{U}^\rho_t$, which implies that $D_j \cap \mathbb{H} \subset U^\rho_t$.

From (i) and (ii) we conclude that

- $\gamma(\sigma_j)$ is an endpoint of a connected component of $\rho \cap H_{\sigma_j}$ that disconnects $D_j \cap \mathbb{H}$ from ∞ in H_{σ_j}. Let this crosscut be denoted by $\widetilde{\rho}_{\sigma_j}$.

- $D(z_j, r_j) \cap \mathbb{H} \subset U^\rho_{\sigma_j}$.

Following the proof in Case 4 with $\widetilde{\rho}_{\sigma_j}$ and $U^\rho_{\sigma_j}$ in place of ρ_{σ_j} and $\widehat{U}^\rho_{\sigma_j}$, respectively, we conclude that (A.3) holds for $\iota = (j, j)$, $1 \le j \le n_0 - 1$. The proof is now complete. $\square$

Let Ξ be a family of mutually disjoint circles with center in $\overline{\mathbb{H}}$, each of which does not pass through or enclose 0. Define a partial order on Ξ such that $\xi_1 < \xi_2$ if ξ_2 is enclosed by ξ_1. One should keep in mind that a smaller element in Ξ has bigger radius, but will be visited earlier (if it happens) by a curve started from 0.

Suppose that Ξ has a partition $\{\Xi_e\}_{e \in \mathcal{E}}$ with the following properties:

- For each $e \in \mathcal{E}$, the elements in Ξ_e are concentric circles with radii forming a geometric sequence with common ratio $1/4$. We denote the common center z_e, the biggest radius R_e, and the smallest radius r_e, and let $y_e = \operatorname{Im} z_e$.

- Let $A_e = \{r_e \le |z - z_0| \le R_e\}$ be the closed annulus associated with Ξ_e, which is a single circle if $R_e = r_e$, i.e., $|\Xi_e| = 1$. Then the annuli A_e, $e \in \mathcal{E}$, are mutually disjoint.

Note that every Ξ_e is a totally ordered set w.r.t. the partial order on Ξ.

Lemma A.4. *Suppose that J_1 and J_2 are disjoint Jordan curves in $\mathbb{C}$, which are disjoint from all $\xi \in \Xi$. Suppose that 0 is not contained in or enclosed by J_1, J_1 is enclosed by J_2, and that every $\xi \in \Xi$ that lies in the doubly connected domain bounded by J_1 and J_2 disconnects J_1 from J_2. Suppose $\xi_a < \xi_b \in \Xi$ are both enclosed by J_1, and $\xi_c \in \Xi$ neither encloses J_2, or is enclosed by J_2. Let E denote the event that $\tau_\xi < \infty$ for all $\xi \in \Xi$, and $\tau_{\xi_a} < \tau_{\xi_c} < \tau_{\xi_b}$. Then*

$$\mathbb{P}[E] \le C_{|\mathcal{E}|} e^{-\frac{\alpha}{4|\mathcal{E}|}\pi d_{\mathbb{C}}(J_1,J_2)} \prod_{e\in\mathcal{E}} \frac{P_{y_e}(r_e)}{P_{y_e}(R_e)},$$

where $C_{|\mathcal{E}|} \in (0,\infty)$ depends only on κ and $|\mathcal{E}|$.

Discussion. From [18, Theorem 3.2], we know that $\mathbb{P}[\tau_\xi < \infty, \xi \in \Xi] \le C_{|\mathcal{E}|} \prod_{e\in\mathcal{E}} \frac{P_{y_e}(r_e)}{P_{y_e}(R_e)}$. Now we need to derive the additional factor $e^{-\alpha\pi d_{\mathbb{C}}(J_1,J_2)/2}$ using the condition $\tau_{\xi_a} < \tau_{\xi_c} < \tau_{\xi_b}$.

Proof. We write $\mathbb{N}_n$ for $\{k \in \mathbb{N} : k \le n\}$. Let S denote the set of bijections $\sigma : \mathbb{N}_{|\Xi|} \to \Xi$ such that $\xi_1 < \xi_2$ implies that $\sigma^{-1}(\xi_1) < \sigma^{-1}(\xi_2)$, and $\sigma^{-1}(\xi_a) < \sigma^{-1}(\xi_c) < \sigma^{-1}(\xi_b)$. Let

$$E^\sigma = \{\tau_{\sigma(1)} < \tau_{\sigma(2)} < \cdots < \tau_{\sigma(|\Xi|)} < \infty\}, \quad \sigma \in S.$$

Then we have

$$E = \bigcup_{\sigma\in S} E^\sigma. \tag{A.6}$$

We will derive an upper bound of $\mathbb{P}[E^\sigma]$ in (A.11).

Fix $\sigma \in S$. For $e \in \mathcal{E}$, if there is no $\xi \in \Xi$ such that $\xi > \max \Xi_e$, then we say that e is a maximal element in E. In this case, we define $\widehat{\Xi}_e = \Xi_e$ and $\xi_e^* = \max \Xi_e$. If e is not a maximal element in E, let ξ_e^* denote the first $\xi > \max \Xi_e$ that is visited by γ on the event E^σ, and define $\widehat{\Xi}_e = \Xi_e \cup \xi_e^*$. This definition certainly depends on σ. Label the elements of $\widehat{\Xi}_e$ by $\xi_0^e < \cdots < \xi_{N_e}^e = \xi_e^*$, where $N_e = |\widehat{\Xi}_e| - 1$.

For $e \in E$, define

$$J_e = \{1 \le n \le N_e : \sigma^{-1}(\xi_n^e) > \sigma^{-1}(\xi_{n-1}^e) + 1\}.$$

Roughly speaking, $n \in J_e$ means that between $\tau_{\xi_{n-1}^e}$ and $\tau_{\xi_n^e}$, γ visits other element in Ξ that it has not visited before on the event E_σ.

Order the elements of $J_e \cup \{0\}$ by $0 = s_e(0) < \cdots < s_e(M_e)$, where $M_e = |J_e|$. Set $s_e(M_e+1) = N_e + 1$. Every $\widehat{\Xi}_e$ can be partitioned into $M_e + 1$ subsets:

$$\widehat{\Xi}_{(e,j)} = \{\xi_n^e : s_e(j) \le n \le s_e(j+1) - 1\}, \quad 0 \le j \le M_e.$$

The meaning of the partition is that, after γ visits the first element in $\widehat{\Xi}_{(e,j)}$, which must be $\xi_{s_e(j)}^e$, it then visits all elements in $\widehat{\Xi}_{(e,j)}$ without visiting any other circles in Ξ that it has not visited before. Let $I = \{(e,j) : e \in \mathcal{E}, 0 \le j \le M_e\}$. Then $\{\widehat{\Xi}_\iota : \iota \in I\}$ is a cover of Ξ. Note that every $\sigma^{-1}(\widehat{\Xi}_\iota)$, $\iota \in I$, is a connected subset of $\mathbb{Z}$.

For $\iota \in I$, let e_ι denote the first coordinate of ι, $z_\iota = z_{e_\iota}$ and $y_\iota = \operatorname{Im} z_\iota$. Define P_ι for each $\iota \in I$. If $\max \widehat{\Xi}_\iota \in \Xi_{e_\iota}$, define $P_\iota = \frac{P_{y_\iota}(R_{\max \widehat{\Xi}_\iota})}{P_{y_\iota}(R_{\min \widehat{\Xi}_\iota})}$, where we use R_ξ to denote the radius of ξ. If $\max \widehat{\Xi}_\iota \notin \Xi_{e_\iota}$, which means $\max \widehat{\Xi}_\iota = \xi^*_{e_\iota} > \max \Xi_{e_\iota}$, then we consider two subcases. If $\widehat{\Xi}_\iota$ contains only one element (i.e., $\xi^*_{e_\iota}$) or two elements (i.e., $\xi^*_{e_\iota}$ and $\max \Xi_{e_\iota}$), then let $P_\iota = 1$; otherwise let $P_\iota = \frac{P_{y_\iota}(R_{\max \Xi_{e_\iota}})}{P_{y_\iota}(R_{\min \widehat{\Xi}_\iota})}$. From the one-point estimate, we get

$$\mathbb{P}[\tau_{\max \widehat{\Xi}_\iota} < \infty | \mathcal{F}_{\min \widehat{\Xi}_\iota}] \le C P_\iota, \quad \iota \in I. \tag{A.7}$$

Let $P_e = \frac{P_{y_e}(r_e)}{P_{y_e}(R_e)}$, $e \in \mathcal{E}$. From Lemma 2.1 we get

$$\prod_{j=0}^{M_e} P_{(e,j)} \le 4^{\alpha M_e} P_e, \quad e \in \mathcal{E}. \tag{A.8}$$

We have $|I| = \sum_{e \in \mathcal{E}} (M_e + 1)$. Considering the order that γ visits $\widehat{\Xi}_\iota$, $\iota \in I$, we get a bijection map $\sigma_I : \mathbb{N}_{|I|} \to I$ such that $n_1 < n_2$ implies that $\max \sigma^{-1}(\widehat{\Xi}_{\sigma_I(n_1)}) \le \min \sigma^{-1}(\widehat{\Xi}_{\sigma_I(n_2)})$, and $n_1 = n_2 - 1$ implies that $\min \sigma^{-1}(\widehat{\Xi}_{\sigma_I(n_2)}) - \max \sigma^{-1}(\widehat{\Xi}_{\sigma_I(n_1)}) \in \{0, 1\}$. The difference may take value 0 if $\max \widehat{\Xi}_{\sigma_I(n_1)} = \xi^*_e \notin \Xi_e$ for $e = e_{\sigma_I(n_1)}$. We may express E^σ as

$$E^\sigma = \{\tau_{\min \widehat{\Xi}_{\sigma_I(1)}} \le \tau_{\max \widehat{\Xi}_{\sigma_I(1)}} \le \tau_{\min \widehat{\Xi}_{\sigma_I(2)}} \le \cdots \le \tau_{\min \widehat{\Xi}_{\sigma_I(|I|)}} < \tau_{\max \widehat{\Xi}_{\sigma_I(|I|)}} < \infty\}.$$

Fix $e_0 \in \mathcal{E}$. Let $n_j = \sigma_I^{-1}((e_0, j))$, $0 \le j \le M_{e_0}$. Then $n_{j+1} \ge n_j + 2$, $0 \le j \le M_{e_0} - 1$. Fix $0 \le j \le M_{e_0} - 1$. Let $m = n_{j+1} - n_j - 1$. If $\max \widehat{\Xi}_{\sigma_I(n_j+k)}$ and $\min \widehat{\Xi}_{\sigma_I(n_j+k)}$ are concentric for $1 \le k \le m$, applying Lemma A.3 with $\widehat{J}_0 = \min \Xi_{e_0}$, $J_0 = \max \widehat{\Xi}_{(e_0,j)} = \max \widehat{\Xi}_{\sigma_I(n_j)}$, $J_0' = \min \widehat{\Xi}_{(e_0,j+1)} = \min \widehat{\Xi}_{\sigma_I(n_{j+1})}$, $\{|z - z_k| = R_k\} = \min \widehat{\Xi}_{\sigma_I(n_j+k)}$ and $\{|z - z_k| = r_k\} = \max \widehat{\Xi}_{\sigma_I(n_j+k)}$, $1 \le k \le m$, we get

$$\mathbb{P}[E^\sigma_{[\max \widehat{\Xi}_{\sigma_I(n_j)}, \min \widehat{\Xi}_{\sigma_I(n_{j+1})}]} | \mathcal{F}_{\tau_{\max \widehat{\Xi}_{\sigma_I(n_j)}}}] \le C^m 4^{-\alpha/4(s_{e_0}(j+1)-1)} \prod_{n=n_j+1}^{n_{j+1}-1} P_{\sigma_I(n)}, \tag{A.9}$$

where $E^\sigma_{[\max \widehat{\Xi}_{\sigma_I(n_j)}, \min \widehat{\Xi}_{\sigma_I(n_{j+1})}]}$ is the event

$$\{\tau_{\max \widehat{\Xi}_{\sigma_I(n_j)}} \le \tau_{\min \widehat{\Xi}_{\sigma_I(n_j+1)}} \le \tau_{\max \widehat{\Xi}_{\sigma_I(n_j+1)}} \le \cdots \le \tau_{\max \widehat{\Xi}_{\sigma_I(n_j+m)}} \le \tau_{\min \widehat{\Xi}_{\sigma_I(n_{j+1})}} < \infty\}.$$

Because of the definition of P_ι, $\iota \in I$, the above estimate still holds in the general case, i.e., there may be some $1 \le k \le n$ such that $\max \widehat{\Xi}_{\sigma_I(n_j+k)} = \xi^*_e \notin \Xi_e$, where $e = e_{\sigma_I(n_j+k)}$.

We say that γ makes a (J_1, J_2) jump during $[\max \widehat{\Xi}_{\sigma_I(n_j)}, \min \widehat{\Xi}_{\sigma_I(n_{j+1})}]$ if $\min \Xi_{e_0}$ is enclosed by J_1, and there is at least one $k_0 \in \mathbb{N}_m$ such that $\min \widehat{\Xi}_{\sigma_I(n_j+k_0)}$ is not enclosed by J_2. In this case, applying Lemma A.3 with $J_0 = J_1$ and $\widehat{J}_0 = J_2$, we get

$$\mathbb{P}[E^\sigma_{[\max \widehat{\Xi}_{\sigma_I(n_j)}, \min \widehat{\Xi}_{\sigma_I(n_{j+1})}]} | \mathcal{F}_{\tau_{\max \widehat{\Xi}_{\sigma_I(n_j)}}}] \le C^m e^{-\alpha\pi d_{\mathbb{C}}(J_1, \widehat{J}_2)/2} \prod_{n=n_j+1}^{n_{j+1}-1} P_{\sigma_I(n)}.$$

Combining this with (A.9), we get

$$\mathbb{P}[E^{\sigma}_{[\max\widehat{\Xi}_{\sigma_I(n_j)},\min\widehat{\Xi}_{\sigma_I(n_{j+1})}]}|\mathcal{F}_{\tau_{\max\widehat{\Xi}_{\sigma_I(n_j)}}}] \le C^m e^{-\frac{\alpha}{4}\pi d_{\mathbb{C}}(J_1,\widehat{J}_2)} 4^{-\frac{\alpha}{8}(s_{e_0}(j+1)-1)} \prod_{n=n_j+1}^{n_{j+1}-1} P_{\sigma_I(n)}. \quad (A.10)$$

Letting j vary between 0 and $M_{e_0}-1$ and using (A.7) and (A.9), we get

$$\mathbb{P}[E^{\sigma}] \le C^{|I|} 4^{-\alpha/4\sum_{j=1}^{M_{e_0}}(s_{e_0}(j)-1)} \prod_{\iota\in I} P_{\iota}.$$

Using (A.8) and $|I| = \sum_e (M_e+1)$, we find that

$$\mathbb{P}[E^{\sigma}] \le C^{|\mathcal{E}|} C^{\sum_{e\in\mathcal{E}} M_e} 4^{-\frac{\alpha}{4}\sum_{j=1}^{M_{e_0}} s_{e_0}(j)} \prod_{e\in\mathcal{E}} P_e.$$

Since $\sigma^{-1}(\xi_a) < \sigma^{-1}(\xi_c) < \sigma^{-1}(\xi_b)$, $\xi_a < \xi_b$ are enclosed by J_1, and ξ_c is not enclosed by J_2, there must exist some $e_0 \in \mathcal{E}$ and some $j \in [0, M_{e_0}-1]$ such that γ makes a (J_1, J_2) jump during $[\max\widehat{\Xi}_{\sigma_I(n_j)}, \min\widehat{\Xi}_{\sigma_I(n_{j+1})}]$. In that case, using (A.7), (A.9), and (A.10), we get

$$\mathbb{P}[E^{\sigma}] \le C^{|\mathcal{E}|} C^{\sum_{e\in\mathcal{E}} M_e} e^{-\frac{\alpha}{4}\pi d_{\mathbb{C}}(J_1,\widehat{J}_2)} 4^{-\frac{\alpha}{8}\sum_{j=1}^{M_{e_0}} s_{e_0}(j)} \prod_{e\in\mathcal{E}} P_e.$$

Taking a geometric average of the above upper bounds for $\mathbb{P}[E^{\sigma}]$ over $e_0 \in \mathcal{E}$, we get

$$\mathbb{P}[E^{\sigma}] \le C^{|\mathcal{E}|} C^{\sum_{e\in\mathcal{E}} M_e} e^{-\frac{\alpha}{4|\mathcal{E}|}\pi d_{\mathbb{C}}(J_1,\widehat{J}_2)} 4^{-\frac{\alpha}{8|\mathcal{E}|}\sum_{e\in\mathcal{E}}\sum_{j=1}^{M_e} s_e(j)} \prod_{e\in\mathcal{E}} P_e. \quad (A.11)$$

So far we have omitted the σ on I, M_e, $s_e(j)$ and etc; we will put σ on the superscript if we want to emphasize the dependence on σ. From (A.6) and (A.11), we get

$$\mathbb{P}[E] \le C^{|\mathcal{E}|} \sum_{(M_e;(s_e(j))_{j=0}^{M_e})_{e\in\mathcal{E}}} |S_{(M_e,(s_e(j)))}| C^{\sum_{e\in\mathcal{E}} M_e} e^{-\frac{\alpha}{4|\mathcal{E}|}\pi d_{\mathbb{C}}(J_1,\widehat{J}_2)} 4^{-\frac{\alpha}{8|\mathcal{E}|}\sum_{e\in\mathcal{E}}\sum_{j=1}^{M_e} s_e(j)} \prod_{e\in\mathcal{E}} P_e, \quad (A.12)$$

where

$$S_{(M_e,(s_e(j)))} := \{\sigma \in S : M_e^{\sigma} = M_e, s_e^{\sigma}(j) = s_e(j), 0 \le j \le M_e, e \in \mathcal{M}\},$$

and the first summation in (A.12) is over all possible $(M_e; (s_e(j))_{j=0}^{M_e})_{e\in\mathcal{E}}$, namely, $M_e \ge 0$ and $0 = s_e(0) < s_e(1) < \cdots s_e(M_e) \le N_e$ for every $e \in \mathcal{E}$. It now suffices to show that

$$\sum_{(M_e;(s_e(j))_{j=1}^{M_e})_{e\in\mathcal{E}}} |S_{(M_e,(s_e(j)))}| C^{\sum_{e\in\mathcal{E}} M_e} 4^{-\frac{\alpha}{8|\mathcal{E}|}\sum_{e\in\mathcal{E}}\sum_{j=1}^{M_e} s_e(j)} \le C_{|\mathcal{E}|}, \quad (A.13)$$

for some $C_{|\mathcal{E}|} < \infty$ depending only on $|\mathcal{E}|$ and κ.

We now bound the size of $S_{(M_e,(s_e(j)))}$. Note that when M_e^σ and $s_e^\sigma(j)$, $0 \le j \le M_e^\sigma$, $e \in \mathcal{E}$, are given, σ is then determined by $\sigma_I : \mathbb{N}_{|I^\sigma|} \to I^\sigma$, which is in turn determined by $e_{\sigma_I(n)}$, $1 \le n \le |I^\sigma| = \sum_{e\in\mathcal{E}}(M_e^\sigma + 1)$. Since each $e_{\sigma_I(n)}$ has at most $|\mathcal{E}|$ possibilities, we have $|S_{(M_e,(s_e(j)))}| \le |\mathcal{E}|^{\sum_{e\in\mathcal{E}}(M_e+1)}$. Thus, the left-hand side of (A.13) is bounded by

$$
\begin{aligned}
&|\mathcal{E}|^{|\mathcal{E}|} \sum_{(M_e;(s_e(j))_{j=0}^{M_e})_{e\in\mathcal{E}}} \prod_{e\in\mathcal{E}} (C|\mathcal{E}|)^{M_e} 4^{-\frac{\alpha}{8|\mathcal{E}|}\sum_{j=1}^{M_e} s_e(j)} \\
=&|\mathcal{E}|^{|\mathcal{E}|} \prod_{e\in\mathcal{E}} \sum_{M_e=0}^{N_e} (C|\mathcal{E}|)^{M_e} \sum_{0=s_e(0)<\cdots<s_e(M_e)\le N_e} 4^{-\frac{\alpha}{8|\mathcal{E}|}\sum_{j=1}^{M_e} s_e(j)} \\
\le&|\mathcal{E}|^{|\mathcal{E}|} \prod_{e\in\mathcal{E}} \sum_{M=0}^{\infty} (C|\mathcal{E}|)^{M} \sum_{s(1)=1}^{\infty} \cdots \sum_{s(M)=M}^{\infty} 4^{-\frac{\alpha}{8|\mathcal{E}|}\sum_{j=1}^{M} s(j)} \\
\le&|\mathcal{E}|^{|\mathcal{E}|} \prod_{e\in\mathcal{E}} \sum_{M=0}^{\infty} (C|\mathcal{E}|)^{M} \prod_{j=1}^{M} \sum_{s(j)=j}^{\infty} 4^{-\frac{\alpha}{8|\mathcal{E}|} s(j)} \\
=&\left[|\mathcal{E}| \sum_{M=0}^{\infty} \left(\frac{C|\mathcal{E}|}{1 - 4^{-\frac{\alpha}{8|\mathcal{E}|}}} \right)^M 4^{-\frac{\alpha}{16|\mathcal{E}|} M(M+1)} \right]^{|\mathcal{E}|}.
\end{aligned}
$$

The conclusion now follows since the summation inside the square bracket equals to a finite number depending only on κ and $|\mathcal{E}|$. $\square$

Proof of Theorem 3.1. By (2.7), we may change the order of the points $z_1, \dots, z_n$. Thus, it suffices to show that

$$
\mathbb{P}[\tau_{r_j}^{z_j} < \infty, 1 \le j \le n; \tau_{s_1}^{z_1} < \tau_{r_2}^{z_2} < \tau_{r_1}^{z_1}] \le C_n \prod_{j=1}^{n} \frac{P_{y_j}(r_j)}{P_{y_j}(l_j)} \cdot \Big(\frac{s_1}{|z_1 - z_2| \wedge |z_1|} \Big)^{\frac{\alpha}{32n^2}}, \qquad \text{(A.14)}
$$

for any distinct points $z_1, \dots, z_n \in \overline{\mathbb{H}} \backslash \{0\}$, $r_j \in (0, d_j)$, $1 \le j \le n$, and $s_1 \in (r_1, |z_1 - z_2| \wedge |z_1|)$, where y_j, l_j, d_j are defined by (2.3).

We want to deduce the theorem from Lemma A.4, so we want to construct a family Ξ of mutually disjoint circles and Jordan curves J_1, J_2.

Suppose $4^{h_j} r_j \le l_j \le 4^{h_j+1} r_j$ for some $h_j \in \mathbb{N}$, $1 \le j \le n$. By increasing the value of s_1, we may assume that $s_1 = 4^{\widetilde{h}_1} r_1$, where $\widetilde{h}_1 \in \mathbb{N}$ and $\widetilde{h}_1 > h_1$. Define

$$
\xi_j^s = \{|z - z_j| = 4^{h_j - s} r_j\}, \quad 1 \le j \le n, \quad 1 \le s \le h_j.
$$

The family $\{\xi_j^s : 1 \le j \le n, \quad 1 \le s \le h_j\}$ may not be mutually disjoint. So we can not define Ξ to be this family. To solve this issue, we will remove some circles as follows. For $1 \le j < k \le n$, let $D_k = \{|z - z_k| \le l_k/4\}$, which contains every ξ_k^r, $1 \le r \le h_k$, and

$$
I_{j,k} = \{\xi_j^s : 1 \le s \le h_j, \xi_j^s \cap D_k \neq \emptyset\}. \qquad \text{(A.15)}
$$

Then $\Xi := \{\xi_j^s : 1 \le j \le n, 1 \le s \le h_j\} \setminus \bigcup_{1\le j<k\le n} I_{j,k}$ is mutually disjoint. If $\operatorname{dist}(\gamma, z_j) \le r_j$, then γ intersects every ξ_j^s, $1 \le s \le h_j$. So we get

$$\mathbb{P}[\operatorname{dist}(\gamma, z_j) \le r_j, 1 \le j \le n] \le \mathbb{P}\Big[\bigcap_{j=1}^{n}\bigcap_{s=1}^{h_j}\{\gamma \cap \xi_j^s \neq \emptyset\}\Big] \le \mathbb{P}\Big[\bigcap_{\xi\in\Xi}\{\gamma \cap \xi \neq \emptyset\}\Big]. \tag{A.16}$$

Next, we construct a partition $\{\Xi_e : e \in \mathcal{E}\}$ of Ξ. We introduce some notation: if e is a family of circles centered at $z_0 \in \overline{\mathbb{H}}$ with biggest radius R and smallest radius r, then we define $A_e = \{r \le |z - z_0| \le R\}$ and $P_e = \frac{P_{\operatorname{Im} z_0}(r)}{P_{\operatorname{Im} z_0}(R)}$.

First, Ξ has a natural partition Ξ_j, $1 \le j \le n$, such that Ξ_j is composed of circles centered at z_j. For each j, we construct a graph G_j, whose vertex set is Ξ_j, and $\xi_1 \neq \xi_2 \in \Xi_j$ are connected by an edge iff the bigger radius is 4 times the smaller one, and the open annulus between them does not contain any other circle in Ξ. Let $\mathcal{E}_j$ denote the set of connected components of G_j. Then we partition Ξ_j into Ξ_e, $e \in \mathcal{E}_j$, such that every Ξ_e is the vertex set of $e \in \mathcal{E}_j$. Then the circles in every Ξ_e are concentric circles with radii forming a geometric sequence with common ratio $1/4$, and the closed annuli A_e associated with Ξ_e, $e \in \mathcal{E}_j$, are mutually disjoint. From the construction we also see that for any $j < k$, and $e \in \mathcal{E}_j$, A_e does not intersect D_k, which contains every A_e with $e \in \mathcal{E}_k$. Let $\mathcal{E} = \bigcup_{j=1}^n \mathcal{E}_j$. Then A_e, $e \in \mathcal{E}$, are mutually disjoint. Thus, $\{\Xi_e : e \in \mathcal{E}\}$ is a partition of Ξ that satisfies the properties before Lemma A.4.

We observe that for $j < k$, $\bigcup_{\xi\in\Xi_k} \xi \subset D_k$ can be covered by an annulus centered at z_j with ratio less than 4 because

$$\frac{\max_{z\in D_k}\{|z - z_j|\}}{\min_{z\in D_k}\{|z - z_j|\}} \le \frac{|z_j - z_k| + l_k/4}{|z_j - z_k| - l_k/4} \le \frac{l_k + l_k/4}{l_k - l_k/4} < 4.$$

Thus, every $I_{j,k}$ defined in (A.15) contains at most one element. We also see that, for $j < k$, $\bigcup_{\xi\in\Xi_k} \xi \subset D_k$ intersects at most 2 annuli from $\{4^{h_j - s} r_j \le |z - z_j| \le 4^{h_j - s+1} r_j\}$, $2 \le s \le h_j$. If $j > k$, by construction, $\bigcup_{\xi\in\Xi_k} \xi$ is disjoint from the annuli $\{4^{h_j - s} r_j \le |z - z_j| \le 4^{h_j - s+1} r_j\}$, $2 \le s \le h_j$, which are contained in D_j.

From [18, Theorem 1.1], we have $\mathbb{P}[\tau_{r_j}^{z_j} < \infty, 1 \le j \le n] \le C_n \prod_{j=1}^n \frac{P_{y_j}(r_j)}{P_{y_j}(l_j)}$. So we may assume that $|z_2 - z_1| \wedge |z_1| > 4^{4n+1} s_1$. Since for $k \ge 2$, $\bigcup_{\xi\in\Xi_k} \xi \subset D_k$ can be covered by an annulus centered at z_1 with ratio less than 4, by pigeon hole principle, we can find a closed annulus centered at z_1 with two radii $r < R$ satisfying $s_1 \le r < R \le |z_2 - z_1| \wedge |z_1|$ and $R/r \le (\frac{|z_2 - z_1| \wedge |z_1|}{s_1})^{1/2n}$ that is disjoint from all $\bigcup_{\xi\in\Xi_k} \xi \subset D_k$, $k \ge 2$. Moreover, we may choose R and r such that the boundary circles are disjoint from every $\xi \in \Xi$. Applying Lemma A.4 with $J_1 = \{|z - z_1| = r\}$, $J_2 = \{|z - z_1| = R\}$, $\xi_a = \{|z - z_1| = s_1\}$, $\xi_b = \{|z - z_1| = r_1\}$, $\xi_c = \{|z - z_2| = r_2\}$, and $\{\Xi_e : e \in \mathcal{E}\}$, we find that

$$\mathbb{P}[\tau_{r_j}^{z_j} < \infty, 1 \le j \le n; \tau_{s_1}^{z_1} < \tau_{r_2}^{z_2} < \tau_{r_1}^{z_1}] \le C_{|\mathcal{E}|}\Big(\frac{s_1}{|z_1 - z_2| \wedge |z_1|}\Big)^{\frac{\alpha}{16n|\mathcal{E}|}} \prod_{j=1}^{n}\prod_{e\in\mathcal{E}_j} P_e. \tag{A.17}$$

Here we set $\prod_{e\in\mathcal{E}_j} P_e = 1$ if $\mathcal{E}_j = \emptyset$. We will finish the proof by proving that $|\mathcal{E}| \le 2n$ and $\prod_{e\in\mathcal{E}} P_e \le C_n \frac{P_{y_j}(r_j)}{P_{y_j}(l_j)}$.

We now bound $|\mathcal{E}| = \sum_{j=1}^n |\mathcal{E}_j|$. For $1 \le m \le n$, we use $\mathcal{E}_j^{(m)}$, $1 \le j \le m$, to denote the set of connected components of the graph $G_j^{(m)}$ obtained by removing the circles in $I_{j,k}$, $j < k \le m$, from Ξ_j. Let $\mathcal{E}^{(m)} = \bigcup_{j=1}^m \mathcal{E}_j^{(m)}$. Then $\mathcal{E} = \mathcal{E}^{(n)}$. For $2 \le m \le n$, and $1 \le j \le m-1$, we may define a map $f_m : \bigcup_{j=1}^{m-1} \mathcal{E}_j^{(m)} \to \mathcal{E}^{(m-1)}$ such that for every $e \in \mathcal{E}_j^{(m)}$, $1 \le j \le m-1$, $f_m(e)$ is the unique element in $\mathcal{E}_j^{(m-1)}$ that contains e. Then each $e \in \mathcal{E}^{(m-1)}$ has at most 2 preimages, and $e \in \mathcal{E}^{(m-1)}$ has exactly 2 preimages iff D_m is contained in the interior of A_e. Since the annuli A_e, $e \in \mathcal{E}^{(m-1)}$, are mutually disjoint, at most one of them has two preimages. Since $\mathcal{E}_m^{(m)}$ contains only one element, we find that $|\mathcal{E}^{(m)}| \le |\mathcal{E}^{(m-1)}| + 2$. From $|\mathcal{E}^{(1)}| = 1$ and $\mathcal{E} = \mathcal{E}^{(n)}$, we get $|\mathcal{E}| \le 2n-1$.

To estimate $\prod_{e\in\mathcal{E}} P_e$, we introduce S_j to be the family of pairs of circles $\{\{|z-z_j| = 4^s r_j\}, \{|z-z_j| = 4^{s-1} r_j\}\}$, $s \in \mathbb{N}$. Let $S_j^{(m)}$ denote the set of $e' \in S_j$ such that $A_{e'} \subset \bigcup_{e\in\mathcal{E}_j^{(m)}} A_e$. Then $\prod_{e\in\mathcal{E}_j^{(m)}} P_e = \prod_{e'\in S_j^{(m)}} P_{e'}$. Note that, for $m > j$, $A_{e'}$, $e' \in S_j^{(m)}$ can be obtained from $A_{e'}$, $e' \in S_j^{(m-1)}$, by removing the annuli in the latter group that intersects D_m. Since D_m can be covered by an annulus centered at z_j with ratio less than 4, it can intersect at most two of $A_{e'}$, $e' \in S_j$. Using Lemma 2.1, we find that $\prod_{e\in\mathcal{E}_j^{(m)}} P_e \le 4^{2\alpha} \prod_{e\in\mathcal{E}_j^{(m-1)}} P_e$. Since $l_j \le 4^{h_j+1} r_j$, we get $\prod_{e\in\mathcal{E}_j^{(j)}} P_e = \frac{P_{y_j}(r_j)}{P_{y_j}(4^{h_j} r_j)} \le 4^\alpha \frac{P_{y_j}(r_j)}{P_{y_j}(l_j)}$. Thus, $\prod_{e\in\mathcal{E}_j^{(n)}} P_e \le 4^{\alpha(2n-2j+1)} \frac{P_{y_j}(r_j)}{P_{y_j}(l_j)}$, which implies that

$$\prod_{e\in\mathcal{E}^{(n)}} P_e = \prod_{j=1}^n \prod_{e\in\mathcal{E}_j^{(n)}} P_e \le \prod_{j=1}^n 4^{\alpha(2n-2j+1)} \frac{P_{y_j}(r_j)}{P_{y_j}(l_j)} = 4^{\alpha n^2} \prod_{j=1}^n \frac{P_{y_j}(r_j)}{P_{y_j}(l_j)}.$$

The proof is now complete. □

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