1. Let \( u, v, \) and \( w \) be vectors in \( \mathbb{R}^3 \). For each of the following, are the quantities vectors, scalars, or undefined?

(a) (2 points) \( (u \times v) \cdot w \) \hspace{1cm} \text{Scalar} \hspace{6cm} \text{undefined}

(b) (2 points) \( (u \cdot v) \cdot w \) \hspace{1cm} \text{undefined}

(c) (2 points) \( (u \times v) \cdot (w \times u) \) \hspace{1cm} \text{Scalar} \hspace{6cm} \text{vector}

(d) (2 points) \( u \times (w \times u) \) \hspace{1cm} \text{undefined}

(e) (2 points) \( (u \cdot v) \times (w \cdot u) \) \hspace{1cm} \text{undefined}

2. Consider the function \( f(x, y) = 10 - x^2 - 2y^3 \).

(a) (8 points) Find the linearization of the function \( f \) at the point \( (3, 2) \).

\[
\begin{align*}
\frac{\partial f}{\partial x} &= -2x \\
\frac{\partial f}{\partial y} &= -6y^2 \\
 f_x(3,2) &= -6 \\
 f_y(3,2) &= -24 \\
 f(3,2) &= 10 - 9 - 2(8) = -15 \\
 L(x,y) &= -15 - 6(x-3) - 24(y-2)
\end{align*}
\]

(b) (2 points) Use the linearization to estimate \( f(3.1, 1.9) \).

\[
\begin{align*}
 f(3.1, 1.9) &\approx L(3.1,1.9) = -15 - 6(3.1-3) - 24(1.9-2) \\
 &= -15 - 6(0.1) + 24(0.1) \\
 &= -15 - 1.8 = -13.2
\end{align*}
\]
3. Acceleration function of a particle is given by

\[ a(t) = \langle 0, 0, -10 \rangle. \]

If the initial velocity of the particle is \( v(0) = \langle 12, 5, 80 \rangle \) and the initial position is \( r(0) = \langle 0, 0, 30 \rangle \), find:

(a) (6 points) The velocity function \( v(t) \).

\[ v(t) = \int a(t) = \langle 0, 0, -10 \rangle t + \langle 12, 5, 80 \rangle \]

(b) (6 points) The position function \( r(t) \).

\[ r(t) = \int v(t) = \langle 0, 0, -10 \rangle \frac{t^2}{2} + \langle 12, 5, 80 \rangle t + \langle 0, 0, 30 \rangle \]

(c) (8 points) Find its maximum height. In other words, maximize the \( z \)-coordinate of \( r(t) \).

\[ z(t) = \frac{-10t^2}{2} + 80t + 30 = -5t^2 + 80t + 30 \]

\[ z'(t) = -10t + 80 = 0 \quad \therefore \quad t = 8 \]

\[ z(8) = -5(8)^2 + 80(8) + 30 \] is the maximum height.
4. (6 points) Find the arc length of \( r(t) = (\cos(4t), 2t - 10, \sin(4t)) \) from \( t = 0 \) to \( t = 5 \).

\[
\mathbf{r}'(t) = \langle -4 \sin(4t), 2, 4 \cos(4t) \rangle
\]

\[
\int_0^5 \sqrt{4 \cos^2(4t) + 2^2 + 4 \sin^2(4t)} \, dt = \int_0^5 \sqrt{20} \, dt
\]

\[
= 5 \sqrt{20}
\]

5. (8 points) Find a vector normal to the plane passing through the points \( P(1, 0, 1) \), \( Q(2, 1, 1) \), and \( R(1, 1, 0) \).

\[
\mathbf{u} = \overrightarrow{PQ} = \langle 1, 1, 0 \rangle \quad \mathbf{u} = \overrightarrow{PR} = \langle 0, 1, 1 \rangle
\]

\[
\mathbf{n} = \mathbf{u} \times \mathbf{v} = (\mathbf{i} + \mathbf{j}) \times (\mathbf{j} - \mathbf{k}) = \mathbf{i} \times \mathbf{j} - \mathbf{i} \times \mathbf{k} + \mathbf{j} \times \mathbf{i} - \mathbf{j} \times \mathbf{k}
\]

\[
= \mathbf{k} + \mathbf{j} + \mathbf{0} - \mathbf{i} = \langle -1, 1, 1 \rangle
\]

6. (6 points) Consider the function \( F(x, y, z) = x^3 - xz + y^4 \). Find the equation of the tangent plane to the level surface \( F(x, y, z) = k \) at the point \((-2, 0, 3) = \mathbf{P}\).

\[
F_x = 3x^2 - z \quad F_x(\mathbf{P}) = 9
\]

\[
F_y = 4y^3 \quad F_y(\mathbf{P}) = 0
\]

\[
F_z = -x \quad F_z(\mathbf{P}) = 2
\]

\[
9(x + 2) + 2(z - 3) = 0
\]
7. (3 points) Find the partial derivative \( \frac{\partial V}{\partial t} \) for

\[
V = x^2y - xy^3, \quad x = \frac{s^2}{t}, \quad y = \ln(st).
\]

Your final answer should include only the variable \( s \) and \( t \).

\[
\frac{\partial V}{\partial t} = \frac{\partial V}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial V}{\partial y} \frac{\partial y}{\partial t}
\]

\[
V_x = 2xy - y^3 = 2 \frac{s^2}{t} \ln(st) - \left(\ln(st)\right)^3
\]

\[
V_y = x^2 - 3xy^2 = \left(\frac{s^2}{t}\right)^2 - 3 \frac{s^2}{t} \left(\ln(st)\right)^2
\]

\[
x_t = -\frac{s^2}{t^2}, \quad y_t = \frac{1}{st}, \quad s = \frac{1}{t}
\]

\[
\frac{\partial V}{\partial t} = \left(2 \frac{s^2}{t} \ln(st) - \left(\ln(st)\right)^3\right)\frac{1}{t}
\]

8. (4 points) Find the maximum rate of change of \( f(x, y) = x^2 \sqrt{y} \) at the point \((-1, 4)\). In which direction does it occur?

\[
\nabla f = \langle f_x, f_y \rangle = \langle 2x \sqrt{y}, \frac{x^2}{2} \sqrt{y} \rangle
\]

\[
\nabla f (-1, 4) = \langle -2 \cdot 2, \frac{1}{2} \cdot \frac{1}{2} \rangle = \langle -4, \frac{1}{4} \rangle
\]

The maximum rate of change is \( |\langle -4, \frac{1}{4} \rangle|\)

\[
= \sqrt{4^2 + \frac{1}{4^2}}
\]

In the direction of \( \nabla f(-1, 4) \) which is \( \langle -4, \frac{1}{4} \rangle \)
9. Evaluate the following limits, if they exist. If not, state the reason.

(a) (5 points)

\[ \lim_{{(x,y) \to (0,0)}} \frac{5x^2}{x^2 + 2xy + y^2} \]

\(\begin{align*}
&0 \land x = 0, \quad \frac{5x^2}{x^2 + 2xy + y^2} = 0 \to 0 \text{ as } y \to 0 \\
&0 \land y = 0, \quad \frac{5x^2}{x^2 + 2xy + y^2} = \frac{5x^2}{x^2} = 5 \to 5 \text{ as } x \to 0
\end{align*} \)

So the limit ONE as it depends on the path.

(b) (5 points)

\[ \lim_{{(x,y) \to (0,0)}} \frac{5x^3}{x^2 + y^2} \]

\[0 \leq \left| \frac{5x^3}{x^2 + y^2} \right| \leq \frac{5|1|x^2}{x^2} \leq \frac{5|1|x^2}{x^2} = 5|1| \to 0\]

as \((x, y) \to 0\)

So by the Squeeze Theorem,

\[ \lim_{{(x,y) \to (0,0)}} \frac{5x^3}{x^2 + y^2} = 0 \]
10. (12 points) Find and classify all the critical points of \( f(x, y) = x^3 - x^2 - y^2 \).

\[ f_x = 3x^2 - 2x = 0 \]
\[ f_y = -2y = 0 \quad y = 0 \]
\[ x(3x-2) = 0 \quad \text{so either } x = 0 \]
\[ \text{or } x = \frac{2}{3} \quad \Rightarrow (0,0) \text{ and } \left(\frac{2}{3},0\right) \text{ are crit. pts.} \]
\[ f_{xx} = 6x - 2 \quad f_{yy} = -2 \quad f_{xy} = 0 \]
\[ D(0,0) = \begin{vmatrix} -2 & 0 \\ 0 & -2 \end{vmatrix} = 4 > 0 \]
\[ f_{xx}(0,0) = -2 < 0 \quad \Rightarrow (0,0) \text{ is a local max.} \]
\[ D\left(\frac{2}{3},0\right) = \begin{vmatrix} 4 & 0 \\ 0 & -2 \end{vmatrix} = -8 < 0 \]
\[ \Rightarrow \left(\frac{2}{3},0\right) \text{ is a saddle pt.} \]

11. (6 points) Find the directional derivative of \( h(x, y) = xy^2 \) at the point \((0,2)\) in the direction of \( \langle 1, 3 \rangle \).

\[ \nabla h = \langle h_x, h_y \rangle = \langle y^2, 2xy \rangle \]
\[ \nabla h(0,2) = \langle 4, 0 \rangle \]
\[ u = \frac{\langle 1, 3 \rangle}{\sqrt{10}} \]
\[ D_u h = \langle 4, 0 \rangle \cdot \left(\frac{1}{\sqrt{10}} \langle 1, 3 \rangle\right) = \frac{1}{\sqrt{10}} \cdot 4 \]