RESEARCH STATEMENT

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1. Introduction

My research mainly lies in the areas of numerical analysis, probability, mathematical modeling, and geometry. My current research interests focus on matrix theory and its applications, especially on duality and lower rank approximation, and compressed sensing, which is a technique for recovering sparse or compressible signals, as well as phase retrieval, which is a process to recover the phase of a signal from the magnitudes of its linear measurements. In random matrices, I gave the probabilistic estimates on the generalized singular value of pregaussian random matrices. These theoretic results can be applied to compressed sensing. I have been working on non-convex optimization, particularly the sparest solution to under-determined multiple linear equations. The uniqueness of the solution to the optimization problem can be characterized by the null space property of the sensing matrix. In this aspect, I gave the simpler condition for the uniqueness to multiple measurement vectors recovery problem. In mathematical modeling, I worked on the simulation of the aggregation process of marine particles and the decomposition of complex dynamic ecological network systems. I am also interested in convex geometry, particularly the cosine transform of functions defined on the topological space consisting of all subspaces of a vector space of any given dimension.

2. Random Matrices and Compressed Sensing

The extremal singular values of random matrices in $\ell_2$-norm, including Gaussian random matrices, Bernoulli random matrices, subgaussian random matrices, etc., have attracted major research interest in recent years. We study the $q$-singular values, defined in terms of the $\ell_q$-quasinorm, of pregaussian random matrices. We give the upper tail probability estimate on the largest $q$-singular value of pregaussian random matrices for $0 < q \leq 1$, and also the lower tail probability estimate. Particularly, these estimates show that the largest $q$-singular value is of order $m^{1/q}$ with high probability for pregaussian random matrices of size $m$ by $m$. Moreover, we also give probabilistic estimates for the smallest $q$-singular value of pregaussian random matrices. In addition, we also present some results on the largest $p$-singular value for $p > 1$, and some numerical-experimental results as well.

Compressed sensing, a technique for recovering sparse signals, has also been an active research topic recently. The extremal singular values of random matrices have applications in compressed sensing, mainly because the restricted isometry constant of sensing matrices depends on them. We prove that the pregaussian random matrices with $m$ much less than $N$ but much larger than $N^{q/2}$ have the $q$-modified restricted isometry property for $0 < q \leq 1$ with overwhelming probability. As a result, we show that every sparse vector can be recovered as a solution to the $\ell_q$-minimization problem with overwhelming probability if $m$ is much less than $N$ but much larger than $N^{q/2}$.

In compressed sensing, we also show that the real and complex null space properties (NSP) are equivalent for the sparse recovery by $\ell_q$-minimization and more generally for
the NSP for the joint-sparse recovery from multiple measurements via $\ell_q$-minimization. These results answer the open questions raised by Foucart and Gribonval in [8]. We also extend Berg and Friedlander’s theorem on NSP for recovery from multiple measurements. As a consequence of the equivalence on NSP and the extension, we give a necessary and sufficient condition for the uniqueness of the solution to the multiple-measurement-vector non-convex optimization problem.

Precisely, in compressed sensing, we want to recover a sparse or compressible signal via solving the minimization problem,

$$\min_{x \in \mathbb{R}^N} \|x\|_0 \quad \text{subject to} \quad Ax = b,$$

(2.1)

in which $\|x\|_0$ is the the number of non-zero entries of the vector $x$, namely the sparsity of $x$. Since $\|x\|_0$ can be approximated by $\|x\|_q$, one can use the $\ell_q$-approach (see [12] and [9]), which is considering the following $\ell_q$-minimization problem with $0 < q \leq 1$,

$$\min_{x \in \mathbb{R}^N} \|x\|_q \quad \text{subject to} \quad Ax = b.$$

(2.2)

Instead of single measurement vector, we study the multiple-measurement-vector (MMV) non-convex optimization problem, that is

$$\min \|X\|_{q,p} \quad \text{subject to} \quad AX = B,$$

(2.3)

in which $A$, $X$ and $B$ are matrices, and

$$\|X\|_{q,p} := \left( \sum_{j=1}^N \|X^{j\rightarrow}\|_p^q \right)^{\frac{1}{q}},$$

(2.4)

where $X^{j\rightarrow}$ is the $j$-th row of $X$, for $0 < q \leq 1$. Furthermore, the largest $q$-singular value defined as

$$s_1^{(q)}(A) := \sup \left\{ \frac{\|Ax\|_q}{\|x\|_q} : x \in \mathbb{R}^N \text{ with } x \neq 0 \right\},$$

(2.5)

and the smallest $q$-singular value defined accordingly (see for instance [22]) become important to the $\ell_q$-approach, because of the restricted isometry property (see [5] and [10]). Particularly for the linear system of equations involving pregaussian matrices, we study the probabilistic estimates on the largest and smallest $q$-singular values of pregaussian random matrices.

In compressed sensing and optimization, we have obtained a generalized version of the perimeter lemma by Foucart and Gribonval in [8], the lemma of $q$-perimeter of polygon under linear map, and have given a simple new proof. Furthermore, we show that the real null space and complex null space property are equivalent for the sparse recovery achieved by $\ell_q$-minimization and more generally that for the joint-sparse recovery from multiple measurements. These results have answered the open questions raised in [8]. We have also extended Berg and Friedlander’s theorem in [1] on null space property for recovery from multiple measurements to $0 < q \leq 1$. As a consequence of the extension, we give a necessary and sufficient condition for the uniqueness of the solution to the multiple-measurement-vector (MMV) non-convex optimization problem. In fact, we consider a joint recovery from multiple measurement vectors via

$$\min \sum_{j=1}^N \left( \sqrt{x_{1,j}^2 + \cdots + x_{r,j}^2} \right)^q : x \in \mathbb{R}^N$$

subject to $A\mathbf{x}^{(1)} = \mathbf{b}^{(1)}, \ldots, A\mathbf{x}^{(r)} = \mathbf{b}^{(r)}$

(2.6)

for a given $0 < q \leq 1$, where $\mathbf{x}^{(k)} = (x_{k,1}, \cdots, x_{k,N})^T \in \mathbb{R}^N$ for all $k = 1, \cdots, r$, and this is actually (2.3) for when $p = 2$.
Written as equivalent conditions, the theorem on the exact recovery we mainly prove in \cite{16} is the following

**Theorem 1** (Null space property for MMV recovery). Let $A$ be a real matrix of size $m \times N$ and $S \subset \{1, 2, \ldots, N\}$ be a fixed index set. Fix $p \in (0, 1]$ and $r \geq 1$. Then the following conditions are equivalent:

(a) All $x^{(k)}$ with support in $S$ for $k = 1, \ldots, r$ can be uniquely recovered using (2.6);

(b) For all vectors $(u^{(1)}, \ldots, u^{(r)}) \in (N(A))^r \setminus \{(0, 0, \ldots, 0)\}$
\[
\sum_{j \in S} \left( \sqrt{u_{i,j}^2 + \cdots + u_{r,j}^2} \right)^q < \sum_{j \in S^c} \left( \sqrt{u_{i,j}^2 + \cdots + u_{r,j}^2} \right)^q ;
\]  \hspace{1cm} (2.7)

(c) For all vector $z \in N(A)$ with $z \neq 0$,
\[
\sum_{j \in S} |z_j|^q < \sum_{j \in S^c} |z_j|^q,
\]  \hspace{1cm} (2.8)

where $z = (z_1, \ldots, z_N)^T \in \mathbb{R}^N$.

The above theorem characterizes the uniqueness of the solution to the MMV non-convex optimization problem (2.3). Therefore, a future direction that I may explore is to develop an efficient algorithm to find the solution to MMV non-convex optimization problem (2.3).

For joint-sparse recovery from multiple measurements, it can also be achieved by mixed $\ell_{q,2}$-minimization for $0 < q \leq 1$, for which one can see \cite{9} and \cite{1}. The above theorem allows us to solve multiple $\ell_q$-minimizations instead, that will simplify the program for the mixed $\ell_{q,2}$-minimization. Furthermore, multiple measurements have been used in many fields of technology, for instance, neuromagnetic imaging \cite{6} and communication channels (see e.g. \cite{7}), as pointed out in \cite{1}, so our results may have real applications in these fields.

Random matrices are often used as the sensing matrices in the optimization problem for compressed sensing, and the extreme singular values, including the largest singular value and smallest singular value, of random matrices become important for the restricted isometry property introduced in \cite{5} by Candes and Tao (see also \cite{4}) and generalized restricted property defined in \cite{10} by Foucart and Lai.

One of the results in random matrices we get is that the pre-Gaussian random matrices, whose entries are independent and identically-distributed pre-Gaussian random variables (see e.g. \cite{3} and \cite{2}), with $N^{1/2} \ll m \ll N$ have the $q$-modified restricted isometry property, $0 < q \leq 1$, introduced by Foucart and Lai in \cite{10}, with overwhelming probability. We also prove that the sparsest solution to the $\ell_q$-minimization problem with $0 < q \leq 1$ can be recovered if $N^{1/2} \ll m \ll N$.

The singular values of random matrices in the $\ell_2$-norm have been studied in recent years, including the largest one and the smallest, for Gaussian matrices, Bernoulli matrices, sub-Gaussian matrices, etc (see for instance, \cite{21} and \cite{23}). In \cite{17}, we study the $q$-singular values of random matrices whose entries are independent and identically-distributed copies of a pregauasian random variable, defined in terms of the $\ell_q$-quasinorm. We are able to obtain the decay on the upper tail probability of the largest $q$-singular value $s_1^{(q)}$ for all $0 < q \leq 1$ when the number of rows of the matrices becomes very large, though the $\ell_q$-quasinorm is non-convex. This result is stated as

**Theorem 2** (Upper tail probability of the largest $q$-singular value, $0 < q < 1$). Let $\xi$ be a pre-Gaussian variable normalized to have variance 1 and $A$ be an $m \times m$ matrix with i.i.d. copies of $\xi$ in its entries. Then for every $0 < q < 1$,
\[
P \left( s_1^{(q)}(A) \geq Cm^{1/2} \right) \leq \exp \left( -C'm \right)
\]  \hspace{1cm} (2.9)

for some $C, C' > 0$ only dependent on the pre-Gaussian variable $\xi$. 

We have also obtained the lower tail probability of the largest $q$-singular value $s_{1}^{(q)}$ for all $0 < q \leq 1$, that is based on a linear bound for partial binomial expansion.

**Theorem 3** (Lower tail probability of the largest $q$-singular value, $0 < q < 1$). Let $\xi$ be a pre-Gaussian variable normalized to have variance 1 and $A$ be an $m \times N$ matrix with i.i.d. copies of $\xi$ in its entries. Then there exists a constant $c > 0$ such that

$$P \left( s_{1}^{(1)}(A) \leq K m^{\frac{1}{2}} \right) \leq c^{N}$$

(2.10)

for some $0 < c < 1$, where $K$ only depends on $\varepsilon$ and the pre-Gaussian variable $\xi$.

More generally, we have

**Theorem 4.** Let $\xi$ be a random variable with non-zero variance and $A$ be an $m \times N$ matrix with i.i.d. copies of $\xi$ in its entries. Then there exists a constant $c > 0$ such that

$$P \left( s_{1}^{(1)}(A) \leq K m^{\frac{1}{2}} \right) \leq c^{N}$$

(2.11)

for some $0 < c < 1$, where $K$ only depends on $\varepsilon$ and the random variable $\xi$.

In particular, these estimates show $s_{\max}^{(q)}(A) \sim m^{\frac{1}{2}}$ asymptotically with very high probability for $m \times m$ pre-Gaussian matrix $A$.

For the smallest $q$-singular value of an $n \times n$ pre-Gaussian matrix, we have also obtained the estimate on the lower tail probability.

**Theorem 5** (Lower tail probabilistic estimate on the smallest $q$-singular value). Given any $0 < q \leq 1$, and let $\xi$ be the pre-Gaussian random variable with variance 1 and $A$ be an $n \times n$ matrix with i.i.d. copies of $\xi$ in its entries. Then for any $\varepsilon > 0$, there exists some $\gamma > 0$ such that

$$P \left( s_{n}^{(q)}(A) < \gamma n^{-\frac{1}{2}} \right) < \varepsilon,$$

(2.12)

where $\gamma$ only depends on $q$, $\varepsilon$ and the pre-Gaussian variable $\xi$.

On the upper tail probability of the smallest $q$-singular value, we have

**Theorem 6** (Upper tail probabilistic estimate on the smallest $q$-singular value). Given any $0 < q \leq 1$, and let $\xi$ be the pre-Gaussian random variable with variance 1 and $A$ be an $n \times n$ matrix with i.i.d. copies of $\xi$ in its entries. Then for any $K > e$, there exist some $C > 0$, $0 < c < 1$, and $\alpha > 0$ only dependent on pre-Gaussian variable $\xi$, $q$, such that

$$P \left( s_{n}^{(q)}(A) > Kn^{-\frac{1}{2}} \right) \leq \frac{C (\ln K)^{\alpha}}{K^{\alpha}} + c^{n}.$$

(2.13)

In particular, for any $\varepsilon > 0$, there exist some $K > 0$ and $n_{0}$, such that

$$P \left( s_{n}^{(q)}(A) > Kn^{-\frac{1}{2}} \right) < \varepsilon$$

(2.14)

for all $n \geq n_{0}$.

To verify the theoretical results, we have done some numerical experiments in MATLAB. For rectangular matrices, we also plot the largest $\frac{1}{4}$-singular value of Gaussian random matrices of size $m \times n$, where $m$ and $n$ run from 1 through 100. See Figure 2.1. This graph shows that the $\frac{1}{4}$-singular value is approximately $O(m^{\frac{1}{4}})$, as estimated in the above theorems.

We plot the largest 4-singular value of Gaussian random matrices and Bernoulli random matrices of size $m \times n$, where $m$ and $n$ run from 1 through $3^{4}(= 81)$. See Figure 2.2.

Also, we plot the largest 4-singular value of Bernoulli random matrices of size $m \times n$, where $m$ and $n$ run from 1 through $3^{4}(= 81)$. See Figure 2.3.
3. MATHEMATICAL MODELING AND INTERDISCIPLINARY RESEARCH

3.1. Mathematical Modeling for the Aggregation of Marine Particles. I had valuable internship experience of working with Prof. Adrian Burd in marine science on using mathematical model to simulate the aggregation process of particles in the marine environment. The reasons we study this are

- The aggregation process of marine particles can affect the ecological system in a certain region of the ocean and the environment, such as zooplankton grazing, vertical flux, light penetration, etc.
- The aggregation process of marine particles can affect the climate change in the long run, because it provides a way to get carbon from the surface ocean to the
The mathematical model for the aggregation process can be described by the following integro-differential equation,

\[
\frac{dn(D,t)}{dt} = \frac{a}{2} \int_0^\infty \beta(\bar{v}, v - \bar{v}) n(v - \bar{v}, t)n(\bar{v}, t) d\bar{v} - \alpha n(v, t) \int_0^\infty \beta(v, \bar{v}) n(v, \bar{v}, t) d\bar{v} + \frac{\alpha}{\beta(D)} (G(D)n(D,t)) \tag{3.1}
\]

where \(v\) is the volume of the particles, directly proportional to \(D^3\) for solid particles, and \(\beta(\bar{v}, v - \bar{v})\) is the coagulation kernel, measuring the frequency of particles' coagulation. First we assume the initial sizes of marine particles is in log-normal distribution, due to the fact that mostly the diameter of a particle is determined by multiplicative factors like dynamic of marine ecological system, ocean currents, and aqua-chemical reactions and use the the moment method introduced in [15]. But later we studied it generally for any initial size distribution.

To solve the integro-differential equation (3.1), we use the quadrature method of moments, which is finding the weights and abscissas of the sizes to discretize the integral in the moments of the size distribution of particles. We wrote a program to realize the product-difference algorithm that is used to approximate the moments of size distribution in the moments of the size distribution of particles. We wrote a program to realize the aggregation and growth equation iteratively in a program with loop structure.

In addition to the aggregation and growth processes of marine particles, the sinking process also occurs sometimes in marine environment. Considering the sinking process in the model, we have the following equation

\[
\frac{dn(D,t)}{dt} = \frac{a}{2} \int_0^\infty \beta(\bar{v}, v - \bar{v}) n(v - \bar{v}, t)n(\bar{v}, t) d\bar{v} - \alpha n(v, t) \int_0^\infty \beta(v, \bar{v}) n(v, \bar{v}, t) d\bar{v} + \frac{\alpha}{\beta(D)} (G(D)n(D,t)) - n(D,t) \frac{w_s(D)}{h}, \tag{3.2}
\]

where \(w_s(D)\) is the settling velocity and \(h\) is the depth, and it would be interesting to explore further along this direction.

3.2. Mathematical Modeling for the Functional Decomposition of Ecological Networks. A complex ecological system can be decomposed into simple chains and cycles. We study the functional decomposition of ecological networks from the perspective of probability and show that a complex ecological system can be uniquely decomposed functionally into simple chains and cycles.

**Proposition 7.** Let \(C_1, C_2, \ldots, C_n\) be the compartments of a network system \(S, \mathcal{F}\) be a chain containing compartments \(C_{n_0}\) with input from the environment \(z_{n_0} > 0\) and \(C_{n_1}, C_{n_2}, \ldots, C_{n_k}\) with no inputs from the environment and \(F_{out}\) be the flow out of compartment \(C_{nj}, j = 1, 2, \ldots, k\) in \(\mathcal{F}\), and \(P(\mathcal{F})\) be the probability that a particle goes through \(\mathcal{F}\). Then

\[
P(\mathcal{F}) = \sum_{i=1}^{n} \prod_{j=1}^{k} \frac{F_{out}^{nj}}{T_{out}^{nj}} \tag{3.3}
\]

where \(T_{out}^{nj}\) is throughflow(sum of outflows)out of compartment \(i\).

We have derived the probabilities that a particle goes through cycles in [18].
Theorem 8. Let $C$ be a cycle containing compartments $C_{m_1}, C_{m_2}, \ldots, C_{m_k}$ and $F_{\text{out}}$ be the flow out of compartment $C_{m_j}, j = 1, 2, \ldots, k$ in $F$, and $\mathbb{P}(C)$ be the probability that a particle goes through $C$. Then

$$\mathbb{P}(C) = \prod_{j=1}^{k} \frac{F_{\text{out}}}{T_{\text{out}}^m}.$$  

(3.4)

We define the expected times of a flux appearing on a pathway in term of the expectation as follows

Definition 9. Let $F$ be a flux, $L$ be a random pathway, $\mathbb{P}(L)$ is the probability of pathway $L$ occurring, $n(L, F)$ is the number of times flux $F$ appearing in pathway $L$. Then we can define the Expected Timed of this flux appearing in a pathway by,

$$E(F) := \sum_{\text{path } L} n(L, F) \mathbb{P}(L),$$

in which $\mathbb{P}(L)$ is the product of flow probabilities that make up the pathway, $E(F)$ is a sum over all pathways. It could be an infinite sum, and $n(L, F)$ can be larger than one only if $F$ is a cycle.

We have the following theorem on the expected times of a flux appearing in a pathway in [18].

Theorem 10. Let $F_1, F_2, \ldots, F_n$ be the chains and $C_1, C_2, \ldots, C_m$ be the cycles in the decomposition. Suppose in the network system cycles that are attached to a chain either disjoint or share a flow of the chain, then the expected times that the chains

$$E(F_i) = \mathbb{P}(F_i) \frac{1}{1 - \sum_{C_j \cap F_i \neq \emptyset} \mathbb{P}(C_j)},$$

(3.6)

for all $i$, and

$$E(C_i) = \frac{\mathbb{P}(C_i) \sum_{C_j \cap F_i \neq \emptyset} \mathbb{P}(F_i)}{\left(1 - \sum_{C_j \cap C_i \neq \emptyset} \mathbb{P}(C_j)\right)^2},$$

(3.7)

for all $i$.

Using the above expected times of a flux appearing in a pathway, we can do the functional decomposition for any network system.

Theorem 11 (Functional decomposition of a network system). Let $F_1, F_2, \ldots, F_n$ be the chains and $C_1, C_2, \ldots, C_m$ be the cycles in the decomposition. Then in the functional decomposition, the coefficient of $F_i$,

$$c_i = \frac{E(F_i)}{\mathbb{P}(F_i)} \sum_{i=1}^{n} z_i,$$

(3.8)

for $i = 1, 2, \ldots, n$ and the coefficient of $C_j$,

$$d_j = E(C_j) \sum_{i=1}^{n} z_i,$$

(3.9)

for $j = 1, 2, \ldots, m$. 
4. Convex Geometry

I have worked in the area of convex geometry in Minkowski spaces, which are vector spaces with Minkowski metrics, and studied the Holmes-Thompson volumes of a convex body \( K \), \( HT_k(K) \) in the Minkowski spaces, which are defined as the symplectic volume of the codisc bundle over \( K \) and can be extended to Holmes-Thompson valuations. We start with the real Minkowski spaces and focus on the complex Minkowski spaces. In a complex Minkowski space \( (\mathbb{C}^n, F) \), where \( F \) is the complex Finsler metric, we explore the valuation theory extended from the Holmes-Thompson volume. In the work, [20], we obtain the following theorem on a characterization of complex Minkowski metric \( C_n \) to be Hermitian,

**Theorem 12** (A characterization of Hermitian metric). Suppose that \( (\mathbb{C}^n, F) \) is a complex Minkowski space. Then the Holmes-Thompson valuation, that is extended from the Holmes-Thompson volume on \( (\mathbb{C}^n, F) \), restricted on \( \mathbb{CP}^{n-1} \) is in the range of the cosine transform on \( C(\mathbb{CP}^{n-1}) \) if and only if the complex Minkowski metric \( F \) is Hermitian.

In [19], we study the Lagrangian subspaces of complex Minkowski spaces under the Kähler form

\[
\kappa = \sqrt{-1} \sum_{i,j=1}^n \frac{1}{2} \frac{\partial^2 (F^2)}{\partial \xi_i \partial \bar{\xi}_j} d\xi_i \wedge d\bar{\xi}_j,
\]

(4.1)

for complex Minkowski space \( (\mathbb{C}^n, F) \), and worked on the cases of complex \( L^1 \) and \( L^\infty \) spaces. As the case of \( \mathbb{C}^n \), we would like to find some canonical valuations as a basis in terms of multiple Kähler angle for the space of valuations on \( (\mathbb{C}^n, F) \) as well.

I have worked on the image of cosine transform of distributions, for which there were large computations involved, and have done some programming to make the computations efficient. In [20], we give a parametrization for Grassmannians by using elementary linear algebra, and characterize the image of cosine transforms of distributions in two different ways, using Mathematica to efficiently compute the coefficients in the Legendre series expansion of distributions and to see the patterns.

**Theorem 13** (Image of cosine transform on distributions). The image of the cosine transform on the space \( D(Gr_2(\mathbb{C}^2)) \) that consists of torus invariant distributions on \( Gr_2(\mathbb{C}^2) \) is the space

\[
G := \left\{ \sum c_{k,l} p_{2k}(x) p_{2l}(y) : c_{k,l} \in \mathbb{R} \right\},
\]

(4.2)

where \( p_{2k}(x) \) is the Legendre polynomial of degree \( 2k \).

5. Future Research Plans

5.1. Duality on Generalized Singular Values of Rectangular Matrices. The singular values are important values for a matrix or linear operator. The classic singular values are defined under the \( \ell_2 \) norm of a vector, in particular, the largest classic singular value is the operator norm of a linear operator from \( (\mathbb{R}^n, \| \cdot \|_2) \) to \( (\mathbb{R}^m, \| \cdot \|_2) \). Similarly, the matrix \( p \)-norm of a matrix \( A \) defined by \( s_1^{(p)}(A) = \sup_{x \in \mathbb{R}^n, \| x \|_p = 1} \| Ax \|_p \) is the largest \( p \)-singular value of a matrix \( A \), which is also the operator norm of the linear operator \( A \) from \( (\mathbb{R}^n, \| \cdot \|_p) \) to \( (\mathbb{R}^m, \| \cdot \|_p) \). In the last decade, many research works have been concentrated on estimating their values for \( p \neq 1, 2, \infty \), see for instance, [13] [14]. From the recent results in [13], matrix \( p \)-norms are NP-hard to approximate if \( p \neq 1, 2, \infty \).
By the classic singular values, a rectangular matrix has a singular value decomposition, which has an implication on the duality between its singular values and the singular value of its adjoint. Motivated by the duality on the classic singular values, we study the duality on the duality between the singular values of a linear operator from \((\mathbb{R}^n, \|\cdot\|_p)\) to \((\mathbb{R}^m, \|\cdot\|_p)\) and the singular values of its adjoint operator from \((\mathbb{R}^m, \|\cdot\|_q)\) to \((\mathbb{R}^n, \|\cdot\|_q)\), where \(\frac{1}{p} + \frac{1}{q} = 1\).

The duality on the \(p\)-singular value has applications in estimating the \(p\)-singular value of random matrices and \(\ell_p\) minimization problems in image processing and image recovery.

5.2. Lower Rank Approximation: A Generalization of Schmidt-Mirsky Theorem and Algorithms. The extensions on the lower-rank approximation in different directions were studied. For examples, in [11], explicit solution to the rank-constrained matrix approximation in Frobenius norm, which is a generalization of the classical approximation of an \(m \times n\) matrix \(A\) by a matrix of, at most, rank \(k\), was given. However, we want to generalize the rank-constrained matrix approximation to the matrix \(p\)-norms from the classical \(\ell_2\)-norm in which the singular value decomposition plays a substantial role.

For \(p \leq 1\), let \(B\) be the matrix, the computation of \(q\)-singular value is relatively easier, because we have a good property on the \(q\)-singular value, see [17].

We want to find the solution of the following minimization problem: given a matrix \(A\) of size \(m\) by \(n\), for any \(k \leq \min(m, n)\) and \(p \neq 2\), find the solution matrix \(B\) with \(\text{rank}(B) \leq k\) achieving the minimal

\[
\min_{\text{rank}(B) \leq k} \|A - B\|_p.
\]

(5.1)

For \(k = 0\), the solution matrix \(B\) is simply the zero matrix of size \(m\) by \(n\), and in this case \(\min_{\text{rank}(B) \leq k} \|A - B\|_p = \|A\|_p\); and for \(k = \min(m, n)\), the solution matrix \(B\) is simply \(A\), and in this case \(\min_{\text{rank}(B) \leq k} \|A - B\|_p = 0\). So these two cases are trivial.

We are also able to design algorithms to find the solution to the lower rank approximation problems for some other non-trivial cases, but there are still cases remained to be solved. The algorithms for solving the lower rank approximation problems can be applied to matrix completion and sparse matrix recovery.

5.3. Applications to compressed sensing and neuroimaging. Among of the applications of random matrices, random matrices can be used as the sensing matrices in compressed sensing, which is an active research area in recent years. But the PI is very interested in working on neuroimaging by joint compressed using random matrices. To study the behavioral and learning functions of the brain, one needs to monitor the responses from multiple locations of the brain simultaneously from stimulus, but to efficiently measure the brain or neural signals or the activities of some neurons in the brain, joint sparse recovery in compressed sensing helps improving the efficiency, because it requires fewer number of measurements and has superior advantages in fast computations. Yet, the success of the joint sparse recovery requires some conditions on the ways or perspectives in the process of taking measurements. As such, there are significant computational benefits in achieving successes in signal recovery and especially in neuroimaging. So, it would provide an important tool for scientists to map the activity of the neurons in the human brain efficiently and economically, as the proposed approaches in random matrices based on the approach in [9] would have several advantages in computational speed and recovery rate.


References


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