The partial differential equation transform – a robust method for signal, image and data analysis

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Abstract

A new method for analysing nonlinear and non-stationary data has recently been developed, namely the partial-differential-equation (PDE) transform. The PDE transform is based on the use of coupled arbitrarily high order PDEs to decompose signals and images into various functional frequency modes, which allow perfect reconstruction of the original signals and images. The PDE transform is compared with the Fourier transform, the wavelet transform, and empirical mode decomposition for analysis of various types of data in real time signals and images.

Key words: Mode decomposition; High order evolution equations; High-pass filters; Low-pass filters; Intrinsic mode functions; PDE transform.
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I Introduction
Analysis of data, which can be signals, real time series, or images, is an important research field in both pure and applied mathematics with huge impact in a wide category of physical sciences and engineering. In nowadays’ practices, data for analysis is usually complicated, being non-stationary and nonlinear.

Conventional filtering is performed directly in the frequency domain by operating on the signal’s frequency spectrum via Fourier transform. Fourier or spectral analysis is the classical technique for mode decomposition which still remains to be a powerful tool in signal and data processing. Since the frequency spectrum contains global information about the whole signal, the frequency domain filters always work efficiently, e.g., in image processing. However, modern signal processing usually encounters high-throughput and highly non-stationary data sets which fall out of the optimal capability of the Fourier method since the Fourier analysis is not data adaptive. In many applications in signal processing, one usually desires the information of detailed position and momentum relation, which is also lack in Fourier spectral analysis since it may be difficult to choose a suitable window size for use in Fourier method to satisfy the conflicting requirements of localizing an event in time and resolving its frequency distribution. In addition, most data sets to be processed nowadays are usually statistically very complicated and involve abundantly many frequency modes, each of which has a specific physical meaning and features (i.e. functional modes, by which we mean the components which share same band of frequency as well as same category, e.g. trend, edge, texture, etc), embedded in equally abundant number of modes which can be noise or statistically irrelevant. Therefore, the subsequent analysis or secondary processing of these modes becomes extremely awkward if the mode decomposition is not fully automatic.

Wavelet transform is another popular technique which are applicable to non-stationary signal and image processing. The key to the success of wavelet methods is the use of localized bases, in comparison to the sine and cosine bases used in Fourier methods, for achieving better locality and adaptiveness. I.e., wavelets realize the locality by resorting to localized multiscale bases. Wavelets thus provide a class of filter bank to decompose signal into various frequency sub-bands on different temporal scales of resolutions. Wavelets are usually more efficient than the Fourier methods in terms of smaller number of sub-bands decomposed, and it also has better control of the spatial-temporal localization. Overall, the wavelet methods are more adaptive than Fourier analysis. However, in spite of the many appealing features of this versatile method, especially its extraordinary compression ratio, wavelet analysis is basically a linear analysis and suffers from many limitations inherited from the Fourier analysis since many commonly used wavelets were originated from Fourier analysis. The down sides include uniformly poor resolution and frequency localization, sometimes counter-intuitive interpretation, and non-data adaptive nature as the same wavelet basis is used to analyze all the data.

On the other hand, mode decomposition has been well understood in approximation theory as a projection of signal onto certain orthonormal basis. Fourier method can be viewed as a form of polynomial projection, while wavelets can be constructed by class of spline functions, etc. Recently, various empirical mode decomposition (EMD) algorithms were proposed for analysis of non-stationary and nonlinear data beyond the classical spectrum methods. The filtering is done in the real domain or time domain in comparison to the frequency domain. Such a subtle difference is critical in dealing with non-stationary signal processing. The key idea of the whole class of EMD method is focusing on the instantaneous frequency (IF) instead of global Fourier spectrum. Specifically, the EMD defines and extracts series of IF based on the local characteristic time scales of the data by requiring these intrinsic mode functions (IMF) to have equal (or different by one) number of extrema and zero crossings and to have zero mean value of the envelopes defined by the local extrema. For example, in the the sifting algorithm of EMD methods, the local maxima and minima of the original signal are respectively connected through cubic splines to form upper and lower envelopes, and the average of the two envelopes is then subtracted from the original data. Therefore, the finest local mode is separated from the data based only on the characteristic time scales.

In general, the class of EMD methods admit better behaved Hilbert transforms and leads to meaningful extraction of IFs for nonlinear and non-stationary data. The methods have found important applications in
fields like material texture analysis,\textsuperscript{30,39} handwriting recognition,\textsuperscript{67} geology and seismology,\textsuperscript{23,51} optics and sound waves,\textsuperscript{13} weather and wind,\textsuperscript{21,65} ocean wave and turbulence,\textsuperscript{21,31} and biological and biomedical sciences.\textsuperscript{5,12,38,68} But there still remain many important issues and concerns to be addressed and further improved. To briefly summarize, stability and robustness of EMD algorithms are biggest concerns in practical applications, despite of several progresses along the direction,\textsuperscript{64} while building a mathematical foundation remains a big theoretical challenge in the study of EMD.\textsuperscript{10,22,29} The accuracy of EMD mode decomposition is also not comparable to global spectral methods like Fourier method when the latter is applicable. Physical interpretation can be difficult or irrelevant since too many empirical modes are decomposed using EMD, even though certain type of stopping criterion is used in EMD to improve the physical sense retained in the IMFs. Overall, finding the few important modes out of all is itself quite empirical in some cases. More details regarding the limitations and improvements of EMD can be found in the discussion section.

On the other hand, PDE approaches have been widely used in signal and image processing in the recent decade.\textsuperscript{2,8,24,37,41} Back in 1980’s, Witkin introduced the diffusion equation for image denoising.\textsuperscript{63} The evolution of image under diffusion operator is equivalent to the application of Gaussian low-pass filter. Perona and Malik later improved the efficiency of denoising while preserving edges of images by introducing anisotropic diffusion equation.\textsuperscript{36} High order PDEs have been realized for their important role in improving the efficiency of noise removing.\textsuperscript{7,16,32,48,69}

For more efficient and effective analysis of the non-stationary signals and data, and partially motivated by the iterative filtering decomposition (IFD)\textsuperscript{29,54,71} and arbitrarily high order partial differential equations (PDE) based filters,\textsuperscript{57,60} the PDE transform was recently proposed to achieve controllable localization and ideal band-pass filter features in extracting IMFs.\textsuperscript{53,55} These methods have been shown to perform the desired full-scale mode decomposition for general type of signals with high accuracy and robustness. In addition, with good knowledge of the signal and appropriate tuning of the relevant parameters, the modes decomposed are usually attributed with clear physical interpretation and can be “horizontally” compared,\textsuperscript{53–55} since only those physically meaningful modes are decomposed, compared to the many more but not all physically relevant empirical modes yielded by EMD methods. The PDE transform greatly improve the numerical stability, accuracy and robustness of the algorithm as well as introduce more rigorous mathematical foundation. In this paper, we analyze the PDE transform, and compare it with the other main stream methods such as the Fourier transform, the wavelet transform, and the empirical mode decomposition. This paper also presents rigorous mathematical analysis of the PDE transform for non-stationary signal processing in the filter language. Specifically, this paper proposes practical principles for designing improved EMD algorithms.

The rest of the paper is organized as follows. In section II, we introduce the theory of the PDE transform, and compare it with other major transform algorithms for data analysis. In section III, various algorithms for implementing PDE transform in different applications are discussed. In section IV we conclude.

II Theory of the PDE transform

The PDE transform was recently proposed as analysis tool for general signal and images especially when the data are nonlinear and non-stationary.\textsuperscript{53–55} The PDE transform was partially motivated by a few important ideas and algorithms, in particular the empirical mode decomposition and local spectral analysis. The introduction to the the theory of PDE transform is thus divided into four sections as follows: empirical mode decomposition based on Hilbert-Huang transform, local spectral method and evolution kernel, PDE transform and adaptive algorithms, and various filter design for implementation purpose.

II.A Frequency and scale analysis of non-stationary data

Spectral analysis and Fourier transform Fourier analysis is the most basic technique which remains very useful for both signal and image processing of stationary data. Fourier represent functions as a superposition of sines and cosines, and analyzes a signal in the time domain for its frequency content. If the signal is nonperiodic, the summation of the periodic functions, sine and cosine, does not accurately represent the signal. One could always artificially extend the signal to make it periodic but it would require additional continuity at the endpoints. Therefore, the windowed Fourier transform (WFT) was
proposed to give information about signals simultaneously in the time domain and in the frequency domain. To apply the WFT, one first chops the input signal into sections, each of which is analyzed for its frequency content separately. If the signal has sharp transitions, we window the input data so that the sections converge to zero at the endpoints. This windowing is accomplished via a weight function that places less emphasis near the interval's endpoints than in the middle. The effect of the window is to localize the signal in time. Based on the limited time window-width Fourier spectral analysis, one can get a time-frequency distribution by successively sliding the window along the time axis. Being for sure that such spectrogram method improves upon the conventional Fourier analysis, however, there remains many great challenges in applying the method towards non-stationary data analysis. In order to localize signal in temporal space, the window width must be adequately and sufficiently narrow, which on the other hand limits the frequency resolution. In principle, use of global basis functions in Fourier analysis is inconsistent with the dual spatial-temporal localization.

**Wavelet transform** Being one of the best analysis tool for non-stationary data, the core of wavelet approach is, though, intimately related to, or inherently recognized as an adjustable window Fourier spectral analysis with the following general definition

\[
W(a, b; X, \psi) = |a|^{-1/2} \int_{-\infty}^{\infty} X(t) \psi^* \left( \frac{t - b}{a} \right) dt
\]  

in which \( \psi(t) \) is the basic wavelet function that satisfies certain general conditions. The form of the mother wavelet \( \psi \) must be given before, and is fixed in the analysis. On the contrary to the frequency analysis in Fourier method, wavelet is rather a scale analysis of signal data \( X(t) \) with \( a \) and \( b \) in the above definition being dilation and translation factors, i.e., \( 1/a \) gives frequency scale and \( b \) specifies the temporal location. \( W(a, b; X, \psi) \) thus specifies the time-frequency distribution of the energy of signal \( X(t) \). Same as the Fourier method, wavelet analysis is basically a linear analysis. One of the many dissimilarities between the Fourier and the wavelet transforms is that individual wavelet functions are localized in space while the Fourier sine and cosine functions are not. Such dual time-frequency localizations leads to sparsity of the functions transformed into the wavelet domain. This sparseness, in turn, is critical for many applications including data compression, edge detection, and noise removal.

The time-frequency resolution difference between the two transforms is better understood by looking at the basis function coverage of the time-frequency plane. In the Fourier transform, the window is simply a square wave to truncate the sine or cosine function to fit a window of a particular width. The resolution of the analysis is the same at all locations in the time-frequency plane. In the wavelet transforms the windows vary. Basis functions need to be very short if one wants to isolate signal discontinuities, and need to be very long if one wants to obtain detailed frequency analysis. The wavelet transform achieves this by having short high-frequency basis functions and long low-frequency ones.

In addition, wavelet is characterized with the double-sworded characteristic: it provides a uniform resolution for all the scales as well as associated uniformly poor resolution due to the limit in the size of the basic wavelet function. Nevertheless, wavelet has been very powerful and popular tool in analyzing data with gradual frequency changes, especially in edge detection and image compression. Another great concern regarding the application of wavelet lies in its sometime counter-intuitive interpretation. For example, to detect a local (temporal) event, one has to plead to the signal's high frequency characteristics. Such a detour is especially problematic when the local event is identified primarily as a low-frequency event.

**Instantaneous frequency** The definition and calculation of the instantaneous frequency (IF) have been important for the analysis of non-stationary data. On the contrary to the “global” frequency in Fourier analysis, the concept of IF was initially proposed to particularly address the dual temporal-frequency localization and the corresponding time-frequency distribution of the energy of non-stationary data. Meanwhile, the notion of IF itself has been highly controversial. In the traditional Fourier analysis, the frequency is necessarily defined for the global basis of sine or cosine functions spanning the whole data length in the temporal space (or physical space). Such temporal localization associated with IF is inherently contradicting with the Fourier spectrum, though necessary for the non-stationary data analysis.
Besides the conceptual confusion, practically it is also difficult to uniquely define the IF. But this difficult has been partially eased by introducing the Hilbert transform for such data analysis.

Hilbert transform was first proposed in Hilbert’s 1905 work on the Riemann-Hilbert problem. In 1928, Marcel Riesz generalized the Hilbert transform from $L^2$ space to general $L^p(R)$ for $1 < p < \infty$. The Hilbert transform was a motivating example for Antoni Zygmund and Alberto Calderon during their study of singular integrals which was important for the modern harmonic analysis. The Hilbert transform $\mathcal{H}[X(t)]$ of the signal $X(t)$ is defined as convolution of the $X(t)$ with the signal $1/\pi t$, i.e.,

$$\mathcal{H}[X(t)] = X(t) * \frac{1}{\pi t} = \frac{1}{\pi} \cdot PV \int_{-\infty}^{\infty} X(\tau) \frac{1}{t-\tau} d\tau$$

(2)

where the Cauchy principle value ($PV$) is taken in the integral due to the fact that the signal $1/\pi t$ is not integrable such that Hilbert transform is explicitly defined as

$$\mathcal{H}[X(t)] = \frac{1}{\pi} \lim_{\epsilon \to 0^+} \left( \int_{t-1/\epsilon}^{t+1/\epsilon} \frac{X(\tau)}{t-\tau} d\tau + \int_{t-1/\epsilon}^{t+1/\epsilon} \frac{X(\tau)}{t-\tau} d\tau \right)$$

(3)

By definition, Hilbert transform is improper integral. Therefore, it is by no means obvious that the Hilbert transform is well-defined. However, the Hilbert transform is well-defined and bounded linear operator for a broad class of functions in $L^p(R)$ for $1 < p < \infty$. In addition, since $\mathcal{H}$ preserves the space $L^p(R)$, the Hilbert transform is invertible on $L^p(R)$ and that $\mathcal{H}^{-1} = -\mathcal{H}$. The signal $1/\pi t$ in the Hilbert transform has Fourier transform $-i \cdot \text{sgn}(\omega)$. Therefore, if $X(t)$ has Fourier transform $\hat{X}(\omega)$, the Hilbert transform $\mathcal{H}[X(t)]$ has the Fourier transform $-i \cdot \text{sgn}(\omega) \hat{X}(\omega)$, which implies that the Hilbert transform, when considered in the frequency domain, does not change the magnitude of $\hat{X}(\omega)$. Essentially, Hilbert transform (2) provides an unique way of defining the complex signal $Z(t) = X(t) + i \cdot \mathcal{H}[X(t)]$, where the convolution of $X(t)$ with $1/\pi t$ emphasizes the local (temporal) properties of the signal $X(t)$. It was thus suggested that IF can be defined as

$$\omega = \frac{d}{dt} \tan^{-1} \left( \frac{\mathcal{H}[X(t)]}{X(t)} \right)$$

(4)

The definition (4) is still quite controversial, for the IF thus defined must be a single value function of time, i.e., the original signal $X(t)$ must be mono-component in terms of time-dependent characteristic frequency distribution. Therefore, in order to obtain meaningful instantaneous frequency, restrictive conditions have to be imposed on the data, e.g., the real part of its Fourier transform has to have only positive frequency. Most restrictions are still global, while in analysis of non-stationary data these global restriction conditions have to be translated to local ones.

**Hilbert-Huang transform and empirical mode decomposition** The Hilbert-Huang transform (HHT) was first proposed by Huang et al.\textsuperscript{20-23} The HHT provides a new method of analyzing non-stationary and nonlinear time series data, which are first decomposed into various intrinsic mode function (IMF) by the empirical mode decomposition (EMD) method.\textsuperscript{10,15,22,29,40} Hilbert spectral analysis (HSA) method\textsuperscript{22} is used to obtain instantaneous frequency data from the IMF. In HHT, possibility of decomposing signal into various IMFs is based on the necessary condition that IMFs are symmetric with respect to the local zero mean, and have the same numbers of zero crossings and extrema. By effective constructing splines and repeatedly subtracting them from original signals, IMFs are obtained as follows

$$w_{mn}^k = H_{mn}X_{mn}^k, \quad \forall k = 1, 2, \ldots$$

(5)

where $w_{mn}^k$ is the $k$-th IMF. Here the residue function is given by

$$X_{mn}^k = X_{mn}^1 - \sum_{j=1}^{k-1} w_{mn}^j, \quad \forall k = 2, 3, \ldots$$

(6)
where $X^1_{mn} = X(r)$. Therefore, $X = \sum_{j=0}^{k-1} w^j_{mn} + X^k_{mn}$ is a perfect reconstruction of $X$ in terms of all the mode functions and the last residue. The above process is referred as sifting process if $H_{mn}$ performs as a general low pass filter. Such algorithms lead to a family of empirical mode decomposition (EMD) methods. It is interesting to note that typical spline based or iterative construction of EMD algorithm is essentially a Haar wavelet with adaptive scale and dilation determined by the local minima, except that each mode is of the same size as original signal (and is thus not appropriate for compression purpose).

II.B PDE transform

The PDE transform was recently proposed, partially motivated by the empirical mode decomposition, to perform analysis for general type of data (non-stationary and nonlinear) with efficiency and robustness comparable with wavelet analysis and accuracy comparable with Fourier analysis (when the latter is applicable). The PDE transform was built on the original arbitrarily high order nonlinear PDEs first proposed by Wei group.\(^{57}\)

\[
\frac{\partial u(r,t)}{\partial t} = \sum_j \nabla \cdot [d_j(u(r), |\nabla u(r)|, t)\nabla \nabla^2 u(r,t)] + c(u(r), |\nabla u(r)|, t), \quad q = 0,1,2,\ldots
\]

(7)

where $u(r,t)$ is the image function and $d_j(u(r), |\nabla u(r)|, t)$ and $c(u(r), |\nabla u(r)|, t)$ are edge sensitive diffusion coefficients and enhancement operator respectively. The PDE transform devised for systematic separation of all the mode components, including extremely highly oscillatory signals, images and functions, is thus formulated as follows,

\[
\frac{\partial}{\partial t} \begin{pmatrix} u_m \\ v_n \end{pmatrix} = \begin{pmatrix} \sum_{j=0}^{n-1} \nabla \cdot d_{uj}(|\nabla u_m|)|\nabla \nabla^2 u_m| - c(|\nabla u_m|), c(|\nabla v_n|) \\ \sum_{j=0}^{n-1} \nabla \cdot d_{vj}(|\nabla v_n|)|\nabla \nabla^2 v_n| - c(|\nabla v_n|) \end{pmatrix} \begin{pmatrix} u_m \\ v_n \end{pmatrix}
\]

(8)

where $c(|\nabla u_m|)$ and $c(|\nabla v_n|)$ are edge sensitive coupling coefficients, and $d_{uj}(|\nabla u_m|) \geq 0$ and $d_{vj}(|\nabla v_n|) \geq 0$ when $j$ is even and $d_{uj}(|\nabla u_m|) \leq 0$ and $d_{vj}(|\nabla v_n|) \leq 0$ when $j$ is odd. The magnitudes of $d_{uj}$ and $d_{vj}$ must satisfy the relation $|d_{vj}| >> |d_{uj}| \sim 0$ so that two PDEs evolve with dramatically different time scales. The initial values of Eq. (8) are set to the original data $X$ or its residue as discussed below. One can choose either the Neumann boundary or the periodic boundary, depending on the data $X$. The choice of $m$ and $n$, i.e., the degree of the frequency localization depends on the nature of the application. The different stages of evolutions lead to a band-pass filter. In principle, we let $m \rightarrow \infty$ and particularly $n \rightarrow \infty$ so that the corresponding PDF operator $H_{mn}(r,t)$ becomes a perfect band-pass filter and has the highest frequency localization. Based on the Eq. (8), band-pass filter can be constructed as follows,\(^{53,55,60}\)

\[
w_m(r,t) = u_m(r,t) - v_n(r,t) = H_{mn}(r,t)X(r),
\]

(9)

where $H_{mn}(r,t)$ denotes the nonlinear PDE operator, and signals or images $u_m(r,t)$ and $v_n(r,t)$ are obtained via arbitrarily high order PDE transform.\(^{53,60}\)

In such a framework, the PDE transform can be used to systematically extract IMFs from residues

\[
w^k_{mn} = H_{mn}X^k_{mn}, \quad \forall k = 1,2,\ldots
\]

(10)

where $w^k_{mn}$ is the $k$-th mode function. Here the residue function is given by

\[
X^1_{mn} = X(r)
\]

(11)

and

\[
X^k_{mn} = X^1_{mn} - \sum_{j=1}^{k-1} w^j_{mn}, \quad \forall k = 2,3,\ldots
\]

(12)

IMFs obtained from Eq. (10) depend on $m, n, d_{uj}, d_{vj}$ and the evolution time.\(^{53,55}\) Note that the mode decomposition procedure in Eq. (10) is nonlinear, even if linear PDE operators are used. Similar to the
IHFD, each IMF \( w_{k,m} \) extracts mode with lower frequency than the previous one. Consequently, the PDE transform has a perfect reconstruction as indicated by the equality

\[
X(t) = \sum_{i=1}^{m} F_i + R
\]  

(13)

where \( R \) is the residue and \( m \) is total number of IMFs. In this IHFD, each IMF \( F_k \) extracts mode with lower frequency than the previous one. Finally, the residue retains the trend of \( X \). In addition, each mode contains multiple instantaneous frequency, much like the wavelet transform but with total controllability and spectral-accuracy. With well chosen parameters, each mode contains certain distinctive physical property and is thus called functional mode. It has been illustrated that such functional modes can refer to selective textures in image processing, natural landscape segmentation, multiscale neuron skeleton analysis, etc.

II.C Comparison of various data analysis methods

<table>
<thead>
<tr>
<th>Partial Differential Equation Transform</th>
<th>Hilbert-Huang transform</th>
<th>Wavelet transform</th>
<th>Fourier transform</th>
</tr>
</thead>
<tbody>
<tr>
<td>Only yield the relevant functional modes</td>
<td>Each sub-band width is totally controllable</td>
<td>Each mode contains selected frequency range</td>
<td></td>
</tr>
<tr>
<td>Each mode contains desired frequency range</td>
<td>Each mode function is determined by PDE order and evolution time</td>
<td>Physical domain representation</td>
<td></td>
</tr>
<tr>
<td>Mode is extracted using accurate high order PDEs based band-pass filters</td>
<td>Adjustable dual temporal-frequency localization</td>
<td>Applicable for non-stationary signal and no Gibbs oscillations</td>
<td></td>
</tr>
<tr>
<td>Instantaneous frequency is obtained for characterizing non-stationary data</td>
<td>Dual time-scale analysis</td>
<td>Perfect localization in frequency space</td>
<td></td>
</tr>
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<td></td>
<td>Robust choice of the mother wavelet</td>
<td>Gibbs oscillations</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Dilation and translation are used to capture the local characteristics</td>
<td>Impressive improvements and applications are still occurring</td>
<td></td>
</tr>
</tbody>
</table>

Figure 1: Comparison of PDE transform with the Fourier transform, the wavelet transform and the empirical mode decomposition.

In Figure 1, the PDE transform is schematically compared with the Fourier transform, the wavelet transform and the empirical mode decomposition. Essentially, when data is perfect and stationary, Fourier transform yields the clear and well defined spectrum analysis with high accuracy. The high order linear PDE transform is comparable with the Fourier analysis in this scenario. On the other hand, when data is highly oscillating and non-stationary, wavelet methods provide a scaled time-frequency analysis using different scales of the mother wavelet which goes beyond the windowed Fourier transform. However the physical interpretation and quantitative performance are usually the big concerns. Empirical mode decomposition provides a better analysis for non-stationary data by iteratively separating original data into multiple intrinsic mode functions corresponding to distinctive instantaneous frequencies. In particular, each intrinsic mode function is obtained by applying data-adaptive low pass filters to the previous residue intrinsic mode functions. However, EMD yields too many mode functions, many of which are not physically relevant or representative, and the associated instantaneous frequencies are not always beyond questions. Both the linear and nonlinear PDE transform can decompose the original non-stationary data into various functional modes, where functional mode is the one with characteristic frequency range and physical scales. Original data is iteratively separated into several adequate number of function modes corresponding to the geometric flow of the PDE.

Numerically, the PDE transform is also easier to implement with simpler and higher controllability of
the relevant parameters. In particular, the wavelet transform can be regarded as a filter bank which is capable of yielding multiresolution sub-bands of the original signal. The outputs of the different filter stages are the wavelet and scaling function transform coefficients. Analyzing a signal by passing it through a filter bank is not a new idea and has been around for many years under the name sub-band coding. There are two classes of filter banks used for sub-band coding. One way is to pre-build many bandpass filters to split the spectrum into frequency bands with the advantage that the width of every band can be chosen freely such that the spectrum of the signal to analyze is covered in the places where it might be interesting. The disadvantage is that one must design every filter separately. The other approach is to split the signal spectrum into two equal parts, a low-pass and a high-pass part. Such a division is repeated till satisfaction so that one actually creates an iterated filter bank. The advantage of this scheme is that we have to design only two filters, the disadvantage is that the signal spectrum coverage is fixed. In the PDE transform, one can freely control the PDE propagation time (or equivalent hyper-diffusion coefficients) to achieve arbitrary sub-band, which can be adjusted to yield different frequency band location and width of interest, yet with a perfect reconstruction or secondary processing. In this sense, the PDE transform behaves like an optimized wavelet transform in providing a easy tuning filter bank to split the spectrum into any number of desired sub-bands. More detailed comparisons are given in the Figure 1, where the properties of the PDE transform listed in the upper half of the figure are compared with each of the three transforms as listed on the lower half of the figure.

III Algorithms and Applications

III.A Mother filter design for PDE transform

Filter design has been a classical and fundamental topic in signal processing and applied mathematics. Mathematically, center to all the signal processing and mode decomposition algorithms and algorithm is the design and application of digital filter banks. The accuracy and efficiency of the PDE transform method is closely related to, and can be effectively analyzed through the filters incorporated in the process.

Essentially, digital signal processing is a process of selecting or suppressing certain frequency components of a signal. Therefore, designing appropriate filter banks is key to the signal processing and its applications in many areas like compression of image, video and audio signals, automatic control, robotics, target tracking, and telecommunication. Good digital filters can be used to shape the frequency spectrum of the signal by passing, suppressing or attenuating certain desired frequency bands of the signal by a desired amount. In engineering, science and mathematics, different implications and requirements are imposed on the filter design in terms of the frequency responses. Conventionally, filters can be classified as low-pass, high-pass and band-pass types depending on how the filters remove frequency components from the signal. A more practical way of classification is to specify filters by family and band form. Some common IIR filters include Butterworth filters (which are designed to have as flat a frequency response as possible in the passband), Chebyshev filters (which have a steeper roll-off and more passband or stopband ripple than Butterworth filters), Bessel filters (which is a type of linear filter with a maximal linear phase response), Gaussian filters (whose impulse response is a Gaussian function with no overshoot to a step function input while minimizing the rise and fall time), elliptic filters (which have equalized ripple behavior in both the passband and the stopband), etc.

One important feature and measure of filters is the frequency response (or transfer function), i.e. the spectrum generated via Fourier transform of the signal. Mathematically, design and analysis of digital filters are closely related to the characteristics of the corresponding transfer function which shows the relation between the input and output of a linear time-invariant system in terms of spatial or temporal frequency. The determining factors for the behavior of most filters are the shape, ripples, steepness, order and complexity of the passband, stopband, and transition band of each individual filter. Closely related to the shape of the frequency response is the stability of the filter. In order to analyze non-stationary data containing local phenomena in time duration, width of the filter's frequency response has to be short. On the other hand, accuracy of the mode decomposition is to certain extent determined by the locality of the frequency response curve. Due to the uncertainty principle of the Fourier transform, the product of the width of the filter's impulse response function and the width of its frequency function must
exceed a certain constant. Consequently, it may not be possible to simultaneously meet requirements on the locality of the filter’s frequency function and applicability to non-stationary data. In the section below, we give a brief review and comparison of some common filters and the filters we proposed in the past decade.

The characterization of an ideal filter is related to the features of passband ripple, stopband attenuation and transition band location of the frequency responses. An ideal low-pass filter can be defined in terms of frequency response function with 100% pass in the low frequency region (with frequency lower than certain cut-off value $\omega_0$) and 0% pass in the high frequency region (with frequency larger than $\omega_0$). In practice, as a filter's response becomes closer and closer to the ideal, the cost of the filter, e.g. time delay, number of elements, dollars, power consumption, will increase. There are a number of ways to approximate an ideal response based on different criteria. In this paper, we categorize different filters according to whether the distortion of the signals in the passband is minimal, with tradeoff of increased attenuation in stopband. Other designs may need the fastest transition from passband to stopband and will allow more distortion in the passband to accomplish that aim. We will discuss various filters according to whether the pass-band is smooth or not, i.e. smooth or non-smooth filters.

We first briefly review smooth low-pass filters, which are sometimes called maximally flat finite impulse response digital filters characterized by the monotonic and flat magnitude frequency response. Mathematically, design of such magnitude response characteristics is equivalent to solving a form of Hermite Interpolation problem.\textsuperscript{42,43} The frequency response of such type of filters is flat around the centers of flatness, and is usually monotone.

\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{figure_a.png} \quad \includegraphics[width=0.4\textwidth]{figure_b.png}
\caption{Frequency response of generalized IFD with Hermite kernel. Order $M_h$ is set 2 and 100 for the upper and lower plot respectively. Number of iteration is 100. Black solid line shows the frequency response when Hermite kernel is used as low-pass filter, and red dashed line corresponds to that of high-pass filter.}
\end{figure}

\textbf{Smooth DSC-Hermite kernel} Discrete singular convolution (DSC) algorithm was initially proposed for the numerical computation of singular convolutions which occur commonly in science and engineering. In this paper, we choose Hermite kernel which has a narrow effective frequency band but its low-pass filter better approximates the idea low-pass filter.\textsuperscript{19,27,56,59}

\begin{equation}
\delta_{r,\Delta}(x) = \frac{\Delta}{\sigma} \sum_{n=0}^{M_h} \left( -\frac{1}{4} \right)^n \frac{1}{\sqrt{2\pi n!}} h_{2n} \left( \frac{x}{\sqrt{2}\sigma} \right)
\end{equation}

where $n$-th order Hermite polynomial is given by\textsuperscript{19}

\begin{equation}
h_n(x) = e^{-x^2} H_n(x) = (-1)^n \frac{d^n}{dx^n} e^{-x^2}.
\end{equation}

One usually needs to truncate summation after $M_h$ terms, which yields a numerical implementation of Hermite local spectral kernels. Overall, all the above DSC filters can be used to construct local spectral methods as an efficient numerical tool for solving PDEs.\textsuperscript{46,71}
In Figure 2, frequency responses of Hermite kernel is plotted with order $M_h = 2$ and $M_h = 100$ respectively. The value of the Gaussian window size $\sigma = 50\Delta$ and $1300\Delta$ respectively in the two figures such that the center of transition band is near the frequency $\omega = 1$ comparable to the previous filters. The value of $\sigma$ can easily tuned to adjust the localization of the transition band. The high order Hermite filter can be used as ideal low pass filter with total smoothness in stopband and passband and sharp transition band. Implementation of such kernels in the DFD scheme enables one to accurately separate various modes with highly desirable numerical stability and robustness. If order is too small, the transition band becomes less localized, but its implementation in the G-IFD scheme leads to a greatly improved localization and thus can be used to achieve similar results as those from high order Hermite filters.

Figure 3: Frequency response of low order and high order PDE transform with parameters chosen such that the center of transition band is near $\omega = 1$. Black solid line shows the frequency response when high order PDE transform filter is used as low-pass filter, and red dashed line corresponds to that of high-pass filter.

**Smooth PDE-based filters** Application of PDE in image denoising is based on the idea of evolving an original image under a diffusion process which is formally equivalent to the standard Gaussian filter. The solution to this PDE at a later time is a modified smooth image. The introduction of anisotropic diffusion equation replace the constant diffusion coefficient by a function of image gradients to distinguish between edges and noise. Moreover, the nonlinear anisotropic diffusion equation can further facilitate a more effective PDE algorithm for noise removing, image restoration, edge detection, and image enhancement.

In Figure 3, frequency responses are plotted for the high order PDE transform based filters we proposed earlier,

\[
\frac{\partial v}{\partial t} = \sum_{j=1}^{n} (-1)^{j+1} d_j \partial_x^{2j} v + \epsilon (X - v), \quad t \geq 0,
\]

where $X$ is the initial signal, $\epsilon \sim 0$, and $d_j$ has the physical meaning of propagation time multiplied by hyper-diffusion coefficient. The values of $d_j$ are positive and can be chosen to determine the localization of the transition band, just like the role of $\sigma$ in the Hermite kernels. The exact solution of Eq.(16) in the Fourier representation is given by

\[
\hat{v}(t) = \hat{L} \hat{X}
\]

where $\hat{v}$ and $\hat{X}$ are the Fourier transforms of $v$ and $X$ respectively and low-pass filter $\hat{L}$ in the frequency space can be written in the form

\[
\hat{L}(\epsilon, t, d_1, d_2, \cdots, d_n) = e^{-\left(\sum_{j=1}^{n} d_j w^{2j} + \epsilon\right)t} + \frac{\epsilon}{\sum_{j=1}^{n} d_j w^{2j} + \epsilon} \left(e^{-\left(\sum_{j=1}^{n} d_j w^{2j} + \epsilon\right)t} - 1\right).
\]

For simplicity, we drop the fidelity term in the present analysis and focus our study on the high order
PDE. Consequently, the frequency response function reduces to

\[ \hat{L}(t, d_1, d_2, \ldots, d_n) = e^{-\left(\sum_{j=1}^{n} d_j w^{2j}\right)t} \]  \hspace{1cm} (19)

which is plotted in the Figure 3 with parameters chosen such that the center of transition band is near \( \omega = 1 \) comparable to the other filters’ frequency responses. Similarly, the behavior of the proposed high-pass PDE transforms can be analyzed by

\[ \hat{H}(t, d_1, d_2, \ldots, d_n) = 1 - \hat{L}(t, d_1, d_2, \ldots, d_n) \]  \hspace{1cm} (20)

One can also devise a band-pass filter by the difference of solutions of two high-order PDEs

\[ \hat{B}(t, d_{11}, d_{12}, \ldots, d_{1m}, d_{21}, d_{22}, \ldots, d_{2n}) = e^{-\left(\sum_{j=1}^{m} d_{1j} w^{2j}\right)t} - e^{-\left(\sum_{j=1}^{m} d_{2j} w^{2j}\right)t} \]  \hspace{1cm} (21)

where we choose \( d_{2j} >> d_{1j} \sim 0 \). When implemented in DFD or G-IFD schemes, fast Fourier transform can be used to efficiently perform the filtering. In general, one can perform the filtering via convolution in real space. The features of the frequency responses are very similar to those of the Hermite kernels with comparable robustness as well as accuracy in mode decomposition. Both Hermite kernels and PDE transform based filters are good candidate for DFD and G-IFD scheme.54

**High-pass filters** Image processing PDEs of the Perona-Malik type and total variation type are mostly designed to function as nonlinear low-pass filters. Coupled nonlinear PDEs have been introduced to behave as high-pass filters for image edge detection.60 The essential idea behind these PDE based high-pass filters is that when two Perona-Malik type of PDEs evolve at dramatically different speeds, the difference of their solutions gives rise to image edges. This follows from the fact that the difference between an all-pass filter and a low-pass one is a high-pass filter.60 The speeds of evolution in these equations are controlled by the appropriate selection of the diffusion coefficients. These PDE-based edge detectors have been shown to work extremely well for images with large amount of textures.46,60

In the PDE transform algorithm proposed by us, both low-pass and high-pass filters have been implemented to achieve effective decomposition of physically meaningful modes.

Direct and iterative filtering via the PDE transform We discuss here two major types of schemes for implementing various filters for mode decomposition purpose. The first type is generalized iterative filter decomposition (G-IFD), and the second one is the direct filtering decomposition (DFD) scheme. For better illustration, we will discuss the two schemes using above mentioned PDE transform based and local spectral method based filters.
The iteration converges if the low-pass filter is of maximally flat type, such as the Butterworth filter. Additionally, various local spectral based or DSC kernels\cite{56,58,61,66,72} are also good candidates for iterative filters. For example, we choose to implement the Hermite kernel\cite{71} in the G-IFD scheme, where the kernel can be written in the form

\[
\delta_\sigma(x) = \frac{1}{\sigma} \sum_{n=0}^{M_h/2} \left(-\frac{1}{4}\right)^n \frac{1}{\sqrt{2\pi n!}} h_{2n} \left(\frac{x}{\sqrt{2\sigma}}\right)
\]  

(22)

where $\Delta$ is the grid spacing, $\sigma$ is the Gaussian window size, $h_{2n}$ is the Hermite function of order $2n$ and $M_h/2$ is the order of the Hermite expansion.

Frequency responses of various smooth and non-smooth filters have been plotted and evaluations of various filters are given in the previous Section. These filters are evaluated in the context of implementation in DFD or G-IFD schemes proposed in this paper. In this section, we choose PDE transform (19) and Hermite kernels (22), which are both smooth filters and can be conveniently used as high-pass or low-pass filters, to be implemented in the DFD and G-IFD schemes. By varying the relevant parameters, one can study the effect of filters on the mode decomposition effect. E.g. In Figures 2, frequency response of Hermite kernel in Eq. (22) with $M_h = 2$ and $M_h = 100$ are plotted plotted. In designing direct filtering decomposition, relatively high order filters are implemented, i.e. large $M_h$ for Hermite kernel or large $2n$ for PDE-transform based kernel. On the other hand, relatively low order filters can be used in G-IFD scheme despite of large width of transition band. Large number of iterations can be used to achieve good frequency localization comparable to the that of the high order filters.

Besides the PDE transform and Hermite kernels, many other classical smooth filters can also be implemented in the general framework of IFD and DFD in a similar way. For example, the Butterworth filter with high order is near-ideal (i.e. flat) low-pass filter. It finds certain applicability, but not a preferred type due to the lack of controllability of the center of the transition band. Similar and more conclusion can be analyzed, e.g., Chebyshev filter with small $\epsilon$ value and low polynomial order $n$ (pink circles) is not perfect low-pass filter because of less perfect frequency localization. Increasing polynomial order $n$ increases the localization much better, but increase in the $\epsilon$ value leads to non-flat fluctuations in the low-frequency region. Therefore, Chebyshev filters with large $\epsilon$ values can not be implemented in the IFD scheme since it would blow out in the low frequency domain. Study of IFD and DFD using above two types of filters shows that iterative algorithm could improve the localization of low-pass filters (and similarly for high-pass filters) if the original filter allows perfect pass in the low frequency domain. Such an improvement is comparable to that of DFD scheme using higher orders of PDEs or polynomials.

In addition, we show two more plots in Figure 4 which illustrate the effect of different PDE parameters on the frequency response of the low-pass PDE transform. The Eq. (22) is used in designing the direct filtering decomposition. In Figure 4(a), y-axis denotes frequency and y-axis corresponds to the Fourier transform of the PDE-based low-pass filters. Various values of $M_h$ are used corresponding various orders of PDE. As $M_h$ increases, the frequency response becomes shaper as it changes of $1$ (total pass) to $0$ (no pass). In the limit of large $M_h$, the PDE transform becomes ideal low-pass filter. On the other hand, the separating line of frequency domain (i.e. the threshold frequency of the low-pass filter) is determined by the ratio of $\sigma/\Delta$. As illustrated in Figure 4(b), the larger ratio the lower the frequency cut-off of the low-pass filter. Consequently, if one chooses $\sigma/\Delta = \infty$, the low-pass filter theoretically filter out the whole frequency domain, while if one chooses $\sigma/\Delta = 0$ the all-pass filter is recovered. One can therefore easily control the low-pass filter behavior by varying the two parameters $M_h$ and $\sigma/\Delta$. Since these are the parameters in the PDE, essentially such a PDE transform provides a family of robust and highly accurate high- and low-pass filters.

III.B Analysis of non-stationary real time series

Signal analysis has remained important research field in both mathematical and engineering. Spectrum analysis as introduced by the conventional Fourier analysis has been important tool for classical analysis when signal is stationary. However, most real time series are non-stationary signals such that frequency analysis is not adequate or applicable. In addition, in many cases, locating local spatial (or temporal) events is as desirable as finding local peaks in the frequency domain. Windowed Fourier transform intro-
duced temporal windows, but it suffers from frequency convolution or aliasing problems. Wavelets were introduced as a powerful time-frequency (or more appropriately scale-frequency) analysis. By scaling and dilation of the mother wavelet, one achieves many desirable functions such as compression or power spectrum analysis in dual space of time and frequency. Essentially, wavelets methods are improved adjustable Fourier spectral analysis. Versatile as the wavelet analysis surely are, leakage are usually generated by the limited length of the basic wavelet function, which makes the quantitative definition of the energy-frequency-time distribution difficult. In addition, the interpretation of the wavelet can also be counterintuitive. For example, wavelet analyze a local temporal change using small scale wavelet basis corresponding to high frequency range. But if a local event occurs only in the low-frequency range, one is still forced to look for its effects in the high-frequency range. Empirical mode decomposition based on the Hilbert-Huang transformation aims to study the instantaneous frequency by looking at local fluctuations adaptively. It has achieved great success for analysis of non-stationary real time series in many fields. EMD is more considered as an algorithm, and essentially performs as windowed Fourier analysis with window size fully and automatically determined by the data itself. In another word, EMD introduces a class of low pass filters whose frequency responses are determined in a totally data-adaptive way. Such a feature is double-sworded: on one hand it makes EMD most powerful tool for non-stationary data analysis due to this data adaptiveness, while on the other hand it is thus difficult for EMD to control the local window size or the frequency property of the low pass filters. The PDE transform based decomposition combines the merits of both EMD and wavelets and allows full control of both the time scale and frequency responses of the low pass, high pass, and band pass filter banks realized by the PDE transforms.

![Figure 5: Moving variance analysis of the seasonal averaged sea surface temperature (SST) between the year 1871 and 1996 over the central Pacific. The red spikes show the original SST data, and the statistical moving average is calculated and shown by the green area.](image)

In this section, we compare all the different methods using the real time series recording the seasonal averaged sea surface temperature (SST) between the year of 1871 and that of 1996 over the central Pacific. The SST has been used as a measure of the amplitude of the El Nino Southern Oscillation (ENSO). The values of the real time series are plotted as red spikes in the Figure 5. The SST time series is highly non-stationary with mixed long interdecadal fluctuations and high frequency aperiodic spikes corresponding to the ENSO. The simplest method for analyzing such non-stationary time series would be to compute statistics such as the mean and variance for different time periods and see if they are significantly different. In Figure 5 we plot the running 15-year variance indicated by the green shaded area, as a measure of total power inherent in the signal versus time. It can be observed that ENSO had more variance before 1920 and as well as after 1975, with a relatively quiet period in between. But the results of running or moving variance is quite qualitative with subtle dependence on the choice of the size of moving window and averaging length of years. Moreover, such a moving variance analysis gives
Figure 6: Comparison of the wavelet analysis and the PDE transform based mode decomposition analysis of the SST Nino3 data. In all the sub-figures, power spectrum showing the magnitude of the SST oscillation is plotted in both time and scale (or frequency) dual representation. The PDE transform using different order of PDEs shows different degree of time and frequency localization. Among the different orders of PDE used in the Figures 6(b) through 6(f), the 2nd order PDE based analysis has the finest time localization (i.e., fine peaks are clearly shown along the x-axis of the year), while the 40th order PDE based analysis has the finest frequency localization and thus the least time resolution.

only magnitude but not frequency information of the periodical signal at different time.

One can use a windowed (or running) Fourier transform (WFT) for such non-stationary signal where the conventional FT is not globally applicable. But similar concern still hold, e.g., regarding the choice of the window size. Moreover, at low frequencies there are too few oscillations within the window such that the frequency localization is lost, while at high frequencies there are so many oscillations that the time localization is lost.

Wavelet analysis improves the non-stationary signal analysis by decomposing a time series into
Figure 7: Mode decomposition of the SST signal using empirical mode decomposition (Figure 7(a)) and the PDE transform of order 4 (Figure 7(b)).

time/frequency space simultaneously, i.e., both the amplitude and oscillation of any periodic signals are collected together. Figure 6(a) shows the power of the Morlet-wavelet transform for the same set of SST data in Figure 5. The contour plot of the power is plotted in the scale-frequency plane in which the x-axis shows the year and y-axis shows the scale of the temporal oscillation that is similar to the frequency in the Fourier analysis. The yellow and red areas correspond to the maximal power distribution. For example, Figure 6(a) shows that most energy is distributed in the scale range of half to two years and around the years 1880-1920 and 1960-1980.

The PDE transform can be used to decompose as well analyze the signal in the dual time-Frequency (or scale) space in the similar way as in wavelet analysis. In Figures 6(b) through 6(f), the original real time series of SST data in Figure 5 is decomposed into different modes using the PDE transform of highest PDE order of 2, i.e., diffusion equation. By increasing diffusion coefficient or propagation time, one can sequentially decompose the signal into various sequential modes, all of which are combined into the single contour plot in time-frequency space. Though the results in the Figures 6(b) through 6(f) are different from the wavelet analysis, but the two methods give qualitatively consistent results in showing the dominant power distribution between 1880-1920 and 1960-1980 on the time scale of 1-4 years. By changing the order \(2^n\) of the PDE used in the PDE transform, one achieves different degree of time and frequency dual localization. Among the different orders of PDE used in the Figures 6(b) through 6(f), the 2nd order PDE based analysis has the finest time localization (i.e., fine peaks are clearly shown along the x-axis of the year), while the 40th order PDE based analysis has the finest frequency localization and thus the least time resolution.

Furthermore, the PDE transform is compared with the EMD method for the analysis of the same SST signal. EMD method is essentially windowed Fourier transform with time-dependent-scaled window. Compared to the previous popular methods, EMD is even more fitting for the non-stationary data analysis. Each mode decomposed is complete and in physical space (time or position) rather than scaled and reduced in dimensionality as in the different level of wavelet analysis. The physical interpretation, as well as the choice of parameters, in wavelets are not appropriate sometime. The straight-forward empirical mode decomposition of the original time series is a good complement to the time-scale energy distribution in the wavelet analysis. On the other hand, if the original signal is too complicated and noisy, EMD decomposition is not always the most efficient method since it could generate too many modes, many of which are not physically relevant. Besides, the key to the success of EMD is the use of data-driven low pass filters for successive mode extraction. Such data-driven low pass filter is in general non-uniform, usually realized through spline fitting to the discrete time series, and time dependent such that dual time-frequency localization and non-stationary data auto-adaptation become possible. Consequently, how to choose spline for fitting is open question despite of much great progress made. In addition, one
usually does not have full control over the characteristics of the low pass filter bank to be used in the EMD mode extraction. In Figures 7(a) and 7(b), the intrinsic mode functions obtained by the EMD and the PDE transform are of the similar quality and pattern, and it is thus illustrated that both methods are capable of producing sequential modes with lower frequencies automatically and self-adapting to the non-stationary data.

III.C Image analysis via generalized geometric flows

In the Figure 6, both the wavelet and PDE transforms are applied to study the energy distribution of the SST in the dual time-scale space. In this paper, we propose the application of the PDE transform to decompose the image into different frequency bands (slices) to form a three-dimensional (two of image axes and one of the frequency) density volume. Various isosurfaces corresponding to the different scales of grey levels can be plotted, distribution of which show the “energy” distribution of the 2D image in the 3D space-frequency domain. Such a 3D spectrum, or equivalently 4D contour plots of the image mode grey level can be used for various image analysis functions. In the Figures 8, the PDE transform is applied to the two images of the Barbara 8(a) and camera man 8(d) for 4D spectrum analysis. In the Figure 8(b), multiple slices of images corresponding to the various mode functions of the Barbara image are combined into one 3D volume, in which the slice on the most front is the highest frequency mode. Isosurfaces of different grey levels are extracted and plotted with different color. The isosurface in the blue color corresponds to the smallest amplitude of grey level fluctuation, i.e. the noise modes.
while the isosurface in the red color corresponds to the largest amplitude of the grey level fluctuations, i.e. the edge modes. As shown in the Figure 8(b), most energy concentrates in the high frequency mode on the front and near the edges. Figure 8(c) is the projection of the 3D volume on the x-y plane, i.e. the front view of the duplicated slices of different frequencies. Similar analysis of the camera man image is shown in the Figures 8(e) and 8(f). The PDE transform based 3D spectrum analysis provides additional viewpoint and tool for the image analysis as demonstrated by the comparison of the Barbara and the camera man images. In the Figure 8(b), most energy is concentrated on the front slice of high energy while the energy of low frequency is mainly concentrated in the back slice. In the Figure 8(e), despite of the energy concentration in the high frequency mode of the front slice, there are continuous and extensive noise energy distribution in both the low frequency range (slices on the back) and medium frequency range (slices in the middle along the perpendicular z-axis). Such a difference in the isosurface distribution indicates the fact that the Barbara image has more distinguishable and dominant edges.

The Figure 8 illustrates how one can decompose the given 2D image into various functional frequency
modes and then recombine the modes into a 3D volume of which various isosurfaces can be constructed and plotted. Such a process not only provides a new way to visualize and analyze the energy distribution of the 2D image in the 3D space-frequency domain, but also introduce the Eulerian formalism into the image analysis. In the construction of biomolecular surfaces for molecular dynamics simulation and computational biology in general, either the Lagrangian formalism or the Eulerian formalism is usually employed. In the Lagrangian formalism, surface elements are evolved directly under various driving forces while in the Eulerian formalism the surface is embedded in a hypersurface, which is therefore evolved under prescribed driving forces. The desired surface is obtained from an isosurface extraction procedure. An improved mean curvature flow for the minimal molecular surface has been constructed as
\[
\frac{\partial S}{\partial t} = \Delta_S S = \sqrt{g} \nabla \cdot \left( \frac{\nabla S}{\sqrt{g}} \right)
\]
where \( g = \text{Det}(g_{ij}) \) is the Gram determinant of \( f \) when one considers a \( C^2 \) immersion \( f : U \to \mathbb{R}^{n+1} \) for an open set \( U \subset \mathbb{R}^n \). The scheme of construction the hypersurfaces or the isosurfaces generated in the Figure 8 via the PDE transform therefore enables one to introduce the potential driving geometric flows for the image analysis.

For further illustration, the various edge modes and residue are decomposed from and depicted for the same Barbara image. In Figure 9(a), Daubechies wavelet transform is applied to the Barbara image to obtain standard multiresolution decomposition. In Figures 9(b) through 9(d), the PDE transform using various orders (2nd order, 4th order and 20th order PDE respectively) is applied to obtain three levels of edges and the remaining residue after edge extraction. It is shown that, by varying the relevant parameters, the PDE transform provides a robust method for the image analysis.

IV Conclusion
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Literature cited


