CHAPTER 2 LIMITS AND CONTINUITY

2.1 RATES OF CHANGE AND TANGENTS TO CURVES

1. (a) $\frac{\Delta f}{\Delta x} = \frac{f(3) - f(2)}{3 - 2} = \frac{28 - 9}{1} = 19$ (b) $\frac{\Delta f}{\Delta x} = \frac{f(1) - f(-1)}{1 - (-1)} = \frac{2 - 0}{2} = 1$

3. (a) $\frac{\Delta h}{\Delta t} = \frac{h\left(\frac{3\pi}{4}\right) - h\left(\frac{\pi}{4}\right)}{\frac{3\pi}{4} - \frac{\pi}{4}} = \frac{-1 - 1}{\frac{\pi}{2}} = -\frac{4}{\pi}$ (b) $\frac{\Delta h}{\Delta t} = \frac{h\left(\frac{\pi}{2}\right) - h\left(\frac{\pi}{6}\right)}{\frac{\pi}{2} - \frac{\pi}{6}} = \frac{0 - \sqrt{3}}{\frac{\pi}{3}} = \frac{-3\sqrt{3}}{\pi}$

5. $\frac{\Delta R}{\Delta \theta} = \frac{R(2) - R(0)}{2 - 0} = \frac{\sqrt{8+1} - \sqrt{1}}{2} = \frac{3 - 1}{2} = 1$

7. (a) $\frac{\Delta y}{\Delta x} = \frac{\left((2+h)^2 - 3\right) - \left(2^2 - 3\right)}{h} = \frac{4 + 4h + h^2 - 3 - 1}{h} = \frac{4h + h^2}{h} = 4 + h$. As $h \to 0$, $4 + h \to 4 \Rightarrow$ at $P(2, 1)$ the slope is 4.
(b) $y - 1 = 4(x - 2) \Rightarrow y - 1 = 4x - 8 \Rightarrow y = 4x - 7$

9. (a) $\frac{\Delta y}{\Delta x} = \frac{\left((2+h)^2 - 2(2+h) - 3\right) - \left(2^2 - 2(2) - 3\right)}{h} = \frac{4 + 4h + h^2 - 4 - 2h - 3 - (-3)}{h} = \frac{2h + h^2}{h} = 2 + h$. As $h \to 0$, $2 + h \to 2 \Rightarrow$ at $P(2, -3)$ the slope is 2.
(b) $y - (-3) = 2(x - 2) \Rightarrow y + 3 = 2x - 4 \Rightarrow y = 2x - 7$.

11. (a) $\frac{\Delta y}{\Delta x} = \frac{(2+h)^3 - 2^3}{h} = \frac{8 + 12h + 4h^2 + h^3 - 8}{h} = \frac{12h + 4h^2 + h^3}{h} = 12 + 4h + h^2$. As $h \to 0$, $12 + 4h + h^2 \to 12$, $\Rightarrow$ at $P(2, 8)$ the slope is 12.
(b) $y - 8 = 12(x - 2) \Rightarrow y - 8 = 12x - 24 \Rightarrow y = 12x - 16$.

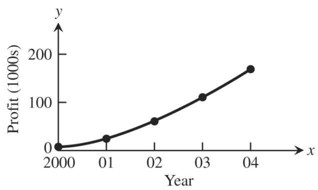
13. (a) $\frac{\Delta y}{\Delta x} = \frac{(1+h)^3 - 12(1+h) - \left(1^3 - 12(1)\right)}{h} = \frac{1 + 3h + 3h^2 + h^3 - 12 - 12h - (-11)}{h} = \frac{-9h + 3h^2 + h^3}{h} = -9 + 3h + h^2$. As $h \to 0$, $-9 + 3h + h^2 \to -9 \Rightarrow$ at $P(1, -11)$ the slope is -9.
(b) $y - (-11) = (-9)(x - 1) \Rightarrow y + 11 = -9x + 9 \Rightarrow y = -9x - 2$.

15. (a)

Q	Slope of PQ $= \frac{\Delta p}{\Delta t}$
$Q_1(10, 225)$	$\frac{650 - 225}{20 - 10} = 42.5$ m/sec
$Q_2(14, 375)$	$\frac{650 - 375}{20 - 14} = 45.83$ m/sec
$Q_3(16.5, 475)$	$\frac{650 - 475}{20 - 16.5} = 50.00$ m/sec
$Q_4(18, 550)$	$\frac{650 - 550}{20 - 18} = 50.00$ m/sec

(b) At $t = 20$, the sportscar was traveling approximately 50 m/sec or 180 km/h.

17. (a)



(b) $\frac{\Delta p}{\Delta t} = \frac{174 - 62}{2004 - 2002} = \frac{112}{2} = 56$ thousand dollars per year
(c) The average rate of change from 2001 to 2002 is $\frac{\Delta p}{\Delta t} = \frac{62 - 27}{20022 - 2001} = 35$ thousand dollars per year.
The average rate of change from 2002 to 2003 is $\frac{\Delta p}{\Delta t} = \frac{111 - 62}{2003 - 2002} = 49$ thousand dollars per year.
So, the rate at which profits were changing in 2002 is approximatley $\frac{1}{2}(35 + 49) = 42$ thousand dollars per year.

19. (a) $\frac{\Delta g}{\Delta x} = \frac{g(2)-g(1)}{2-1} = \frac{\sqrt{2}-1}{2-1} \approx 0.414213$ $\qquad \frac{\Delta g}{\Delta x} = \frac{g(1.5)-g(1)}{1.5-1} = \frac{\sqrt{1.5}-1}{0.5} \approx 0.449489$

$\frac{\Delta g}{\Delta x} = \frac{g(1+h)-g(1)}{(1+h)-1} = \frac{\sqrt{1+h}-1}{h}$

(b) $g(x) = \sqrt{x}$

$1+h$	1.1	1.01	1.001	1.0001	1.00001	1.000001
$\sqrt{1+h}$	1.04880	1.004987	1.0004998	1.0000499	1.000005	1.0000005
$\left(\sqrt{1+h}-1\right)/h$	0.4880	0.4987	0.4998	0.499	0.5	0.5

(c) The rate of change of g(x) at x = 1 is 0.5.

(d) The calculator gives $\lim\limits_{h \to 0} \frac{\sqrt{1+h}-1}{h} = \frac{1}{2}$.

NOTE: Answers will vary in Exercise 21.

21. (a) $[0, 1]$: $\frac{\Delta s}{\Delta t} = \frac{15-0}{1-0} = 15$ mph; $[1, 2.5]$: $\frac{\Delta s}{\Delta t} = \frac{20-15}{2.5-1} = \frac{10}{3}$ mph; $[2.5, 3.5]$: $\frac{\Delta s}{\Delta t} = \frac{30-20}{3.5-2.5} = 10$ mph

(b) At $P\left(\frac{1}{2}, 7.5\right)$: Since the portion of the graph from t = 0 to t = 1 is nearly linear, the instantaneous rate of change will be almost the same as the average rate of change, thus the instantaneous speed at $t = \frac{1}{2}$ is $\frac{15-7.5}{1-0.5} = 15$ mi/hr.

At P(2, 20): Since the portion of the graph from t = 2 to t = 2.5 is nearly linear, the instantaneous rate of change will be nearly the same as the average rate of change, thus $v = \frac{20-20}{2.5-2} = 0$ mi/hr. For values of t less than 2, we have

Q	Slope of PQ $= \frac{\Delta s}{\Delta t}$
$Q_1(1, 15)$	$\frac{15-20}{1-2} = 5$ mi/hr
$Q_2(1.5, 19)$	$\frac{19-20}{1.5-2} = 2$ mi/hr
$Q_3(1.9, 19.9)$	$\frac{19.9-20}{1.9-2} = 1$ mi/hr

Thus, it appears that the instantaneous speed at t = 2 is 0 mi/hr.

At P(3, 22):

Q	Slope of PQ $= \frac{\Delta s}{\Delta t}$	Q	Slope of PQ $= \frac{\Delta s}{\Delta t}$
$Q_1(4, 35)$	$\frac{35-22}{4-3} = 13$ mi/hr	$Q_1(2, 20)$	$\frac{20-22}{2-3} = 2$ mi/hr
$Q_2(3.5, 30)$	$\frac{30-22}{3.5-3} = 16$ mi/hr	$Q_2(2.5, 20)$	$\frac{20-22}{2.5-3} = 4$ mi/hr
$Q_3(3.1, 23)$	$\frac{23-22}{3.1-3} = 10$ mi/hr	$Q_3(2.9, 21.6)$	$\frac{21.6-22}{2.9-3} = 4$ mi/hr

Thus, it appears that the instantaneous speed at t = 3 is about 7 mi/hr.

(c) It appears that the curve is increasing the fastest at t = 3.5. Thus for P(3.5, 30)

Q	Slope of PQ $= \frac{\Delta s}{\Delta t}$	Q	Slope of PQ $= \frac{\Delta s}{\Delta t}$
$Q_1(4, 35)$	$\frac{35-30}{4-3.5} = 10$ mi/hr	$Q_1(3, 22)$	$\frac{22-30}{3-3.5} = 16$ mi/hr
$Q_2(3.75, 34)$	$\frac{34-30}{3.75-3.5} = 16$ mi/hr	$Q_2(3.25, 25)$	$\frac{25-30}{3.25-3.5} = 20$ mi/hr
$Q_3(3.6, 32)$	$\frac{32-30}{3.6-3.5} = 20$ mi/hr	$Q_3(3.4, 28)$	$\frac{28-30}{3.4-3.5} = 20$ mi/hr

Thus, it appears that the instantaneous speed at t = 3.5 is about 20 mi/hr.

2.2 LIMIT OF A FUNCTION AND LIMIT LAWS

1. (a) Does not exist. As x approaches 1 from the right, g(x) approaches 0. As x approaches 1 from the left, g(x) approaches 1. There is no single number L that all the values g(x) get arbitrarily close to as $x \to 1$.

(b) 1 (c) 0 (d) 0.5

3. (a) True (b) True (c) False

(d) False (e) False (f) True

(g) True

5. $\lim_{x \to 0} \frac{x}{|x|}$ does not exist because $\frac{x}{|x|} = \frac{x}{x} = 1$ if $x > 0$ and $\frac{x}{|x|} = \frac{x}{-x} = -1$ if $x < 0$. As x approaches 0 from the left, $\frac{x}{|x|}$ approaches -1. As x approaches 0 from the right, $\frac{x}{|x|}$ approaches 1. There is no single number L that all the function values get arbitrarily close to as $x \to 0$.

7. Nothing can be said about f(x) because the existence of a limit as $x \to x_0$ does not depend on how the function is defined at x_0. In order for a limit to exist, f(x) must be arbitrarily close to a single real number L when x is close enough to x_0. That is, the existence of a limit depends on the values of f(x) for x near x_0, not on the definition of f(x) at x_0 itself.

9. No, the definition does not require that f be defined at $x = 1$ in order for a limiting value to exist there. If f(1) is defined, it can be any real number, so we can conclude nothing about f(1) from $\lim_{x \to 1} f(x) = 5$.

11. $\lim_{x \to -7} (2x + 5) = 2(-7) + 5 = -14 + 5 = -9$

13. $\lim_{t \to 6} 8(t - 5)(t - 7) = 8(6 - 5)(6 - 7) = -8$

15. $\lim_{x \to 2} \frac{x+3}{x+6} = \frac{2+3}{2+6} = \frac{5}{8}$

17. $\lim_{x \to -1} 3(2x - 1)^2 = 3(2(-1) - 1)^2 = 3(-3)^2 = 27$

19. $\lim_{y \to -3} (5 - y)^{4/3} = [5 - (-3)]^{4/3} = (8)^{4/3} = \left((8)^{1/3}\right)^4 = 2^4 = 16$

21. $\lim_{h \to 0} \frac{3}{\sqrt{3h+1}+1} = \frac{3}{\sqrt{3(0)+1}+1} = \frac{3}{\sqrt{1}+1} = \frac{3}{2}$

23. $\lim_{x \to 5} \frac{x-5}{x^2-25} = \lim_{x \to 5} \frac{x-5}{(x+5)(x-5)} = \lim_{x \to 5} \frac{1}{x+5} = \frac{1}{5+5} = \frac{1}{10}$

25. $\lim_{x \to -5} \frac{x^2+3x-10}{x+5} = \lim_{x \to -5} \frac{(x+5)(x-2)}{x+5} = \lim_{x \to -5} (x - 2) = -5 - 2 = -7$

27. $\lim_{t \to 1} \frac{t^2+t-2}{t^2-1} = \lim_{t \to 1} \frac{(t+2)(t-1)}{(t-1)(t+1)} = \lim_{t \to 1} \frac{t+2}{t+1} = \frac{1+2}{1+1} = \frac{3}{2}$

29. $\lim_{x \to -2} \frac{-2x-4}{x^3+2x^2} = \lim_{x \to -2} \frac{-2(x+2)}{x^2(x+2)} = \lim_{x \to -2} \frac{-2}{x^2} = \frac{-2}{4} = -\frac{1}{2}$

31. $\lim_{x \to 1} \frac{\frac{1}{x}-1}{x-1} = \lim_{x \to 1} \frac{\frac{1-x}{x}}{x-1} = \lim_{x \to 1} \left(\frac{1-x}{x} \cdot \frac{1}{x-1}\right) = \lim_{x \to 1} -\frac{1}{x} = -1$

33. $\lim_{u \to 1} \frac{u^4-1}{u^3-1} = \lim_{u \to 1} \frac{(u^2+1)(u+1)(u-1)}{(u^2+u+1)(u-1)} = \lim_{u \to 1} \frac{(u^2+1)(u+1)}{u^2+u+1} = \frac{(1+1)(1+1)}{1+1+1} = \frac{4}{3}$

35. $\lim_{x \to 9} \frac{\sqrt{x}-3}{x-9} = \lim_{x \to 9} \frac{\sqrt{x}-3}{\left(\sqrt{x}-3\right)\left(\sqrt{x}+3\right)} = \lim_{x \to 9} \frac{1}{\sqrt{x}+3} = \frac{1}{\sqrt{9}+3} = \frac{1}{6}$

37. $\lim_{x \to 1} \frac{x-1}{\sqrt{x+3}-2} = \lim_{x \to 1} \frac{(x-1)\left(\sqrt{x+3}+2\right)}{\left(\sqrt{x+3}-2\right)\left(\sqrt{x+3}+2\right)} = \lim_{x \to 1} \frac{(x-1)\left(\sqrt{x+3}+2\right)}{(x+3)-4} = \lim_{x \to 1} \left(\sqrt{x+3}+2\right)$
$= \sqrt{4} + 2 = 4$

39. $\lim_{x \to 2} \frac{\sqrt{x^2+12}-4}{x-2} = \lim_{x \to 2} \frac{\left(\sqrt{x^2+12}-4\right)\left(\sqrt{x^2+12}+4\right)}{(x-2)\left(\sqrt{x^2+12}+4\right)} = \lim_{x \to 2} \frac{(x^2+12)-16}{(x-2)\left(\sqrt{x^2+12}+4\right)}$
$= \lim_{x \to 2} \frac{(x-2)(x+2)}{(x-2)\left(\sqrt{x^2+12}+4\right)} = \lim_{x \to 2} \frac{x+2}{\sqrt{x^2+12}+4} = \frac{4}{\sqrt{16}+4} = \frac{1}{2}$

41. $\lim\limits_{x \to -3} \frac{2-\sqrt{x^2-5}}{x+3} = \lim\limits_{x \to -3} \frac{\left(2-\sqrt{x^2-5}\right)\left(2+\sqrt{x^2-5}\right)}{(x+3)\left(2+\sqrt{x^2-5}\right)} = \lim\limits_{x \to -3} \frac{4-(x^2-5)}{(x+3)\left(2+\sqrt{x^2-5}\right)}$

$= \lim\limits_{x \to -3} \frac{9-x^2}{(x+3)\left(2+\sqrt{x^2-5}\right)} = \lim\limits_{x \to -3} \frac{(3-x)(3+x)}{(x+3)\left(2+\sqrt{x^2-5}\right)} = \lim\limits_{x \to -3} \frac{3-x}{2+\sqrt{x^2-5}} = \frac{6}{2+\sqrt{4}} = \frac{3}{2}$

43. $\lim\limits_{x \to 0}(2\sin x - 1) = 2\sin 0 - 1 = 0 - 1 = -1$

45. $\lim\limits_{x \to 0} \sec x = \lim\limits_{x \to 0} \frac{1}{\cos x} = \frac{1}{\cos 0} = \frac{1}{1} = 1$

47. $\lim\limits_{x \to 0} \frac{1+x+\sin x}{3\cos x} = \frac{1+0+\sin 0}{3\cos 0} = \frac{1+0+0}{3} = \frac{1}{3}$

49. $\lim\limits_{x \to -\pi} \sqrt{x+4}\cos(x+\pi) = \lim\limits_{x \to -\pi} \sqrt{x+4} \cdot \lim\limits_{x \to -\pi} \cos(x+\pi) = \sqrt{-\pi+4} \cdot \cos 0 = \sqrt{4-\pi} \cdot 1 = \sqrt{4-\pi}$

51. (a) quotient rule (b) difference and power rules
(c) sum and constant multiple rules

53. (a) $\lim\limits_{x \to c} f(x)\,g(x) = \left[\lim\limits_{x \to c} f(x)\right]\left[\lim\limits_{x \to c} g(x)\right] = (5)(-2) = -10$
(b) $\lim\limits_{x \to c} 2f(x)\,g(x) = 2\left[\lim\limits_{x \to c} f(x)\right]\left[\lim\limits_{x \to c} g(x)\right] = 2(5)(-2) = -20$
(c) $\lim\limits_{x \to c} [f(x) + 3g(x)] = \lim\limits_{x \to c} f(x) + 3\lim\limits_{x \to c} g(x) = 5 + 3(-2) = -1$
(d) $\lim\limits_{x \to c} \frac{f(x)}{f(x)-g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} f(x) - \lim_{x \to c} g(x)} = \frac{5}{5-(-2)} = \frac{5}{7}$

55. (a) $\lim\limits_{x \to b} [f(x) + g(x)] = \lim\limits_{x \to b} f(x) + \lim\limits_{x \to b} g(x) = 7 + (-3) = 4$
(b) $\lim\limits_{x \to b} f(x) \cdot g(x) = \left[\lim\limits_{x \to b} f(x)\right]\left[\lim\limits_{x \to b} g(x)\right] = (7)(-3) = -21$
(c) $\lim\limits_{x \to b} 4g(x) = \left[\lim\limits_{x \to b} 4\right]\left[\lim\limits_{x \to b} g(x)\right] = (4)(-3) = -12$
(d) $\lim\limits_{x \to b} f(x)/g(x) = \lim\limits_{x \to b} f(x)/\lim\limits_{x \to b} g(x) = \frac{7}{-3} = -\frac{7}{3}$

57. $\lim\limits_{h \to 0} \frac{(1+h)^2-1^2}{h} = \lim\limits_{h \to 0} \frac{1+2h+h^2-1}{h} = \lim\limits_{h \to 0} \frac{h(2+h)}{h} = \lim\limits_{h \to 0} (2+h) = 2$

59. $\lim\limits_{h \to 0} \frac{[3(2+h)-4]-[3(2)-4]}{h} = \lim\limits_{h \to 0} \frac{3h}{h} = 3$

61. $\lim\limits_{h \to 0} \frac{\sqrt{7+h}-\sqrt{7}}{h} = \lim\limits_{h \to 0} \frac{\left(\sqrt{7+h}-\sqrt{7}\right)\left(\sqrt{7+h}+\sqrt{7}\right)}{h\left(\sqrt{7+h}+\sqrt{7}\right)} = \lim\limits_{h \to 0} \frac{(7+h)-7}{h\left(\sqrt{7+h}+\sqrt{7}\right)}$

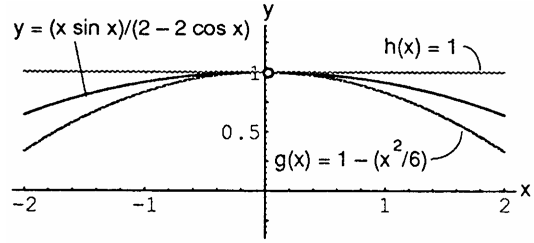
$= \lim\limits_{h \to 0} \frac{h}{h\left(\sqrt{7+h}+\sqrt{7}\right)} = \lim\limits_{h \to 0} \frac{1}{\sqrt{7+h}+\sqrt{7}} = \frac{1}{2\sqrt{7}}$

63. $\lim\limits_{x \to 0} \sqrt{5-2x^2} = \sqrt{5-2(0)^2} = \sqrt{5}$ and $\lim\limits_{x \to 0} \sqrt{5-x^2} = \sqrt{5-(0)^2} = \sqrt{5}$; by the sandwich theorem,

$\lim\limits_{x \to 0} f(x) = \sqrt{5}$

65. (a) $\lim\limits_{x \to 0} \left(1 - \frac{x^2}{6}\right) = 1 - \frac{0}{6} = 1$ and $\lim\limits_{x \to 0} 1 = 1$; by the sandwich theorem, $\lim\limits_{x \to 0} \frac{x \sin x}{2-2\cos x} = 1$

(b) For $x \neq 0$, $y = (x \sin x)/(2 - 2\cos x)$ lies between the other two graphs in the figure, and the graphs converge as $x \to 0$.



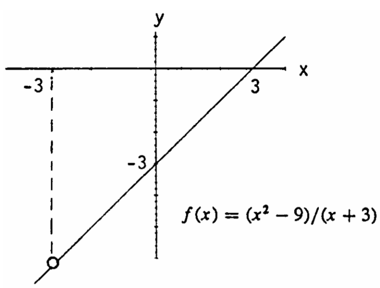
67. (a) $f(x) = (x^2 - 9)/(x + 3)$

x	−3.1	−3.01	−3.001	−3.0001	−3.00001	−3.000001
f(x)	−6.1	−6.01	−6.001	−6.0001	−6.00001	−6.000001

x	−2.9	−2.99	−2.999	−2.9999	−2.99999	−2.999999
f(x)	−5.9	−5.99	−5.999	−5.9999	−5.99999	−5.999999

The estimate is $\lim_{x \to -3} f(x) = -6$.

(b)



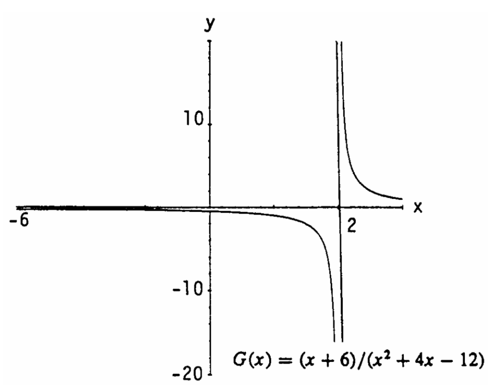
(c) $f(x) = \frac{x^2-9}{x+3} = \frac{(x+3)(x-3)}{x+3} = x - 3$ if $x \neq -3$, and $\lim_{x \to -3} (x - 3) = -3 - 3 = -6$.

69. (a) $G(x) = (x + 6)/\left(x^2 + 4x - 12\right)$

x	−5.9	−5.99	−5.999	−5.9999	−5.99999	−5.999999
G(x)	−.126582	−.1251564	−.1250156	−.1250015	−.1250001	−.1250000

x	−6.1	−6.01	−6.001	−6.0001	−6.00001	−6.000001
G(x)	−.123456	−.124843	−.124984	−.124998	−.124999	−.124999

(b)



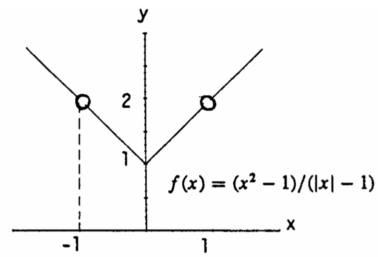
(c) $G(x) = \frac{x+6}{(x^2+4x-12)} = \frac{x+6}{(x+6)(x-2)} = \frac{1}{x-2}$ if $x \neq -6$, and $\lim_{x \to -6} \frac{1}{x-2} = \frac{1}{-6-2} = -\frac{1}{8} = -0.125$.

71. (a) $f(x) = (x^2 - 1) / (|x| - 1)$

x	-1.1	-1.01	-1.001	-1.0001	-1.00001	-1.000001
f(x)	2.1	2.01	2.001	2.0001	2.00001	2.000001

x	$-.9$	$-.99$	$-.999$	$-.9999$	$-.99999$	$-.999999$
f(x)	1.9	1.99	1.999	1.9999	1.99999	1.999999

(b)



(c) $f(x) = \frac{x^2-1}{|x|-1} = \begin{cases} \frac{(x+1)(x-1)}{x-1} = x+1, & x \ge 0 \text{ and } x \ne 1 \\ \frac{(x+1)(x-1)}{-(x+1)} = 1-x, & x < 0 \text{ and } x \ne -1 \end{cases}$, and $\lim_{x \to -1} (1-x) = 1-(-1) = 2.$

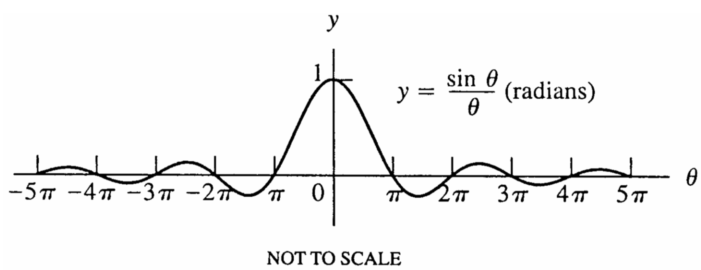
73. (a) $g(\theta) = (\sin \theta)/\theta$

θ	.1	.01	.001	.0001	.00001	.000001
$g(\theta)$	.998334	.999983	.999999	.999999	.999999	.999999

θ	$-.1$	$-.01$	$-.001$	$-.0001$	$-.00001$	$-.000001$
$g(\theta)$	.998334	.999983	.999999	.999999	.999999	.999999

$\lim_{\theta \to 0} g(\theta) = 1$

(b)



75. $\lim_{x \to c} f(x)$ exists at those points c where $\lim_{x \to c} x^4 = \lim_{x \to c} x^2$. Thus, $c^4 = c^2 \Rightarrow c^2(1-c^2) = 0 \Rightarrow c = 0, 1, \text{ or } -1$.

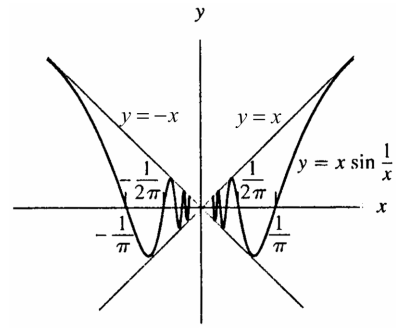
Moreover, $\lim_{x \to 0} f(x) = \lim_{x \to 0} x^2 = 0$ and $\lim_{x \to -1} f(x) = \lim_{x \to 1} f(x) = 1$.

77. $1 = \lim_{x \to 4} \frac{f(x)-5}{x-2} = \frac{\lim_{x\to4} f(x) - \lim_{x\to4} 5}{\lim_{x\to4} x - \lim_{x\to4} 2} = \frac{\lim_{x\to4} f(x) - 5}{4-2} \Rightarrow \lim_{x \to 4} f(x) - 5 = 2(1) \Rightarrow \lim_{x \to 4} f(x) = 2 + 5 = 7.$

79. (a) $0 = 3 \cdot 0 = \left[\lim_{x \to 2} \frac{f(x)-5}{x-2}\right]\left[\lim_{x \to 2} (x-2)\right] = \lim_{x \to 2} \left[\left(\frac{f(x)-5}{x-2}\right)(x-2)\right] = \lim_{x \to 2} [f(x)-5] = \lim_{x \to 2} f(x) - 5$
$\Rightarrow \lim_{x \to 2} f(x) = 5.$

(b) $0 = 4 \cdot 0 = \left[\lim_{x \to 2} \frac{f(x)-5}{x-2}\right]\left[\lim_{x \to 2} (x-2)\right] \Rightarrow \lim_{x \to 2} f(x) = 5$ as in part (a).

81. (a) $\lim_{x \to 0} x \sin \frac{1}{x} = 0$

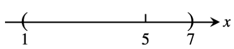


(b) $-1 \le \sin \frac{1}{x} \le 1$ for $x \ne 0$:

$x > 0 \Rightarrow -x \le x \sin \frac{1}{x} \le x \Rightarrow \lim_{x \to 0} x \sin \frac{1}{x} = 0$ by the sandwich theorem;

$x < 0 \Rightarrow -x \ge x \sin \frac{1}{x} \ge x \Rightarrow \lim_{x \to 0} x \sin \frac{1}{x} = 0$ by the sandwich theorem.

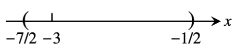
2.3 THE PRECISE DEFINITION OF A LIMIT

1. 

Step 1: $|x - 5| < \delta \Rightarrow -\delta < x - 5 < \delta \Rightarrow -\delta + 5 < x < \delta + 5$

Step 2: $\delta + 5 = 7 \Rightarrow \delta = 2$, or $-\delta + 5 = 1 \Rightarrow \delta = 4$.

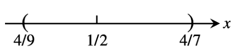
The value of δ which assures $|x - 5| < \delta \Rightarrow 1 < x < 7$ is the smaller value, $\delta = 2$.

3. 

Step 1: $|x - (-3)| < \delta \Rightarrow -\delta < x + 3 < \delta \Rightarrow -\delta - 3 < x < \delta - 3$

Step 2: $-\delta - 3 = -\frac{7}{2} \Rightarrow \delta = \frac{1}{2}$, or $\delta - 3 = -\frac{1}{2} \Rightarrow \delta = \frac{5}{2}$.

The value of δ which assures $|x - (-3)| < \delta \Rightarrow -\frac{7}{2} < x < -\frac{1}{2}$ is the smaller value, $\delta = \frac{1}{2}$.

5. 

Step 1: $\left|x - \frac{1}{2}\right| < \delta \Rightarrow -\delta < x - \frac{1}{2} < \delta \Rightarrow -\delta + \frac{1}{2} < x < \delta + \frac{1}{2}$

Step 2: $-\delta + \frac{1}{2} = \frac{4}{9} \Rightarrow \delta = \frac{1}{18}$, or $\delta + \frac{1}{2} = \frac{4}{7} \Rightarrow \delta = \frac{1}{14}$.

The value of δ which assures $\left|x - \frac{1}{2}\right| < \delta \Rightarrow \frac{4}{9} < x < \frac{4}{7}$ is the smaller value, $\delta = \frac{1}{18}$.

7. Step 1: $|x - 5| < \delta \Rightarrow -\delta < x - 5 < \delta \Rightarrow -\delta + 5 < x < \delta + 5$

Step 2: From the graph, $-\delta + 5 = 4.9 \Rightarrow \delta = 0.1$, or $\delta + 5 = 5.1 \Rightarrow \delta = 0.1$; thus $\delta = 0.1$ in either case.

9. Step 1: $|x - 1| < \delta \Rightarrow -\delta < x - 1 < \delta \Rightarrow -\delta + 1 < x < \delta + 1$

Step 2: From the graph, $-\delta + 1 = \frac{9}{16} \Rightarrow \delta = \frac{7}{16}$, or $\delta + 1 = \frac{25}{16} \Rightarrow \delta = \frac{9}{16}$; thus $\delta = \frac{7}{16}$.

11. Step 1: $|x - 2| < \delta \Rightarrow -\delta < x - 2 < \delta \Rightarrow -\delta + 2 < x < \delta + 2$

Step 2: From the graph, $-\delta + 2 = \sqrt{3} \Rightarrow \delta = 2 - \sqrt{3} \approx 0.2679$, or $\delta + 2 = \sqrt{5} \Rightarrow \delta = \sqrt{5} - 2 \approx 0.2361$; thus $\delta = \sqrt{5} - 2$.

13. Step 1: $|x - (-1)| < \delta \Rightarrow -\delta < x + 1 < \delta \Rightarrow -\delta - 1 < x < \delta - 1$

Step 2: From the graph, $-\delta - 1 = -\frac{16}{9} \Rightarrow \delta = \frac{7}{9} \approx 0.77$, or $\delta - 1 = -\frac{16}{25} \Rightarrow \frac{9}{25} = 0.36$; thus $\delta = \frac{9}{25} = 0.36$.

15. Step 1: $|(x+1)-5| < 0.01 \Rightarrow |x-4| < 0.01 \Rightarrow -0.01 < x-4 < 0.01 \Rightarrow 3.99 < x < 4.01$
Step 2: $|x-4| < \delta \Rightarrow -\delta < x-4 < \delta \Rightarrow -\delta+4 < x < \delta+4 \Rightarrow \delta = 0.01.$

17. Step 1: $\left|\sqrt{x+1}-1\right| < 0.1 \Rightarrow -0.1 < \sqrt{x+1}-1 < 0.1 \Rightarrow 0.9 < \sqrt{x+1} < 1.1 \Rightarrow 0.81 < x+1 < 1.21$
$\Rightarrow -0.19 < x < 0.21$
Step 2: $|x-0| < \delta \Rightarrow -\delta < x < \delta$. Then, $-\delta = -0.19 \Rightarrow \delta = 0.19$ or $\delta = 0.21$; thus, $\delta = 0.19$.

19. Step 1: $\left|\sqrt{19-x}-3\right| < 1 \Rightarrow -1 < \sqrt{19-x}-3 < 1 \Rightarrow 2 < \sqrt{19-x} < 4 \Rightarrow 4 < 19-x < 16$
$\Rightarrow -4 > x-19 > -16 \Rightarrow 15 > x > 3$ or $3 < x < 15$
Step 2: $|x-10| < \delta \Rightarrow -\delta < x-10 < \delta \Rightarrow -\delta+10 < x < \delta+10.$
Then $-\delta+10 = 3 \Rightarrow \delta = 7$, or $\delta+10 = 15 \Rightarrow \delta = 5$; thus $\delta = 5$.

21. Step 1: $\left|\frac{1}{x}-\frac{1}{4}\right| < 0.05 \Rightarrow -0.05 < \frac{1}{x}-\frac{1}{4} < 0.05 \Rightarrow 0.2 < \frac{1}{x} < 0.3 \Rightarrow \frac{10}{2} > x > \frac{10}{3}$ or $\frac{10}{3} < x < 5$.
Step 2: $|x-4| < \delta \Rightarrow -\delta < x-4 < \delta \Rightarrow -\delta+4 < x < \delta+4.$
Then $-\delta+4 = \frac{10}{3}$ or $\delta = \frac{2}{3}$, or $\delta+4 = 5$ or $\delta = 1$; thus $\delta = \frac{2}{3}$.

23. Step 1: $|x^2-4| < 0.5 \Rightarrow -0.5 < x^2-4 < 0.5 \Rightarrow 3.5 < x^2 < 4.5 \Rightarrow \sqrt{3.5} < |x| < \sqrt{4.5} \Rightarrow -\sqrt{4.5} < x < -\sqrt{3.5}$, for x near -2.
Step 2: $|x-(-2)| < \delta \Rightarrow -\delta < x+2 < \delta \Rightarrow -\delta-2 < x < \delta-2.$
Then $-\delta-2 = -\sqrt{4.5} \Rightarrow \delta = \sqrt{4.5}-2 \approx 0.1213$, or $\delta-2 = -\sqrt{3.5} \Rightarrow \delta = 2-\sqrt{3.5} \approx 0.1292$;
thus $\delta = \sqrt{4.5}-2 \approx 0.12$.

25. Step 1: $|(x^2-5)-11| < 1 \Rightarrow |x^2-16| < 1 \Rightarrow -1 < x^2-16 < 1 \Rightarrow 15 < x^2 < 17 \Rightarrow \sqrt{15} < x < \sqrt{17}.$
Step 2: $|x-4| < \delta \Rightarrow -\delta < x-4 < \delta \Rightarrow -\delta+4 < x < \delta+4.$
Then $-\delta+4 = \sqrt{15} \Rightarrow \delta = 4-\sqrt{15} \approx 0.1270$, or $\delta+4 = \sqrt{17} \Rightarrow \delta = \sqrt{17}-4 \approx 0.1231$;
thus $\delta = \sqrt{17}-4 \approx 0.12$.

27. Step 1: $|mx-2m| < 0.03 \Rightarrow -0.03 < mx-2m < 0.03 \Rightarrow -0.03+2m < mx < 0.03+2m \Rightarrow$
$2-\frac{0.03}{m} < x < 2+\frac{0.03}{m}$.
Step 2: $|x-2| < \delta \Rightarrow -\delta < x-2 < \delta \Rightarrow -\delta+2 < x < \delta+2.$
Then $-\delta+2 = 2-\frac{0.03}{m} \Rightarrow \delta = \frac{0.03}{m}$, or $\delta+2 = 2+\frac{0.03}{m} \Rightarrow \delta = \frac{0.03}{m}$. In either case, $\delta = \frac{0.03}{m}$.

29. Step 1: $\left|(mx+b)-\left(\frac{m}{2}+b\right)\right| < c \Rightarrow -c < mx-\frac{m}{2} < c \Rightarrow -c+\frac{m}{2} < mx < c+\frac{m}{2} \Rightarrow \frac{1}{2}-\frac{c}{m} < x < \frac{1}{2}+\frac{c}{m}$.
Step 2: $\left|x-\frac{1}{2}\right| < \delta \Rightarrow -\delta < x-\frac{1}{2} < \delta \Rightarrow -\delta+\frac{1}{2} < x < \delta+\frac{1}{2}$.
Then $-\delta+\frac{1}{2} = \frac{1}{2}-\frac{c}{m} \Rightarrow \delta = \frac{c}{m}$, or $\delta+\frac{1}{2} = \frac{1}{2}+\frac{c}{m} \Rightarrow \delta = \frac{c}{m}$. In either case, $\delta = \frac{c}{m}$.

31. $\lim_{x \to 3} (3-2x) = 3-2(3) = -3$
Step 1: $|(3-2x)-(-3)| < 0.02 \Rightarrow -0.02 < 6-2x < 0.02 \Rightarrow -6.02 < -2x < -5.98 \Rightarrow 3.01 > x > 2.99$ or $2.99 < x < 3.01$.
Step 2: $0 < |x-3| < \delta \Rightarrow -\delta < x-3 < \delta \Rightarrow -\delta+3 < x < \delta+3.$
Then $-\delta+3 = 2.99 \Rightarrow \delta = 0.01$, or $\delta+3 = 3.01 \Rightarrow \delta = 0.01$; thus $\delta = 0.01$.

33. $\lim_{x \to 2} \frac{x^2-4}{x-2} = \lim_{x \to 2} \frac{(x+2)(x-2)}{(x-2)} = \lim_{x \to 2} (x+2) = 2+2 = 4,\ x \neq 2$
Step 1: $\left|\left(\frac{x^2-4}{x-2}\right)-4\right| < 0.05 \Rightarrow -0.05 < \frac{(x+2)(x-2)}{(x-2)}-4 < 0.05 \Rightarrow 3.95 < x+2 < 4.05,\ x \neq 2$
$\Rightarrow 1.95 < x < 2.05,\ x \neq 2.$

Step 2: $|x-2| < \delta \Rightarrow -\delta < x-2 < \delta \Rightarrow -\delta+2 < x < \delta+2.$
Then $-\delta+2 = 1.95 \Rightarrow \delta = 0.05$, or $\delta+2 = 2.05 \Rightarrow \delta = 0.05$; thus $\delta = 0.05$.

35. $\lim_{x\to -3} \sqrt{1-5x} = \sqrt{1-5(-3)} = \sqrt{16} = 4$
Step 1: $\left|\sqrt{1-5x}-4\right| < 0.5 \Rightarrow -0.5 < \sqrt{1-5x}-4 < 0.5 \Rightarrow 3.5 < \sqrt{1-5x} < 4.5 \Rightarrow 12.25 < 1-5x < 20.25$
$\Rightarrow 11.25 < -5x < 19.25 \Rightarrow -3.85 < x < -2.25.$
Step 2: $|x-(-3)| < \delta \Rightarrow -\delta < x+3 < \delta \Rightarrow -\delta-3 < x < \delta-3.$
Then $-\delta-3 = -3.85 \Rightarrow \delta = 0.85$, or $\delta-3 = -2.25 \Rightarrow 0.75$; thus $\delta = 0.75$.

37. Step 1: $|(9-x)-5| < \epsilon \Rightarrow -\epsilon < 4-x < \epsilon \Rightarrow -\epsilon-4 < -x < \epsilon-4 \Rightarrow \epsilon+4 > x > 4-\epsilon \Rightarrow 4-\epsilon < x < 4+\epsilon.$
Step 2: $|x-4| < \delta \Rightarrow -\delta < x-4 < \delta \Rightarrow -\delta+4 < x < \delta+4.$
Then $-\delta+4 = -\epsilon+4 \Rightarrow \delta = \epsilon$, or $\delta+4 = \epsilon+4 \Rightarrow \delta = \epsilon$. Thus choose $\delta = \epsilon$.

39. Step 1: $\left|\sqrt{x-5}-2\right| < \epsilon \Rightarrow -\epsilon < \sqrt{x-5}-2 < \epsilon \Rightarrow 2-\epsilon < \sqrt{x-5} < 2+\epsilon \Rightarrow (2-\epsilon)^2 < x-5 < (2+\epsilon)^2$
$\Rightarrow (2-\epsilon)^2+5 < x < (2+\epsilon)^2+5.$
Step 2: $|x-9| < \delta \Rightarrow -\delta < x-9 < \delta \Rightarrow -\delta+9 < x < \delta+9.$
Then $-\delta+9 = \epsilon^2-4\epsilon+9 \Rightarrow \delta = 4\epsilon-\epsilon^2$, or $\delta+9 = \epsilon^2+4\epsilon+9 \Rightarrow \delta = 4\epsilon+\epsilon^2$. Thus choose the smaller distance, $\delta = 4\epsilon-\epsilon^2$.

41. Step 1: For $x \neq 1$, $|x^2-1| < \epsilon \Rightarrow -\epsilon < x^2-1 < \epsilon \Rightarrow 1-\epsilon < x^2 < 1+\epsilon \Rightarrow \sqrt{1-\epsilon} < |x| < \sqrt{1+\epsilon}$
$\Rightarrow \sqrt{1-\epsilon} < x < \sqrt{1+\epsilon}$ near $x = 1$.
Step 2: $|x-1| < \delta \Rightarrow -\delta < x-1 < \delta \Rightarrow -\delta+1 < x < \delta+1.$
Then $-\delta+1 = \sqrt{1-\epsilon} \Rightarrow \delta = 1-\sqrt{1-\epsilon}$, or $\delta+1 = \sqrt{1+\epsilon} \Rightarrow \delta = \sqrt{1+\epsilon}-1$. Choose $\delta = \min\left\{1-\sqrt{1-\epsilon}, \sqrt{1+\epsilon}-1\right\}$, that is, the smaller of the two distances.

43. Step 1: $\left|\frac{1}{x}-1\right| < \epsilon \Rightarrow -\epsilon < \frac{1}{x}-1 < \epsilon \Rightarrow 1-\epsilon < \frac{1}{x} < 1+\epsilon \Rightarrow \frac{1}{1+\epsilon} < x < \frac{1}{1-\epsilon}.$
Step 2: $|x-1| < \delta \Rightarrow -\delta < x-1 < \delta \Rightarrow 1-\delta < x < 1+\delta.$
Then $1-\delta = \frac{1}{1+\epsilon} \Rightarrow \delta = 1-\frac{1}{1+\epsilon} = \frac{\epsilon}{1+\epsilon}$, or $1+\delta = \frac{1}{1-\epsilon} \Rightarrow \delta = \frac{1}{1-\epsilon}-1 = \frac{\epsilon}{1-\epsilon}$.
Choose $\delta = \frac{\epsilon}{1+\epsilon}$, the smaller of the two distances.

45. Step 1: $\left|\left(\frac{x^2-9}{x+3}\right)-(-6)\right| < \epsilon \Rightarrow -\epsilon < (x-3)+6 < \epsilon, x \neq -3 \Rightarrow -\epsilon < x+3 < \epsilon \Rightarrow -\epsilon-3 < x < \epsilon-3.$
Step 2: $|x-(-3)| < \delta \Rightarrow -\delta < x+3 < \delta \Rightarrow -\delta-3 < x < \delta-3.$
Then $-\delta-3 = -\epsilon-3 \Rightarrow \delta = \epsilon$, or $\delta-3 = \epsilon-3 \Rightarrow \delta = \epsilon$. Choose $\delta = \epsilon$.

47. Step 1: $x < 1$: $|(4-2x)-2| < \epsilon \Rightarrow 0 < 2-2x < \epsilon$ since $x < 1$. Thus, $1-\frac{\epsilon}{2} < x < 0$;
$x \geq 1$: $|(6x-4)-2| < \epsilon \Rightarrow 0 \leq 6x-6 < \epsilon$ since $x \geq 1$. Thus, $1 \leq x < 1+\frac{\epsilon}{6}$.
Step 2: $|x-1| < \delta \Rightarrow -\delta < x-1 < \delta \Rightarrow 1-\delta < x < 1+\delta.$
Then $1-\delta = 1-\frac{\epsilon}{2} \Rightarrow \delta = \frac{\epsilon}{2}$, or $1+\delta = 1+\frac{\epsilon}{6} \Rightarrow \delta = \frac{\epsilon}{6}$. Choose $\delta = \frac{\epsilon}{6}$.

49. By the figure, $-x \leq x \sin\frac{1}{x} \leq x$ for all $x > 0$ and $-x \geq x \sin\frac{1}{x} \geq x$ for $x < 0$. Since $\lim_{x\to 0}(-x) = \lim_{x\to 0} x = 0$, then by the sandwich theorem, in either case, $\lim_{x\to 0} x \sin\frac{1}{x} = 0$.

51. As x approaches the value 0, the values of g(x) approach k. Thus for every number $\epsilon > 0$, there exists a $\delta > 0$ such that $0 < |x-0| < \delta \Rightarrow |g(x)-k| < \epsilon$.

53. Let $f(x) = x^2$. The function values do get closer to -1 as x approaches 0, but $\lim_{x \to 0} f(x) = 0$, not -1. The function $f(x) = x^2$ never gets <u>arbitrarily</u> <u>close</u> to -1 for x near 0.

55. $|A - 9| \le 0.01 \Rightarrow -0.01 \le \pi\left(\frac{x}{2}\right)^2 - 9 \le 0.01 \Rightarrow 8.99 \le \frac{\pi x^2}{4} \le 9.01 \Rightarrow \frac{4}{\pi}(8.99) \le x^2 \le \frac{4}{\pi}(9.01)$ $\Rightarrow 2\sqrt{\frac{8.99}{\pi}} \le x \le 2\sqrt{\frac{9.01}{\pi}}$ or $3.384 \le x \le 3.387$. To be safe, the left endpoint was rounded up and the right endpoint was rounded down.

57. (a) $-\delta < x - 1 < 0 \Rightarrow 1 - \delta < x < 1 \Rightarrow f(x) = x$. Then $|f(x) - 2| = |x - 2| = 2 - x > 2 - 1 = 1$. That is, $|f(x) - 2| \ge 1 \ge \frac{1}{2}$ no matter how small δ is taken when $1 - \delta < x < 1 \Rightarrow \lim_{x \to 1} f(x) \ne 2$.
(b) $0 < x - 1 < \delta \Rightarrow 1 < x < 1 + \delta \Rightarrow f(x) = x + 1$. Then $|f(x) - 1| = |(x + 1) - 1| = |x| = x > 1$. That is, $|f(x) - 1| \ge 1$ no matter how small δ is taken when $1 < x < 1 + \delta \Rightarrow \lim_{x \to 1} f(x) \ne 1$.
(c) $-\delta < x - 1 < 0 \Rightarrow 1 - \delta < x < 1 \Rightarrow f(x) = x$. Then $|f(x) - 1.5| = |x - 1.5| = 1.5 - x > 1.5 - 1 = 0.5$. Also, $0 < x - 1 < \delta \Rightarrow 1 < x < 1 + \delta \Rightarrow f(x) = x + 1$. Then $|f(x) - 1.5| = |(x + 1) - 1.5| = |x - 0.5|$ $= x - 0.5 > 1 - 0.5 = 0.5$. Thus, no matter how small δ is taken, there exists a value of x such that $-\delta < x - 1 < \delta$ but $|f(x) - 1.5| \ge \frac{1}{2} \Rightarrow \lim_{x \to 1} f(x) \ne 1.5$.

59. (a) For $3 - \delta < x < 3 \Rightarrow f(x) > 4.8 \Rightarrow |f(x) - 4| \ge 0.8$. Thus for $\epsilon < 0.8$, $|f(x) - 4| \ge \epsilon$ whenever $3 - \delta < x < 3$ no matter how small we choose $\delta > 0 \Rightarrow \lim_{x \to 3} f(x) \ne 4$.
(b) For $3 < x < 3 + \delta \Rightarrow f(x) < 3 \Rightarrow |f(x) - 4.8| \ge 1.8$. Thus for $\epsilon < 1.8$, $|f(x) - 4.8| \ge \epsilon$ whenever $3 < x < 3 + \delta$ no matter how small we choose $\delta > 0 \Rightarrow \lim_{x \to 3} f(x) \ne 4.8$.
(c) For $3 - \delta < x < 3 \Rightarrow f(x) > 4.8 \Rightarrow |f(x) - 3| \ge 1.8$. Again, for $\epsilon < 1.8$, $|f(x) - 3| \ge \epsilon$ whenever $3 - \delta < x < 3$ no matter how small we choose $\delta > 0 \Rightarrow \lim_{x \to 3} f(x) \ne 3$.

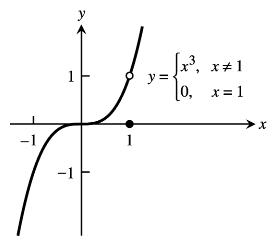
2.4 ONE-SIDED LIMITS

1. (a) True (b) True (c) False (d) True
(e) True (f) True (g) False (h) False
(i) False (j) False (k) True (l) False

3. (a) $\lim_{x \to 2^+} f(x) = \frac{2}{2} + 1 = 2$, $\lim_{x \to 2^-} f(x) = 3 - 2 = 1$
(b) No, $\lim_{x \to 2} f(x)$ does not exist because $\lim_{x \to 2^+} f(x) \ne \lim_{x \to 2^-} f(x)$
(c) $\lim_{x \to 4^-} f(x) = \frac{4}{2} + 1 = 3$, $\lim_{x \to 4^+} f(x) = \frac{4}{2} + 1 = 3$
(d) Yes, $\lim_{x \to 4} f(x) = 3$ because $3 = \lim_{x \to 4^-} f(x) = \lim_{x \to 4^+} f(x)$

5. (a) No, $\lim_{x \to 0^+} f(x)$ does not exist since $\sin\left(\frac{1}{x}\right)$ does not approach any single value as x approaches 0
(b) $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} 0 = 0$
(c) $\lim_{x \to 0} f(x)$ does not exist because $\lim_{x \to 0^+} f(x)$ does not exist

7. (a)



(b) $\lim_{x \to 1^-} f(x) = 1 = \lim_{x \to 1^+} f(x)$

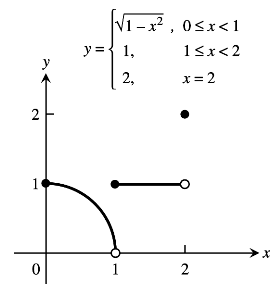
(c) Yes, $\lim_{x \to 1} f(x) = 1$ since the right-hand and left-hand limits exist and equal 1

9. (a) domain: $0 \le x \le 2$
range: $0 < y \le 1$ and $y = 2$

(b) $\lim_{x \to c} f(x)$ exists for c belonging to $(0, 1) \cup (1, 2)$

(c) $x = 2$

(d) $x = 0$



11. $\lim_{x \to -0.5^-} \sqrt{\frac{x+2}{x-1}} = \sqrt{\frac{-0.5+2}{-0.5+1}} = \sqrt{\frac{3/2}{1/2}} = \sqrt{3}$

13. $\lim_{x \to -2^+} \left(\frac{x}{x+1}\right)\left(\frac{2x+5}{x^2+x}\right) = \left(\frac{-2}{-2+1}\right)\left(\frac{2(-2)+5}{(-2)^2+(-2)}\right) = (2)\left(\frac{1}{2}\right) = 1$

15. $\lim_{h \to 0^+} \frac{\sqrt{h^2+4h+5}-\sqrt{5}}{h} = \lim_{h \to 0^+} \left(\frac{\sqrt{h^2+4h+5}-\sqrt{5}}{h}\right)\left(\frac{\sqrt{h^2+4h+5}+\sqrt{5}}{\sqrt{h^2+4h+5}+\sqrt{5}}\right)$
$= \lim_{h \to 0^+} \frac{(h^2+4h+5)-5}{h\left(\sqrt{h^2+4h+5}+\sqrt{5}\right)} = \lim_{h \to 0^+} \frac{h(h+4)}{h\left(\sqrt{h^2+4h+5}+\sqrt{5}\right)} = \frac{0+4}{\sqrt{5}+\sqrt{5}} = \frac{2}{\sqrt{5}}$

17. (a) $\lim_{x \to -2^+} (x+3)\frac{|x+2|}{x+2} = \lim_{x \to -2^+} (x+3)\frac{(x+2)}{(x+2)}$ $\quad (|x+2| = (x+2) \text{ for } x > -2)$
$= \lim_{x \to -2^+} (x+3) = ((-2)+3) = 1$

(b) $\lim_{x \to -2^-} (x+3)\frac{|x+2|}{x+2} = \lim_{x \to -2^-} (x+3)\left[\frac{-(x+2)}{(x+2)}\right]$ $\quad (|x+2| = -(x+2) \text{ for } x < -2)$
$= \lim_{x \to -2^-} (x+3)(-1) = -(-2+3) = -1$

19. (a) $\lim_{\theta \to 3^+} \frac{\lfloor \theta \rfloor}{\theta} = \frac{3}{3} = 1$

(b) $\lim_{\theta \to 3^-} \frac{\lfloor \theta \rfloor}{\theta} = \frac{2}{3}$

21. $\lim_{\theta \to 0} \frac{\sin \sqrt{2}\theta}{\sqrt{2}\theta} = \lim_{x \to 0} \frac{\sin x}{x} = 1$ $\quad$ (where $x = \sqrt{2}\theta$)

23. $\lim_{y \to 0} \frac{\sin 3y}{4y} = \frac{1}{4}\lim_{y \to 0} \frac{3 \sin 3y}{3y} = \frac{3}{4}\lim_{y \to 0} \frac{\sin 3y}{3y} = \frac{3}{4}\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = \frac{3}{4}$ $\quad$ (where $\theta = 3y$)

25. $\lim_{x \to 0} \frac{\tan 2x}{x} = \lim_{x \to 0} \frac{\left(\frac{\sin 2x}{\cos 2x}\right)}{x} = \lim_{x \to 0} \frac{\sin 2x}{x \cos 2x} = \left(\lim_{x \to 0} \frac{1}{\cos 2x}\right)\left(\lim_{x \to 0} \frac{2 \sin 2x}{2x}\right) = 1 \cdot 2 = 2$

27. $\lim_{x \to 0} \frac{x \csc 2x}{\cos 5x} = \lim_{x \to 0} \left(\frac{x}{\sin 2x} \cdot \frac{1}{\cos 5x}\right) = \left(\frac{1}{2}\lim_{x \to 0} \frac{2x}{\sin 2x}\right)\left(\lim_{x \to 0} \frac{1}{\cos 5x}\right) = \left(\frac{1}{2} \cdot 1\right)(1) = \frac{1}{2}$

29. $\lim_{x \to 0} \frac{x + x \cos x}{\sin x \cos x} = \lim_{x \to 0} \left(\frac{x}{\sin x \cos x} + \frac{x \cos x}{\sin x \cos x}\right) = \lim_{x \to 0} \left(\frac{x}{\sin x} \cdot \frac{1}{\cos x}\right) + \lim_{x \to 0} \frac{x}{\sin x}$
$= \lim_{x \to 0} \left(\frac{1}{\frac{\sin x}{x}}\right) \cdot \lim_{x \to 0} \left(\frac{1}{\cos x}\right) + \lim_{x \to 0} \left(\frac{1}{\frac{\sin x}{x}}\right) = (1)(1) + 1 = 2$

31. $\lim_{\theta \to 0} \frac{1-\cos\theta}{\sin 2\theta} = \lim_{\theta \to 0} \frac{(1-\cos\theta)(1+\cos\theta)}{(2\sin\theta\cos\theta)(1+\cos\theta)} = \lim_{\theta \to 0} \frac{1-\cos^2\theta}{(2\sin\theta\cos\theta)(1+\cos\theta)} = \lim_{\theta \to 0} \frac{\sin^2\theta}{(2\sin\theta\cos\theta)(1+\cos\theta)}$
$= \lim_{\theta \to 0} \frac{\sin\theta}{(2\cos\theta)(1+\cos\theta)} = \frac{0}{(2)(2)} = 0$

33. $\lim_{t \to 0} \frac{\sin(1-\cos t)}{1-\cos t} = \lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$ since $\theta = 1 - \cos t \to 0$ as $t \to 0$

35. $\lim_{\theta \to 0} \frac{\sin\theta}{\sin 2\theta} = \lim_{\theta \to 0} \left(\frac{\sin\theta}{\sin 2\theta} \cdot \frac{2\theta}{2\theta}\right) = \frac{1}{2} \lim_{\theta \to 0} \left(\frac{\sin\theta}{\theta} \cdot \frac{2\theta}{\sin 2\theta}\right) = \frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{2}$

37. $\lim_{\theta \to 0} \theta\cos\theta = 0 \cdot 1 = 0$

39. $\lim_{x \to 0} \frac{\tan 3x}{\sin 8x} = \lim_{x \to 0} \left(\frac{\sin 3x}{\cos 3x} \cdot \frac{1}{\sin 8x}\right) = \lim_{x \to 0} \left(\frac{\sin 3x}{\cos 3x} \cdot \frac{1}{\sin 8x} \cdot \frac{8x}{3x} \cdot \frac{3}{8}\right)$
$= \frac{3}{8} \lim_{x \to 0} \left(\frac{1}{\cos 3x}\right)\left(\frac{\sin 3x}{3x}\right)\left(\frac{8x}{\sin 8x}\right) = \frac{3}{8} \cdot 1 \cdot 1 \cdot 1 = \frac{3}{8}$

41. $\lim_{\theta \to 0} \frac{\tan\theta}{\theta^2 \cot 3\theta} = \lim_{\theta \to 0} \frac{\frac{\sin\theta}{\cos\theta}}{\theta^2 \frac{\cos 3\theta}{\sin 3\theta}} = \lim_{\theta \to 0} \frac{\sin\theta \sin 3\theta}{\theta^2 \cos\theta \cos 3\theta} = \lim_{\theta \to 0} \left(\frac{\sin\theta}{\theta}\right)\left(\frac{\sin 3\theta}{3\theta}\right)\left(\frac{3}{\cos\theta\cos 3\theta}\right) = (1)(1)\left(\frac{3}{1 \cdot 1}\right) = 3$

43. Yes. If $\lim_{x \to a^+} f(x) = L = \lim_{x \to a^-} f(x)$, then $\lim_{x \to a} f(x) = L$. If $\lim_{x \to a^+} f(x) \neq \lim_{x \to a^-} f(x)$, then $\lim_{x \to a} f(x)$ does not exist.

45. If f is an odd function of x, then $f(-x) = -f(x)$. Given $\lim_{x \to 0^+} f(x) = 3$, then $\lim_{x \to 0^-} f(x) = -3$.

47. $I = (5, 5+\delta) \Rightarrow 5 < x < 5+\delta$. Also, $\sqrt{x-5} < \epsilon \Rightarrow x - 5 < \epsilon^2 \Rightarrow x < 5 + \epsilon^2$. Choose $\delta = \epsilon^2$
$\Rightarrow \lim_{x \to 5^+} \sqrt{x-5} = 0$.

49. As $x \to 0^-$ the number x is always negative. Thus, $\left|\frac{x}{|x|} - (-1)\right| < \epsilon \Rightarrow \left|\frac{x}{-x} + 1\right| < \epsilon \Rightarrow 0 < \epsilon$ which is always true independent of the value of x. Hence we can choose any $\delta > 0$ with $-\delta < x < 0 \Rightarrow \lim_{x \to 0^-} \frac{x}{|x|} = -1$.

51. (a) $\lim_{x \to 400^+} \lfloor x \rfloor = 400$. Just observe that if $400 < x < 401$, then $\lfloor x \rfloor = 400$. Thus if we choose $\delta = 1$, we have for any number $\epsilon > 0$ that $400 < x < 400 + \delta \Rightarrow |\lfloor x \rfloor - 400| = |400 - 400| = 0 < \epsilon$.
    (b) $\lim_{x \to 400^-} \lfloor x \rfloor = 399$. Just observe that if $399 < x < 400$ then $\lfloor x \rfloor = 399$. Thus if we choose $\delta = 1$, we have for any number $\epsilon > 0$ that $400 - \delta < x < 400 \Rightarrow |\lfloor x \rfloor - 399| = |399 - 399| = 0 < \epsilon$.
    (c) Since $\lim_{x \to 400^+} \lfloor x \rfloor \neq \lim_{x \to 400^-} \lfloor x \rfloor$ we conclude that $\lim_{x \to 400} \lfloor x \rfloor$ does not exist.

2.5 CONTINUITY

1. No, discontinuous at $x = 2$, not defined at $x = 2$

3. Continuous on $[-1, 3]$

5. (a) Yes (b) Yes, $\lim_{x \to -1^+} f(x) = 0$
   (c) Yes (d) Yes

7. (a) No (b) No

9. $f(2) = 0$, since $\lim_{x \to 2^-} f(x) = -2(2) + 4 = 0 = \lim_{x \to 2^+} f(x)$

11. Nonremovable discontinuity at $x = 1$ because $\lim_{x \to 1} f(x)$ fails to exist ($\lim_{x \to 1^-} f(x) = 1$ and $\lim_{x \to 1^+} f(x) = 0$).
Removable discontinuity at $x = 0$ by assigning the number $\lim_{x \to 0} f(x) = 0$ to be the value of $f(0)$ rather than $f(0) = 1$.

13. Discontinuous only when $x - 2 = 0 \Rightarrow x = 2$

15. Discontinuous only when $x^2 - 4x + 3 = 0 \Rightarrow (x - 3)(x - 1) = 0 \Rightarrow x = 3$ or $x = 1$

17. Continuous everywhere. ($|x - 1| + \sin x$ defined for all x; limits exist and are equal to function values.)

19. Discontinuous only at $x = 0$

21. Discontinuous when 2x is an integer multiple of π, i.e., $2x = n\pi$, n an integer $\Rightarrow x = \frac{n\pi}{2}$, n an integer, but continuous at all other x.

23. Discontinuous at odd integer multiples of $\frac{\pi}{2}$, i.e., $x = (2n - 1)\frac{\pi}{2}$, n an integer, but continuous at all other x.

25. Discontinuous when $2x + 3 < 0$ or $x < -\frac{3}{2} \Rightarrow$ continuous on the interval $\left[-\frac{3}{2}, \infty\right)$.

27. Continuous everywhere: $(2x - 1)^{1/3}$ is defined for all x; limits exist and are equal to function values.

29. Continuous everywhere since $\lim_{x \to 3} \frac{x^2 - x - 6}{x - 3} = \lim_{x \to 3} \frac{(x - 3)(x + 2)}{x - 3} = \lim_{x \to 3} (x + 2) = 5 = g(3)$

31. $\lim_{x \to \pi} \sin(x - \sin x) = \sin(\pi - \sin \pi) = \sin(\pi - 0) = \sin \pi = 0$, and function continuous at $x = \pi$.

33. $\lim_{y \to 1} \sec(y \sec^2 y - \tan^2 y - 1) = \lim_{y \to 1} \sec(y \sec^2 y - \sec^2 y) = \lim_{y \to 1} \sec((y - 1)\sec^2 y) = \sec((1 - 1)\sec^2 1)$
$= \sec 0 = 1$, and function continuous at $y = 1$.

35. $\lim_{t \to 0} \cos\left[\frac{\pi}{\sqrt{19 - 3 \sec 2t}}\right] = \cos\left[\frac{\pi}{\sqrt{19 - 3 \sec 0}}\right] = \cos \frac{\pi}{\sqrt{16}} = \cos \frac{\pi}{4} = \frac{\sqrt{2}}{2}$, and function continuous at $t = 0$.

37. $g(x) = \frac{x^2 - 9}{x - 3} = \frac{(x + 3)(x - 3)}{(x - 3)} = x + 3, x \neq 3 \Rightarrow g(3) = \lim_{x \to 3} (x + 3) = 6$

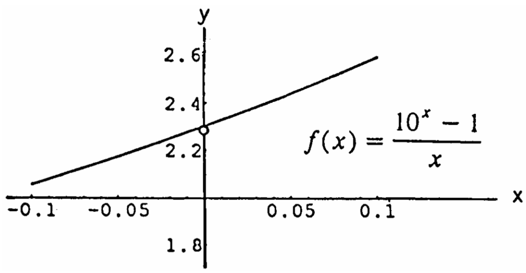
39. $f(s) = \frac{s^3 - 1}{s^2 - 1} = \frac{(s^2 + s + 1)(s - 1)}{(s + 1)(s - 1)} = \frac{s^2 + s + 1}{s + 1}, s \neq 1 \Rightarrow f(1) = \lim_{s \to 1} \left(\frac{s^2 + s + 1}{s + 1}\right) = \frac{3}{2}$

41. As defined, $\lim_{x \to 3^-} f(x) = (3)^2 - 1 = 8$ and $\lim_{x \to 3^+} (2a)(3) = 6a$. For f(x) to be continuous we must have $6a = 8 \Rightarrow a = \frac{4}{3}$.

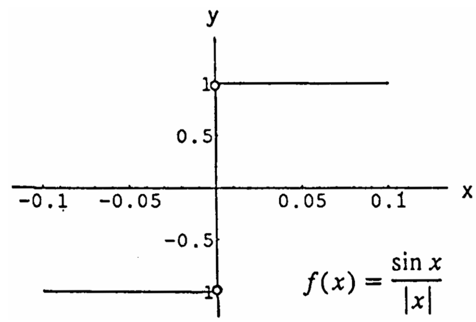
43. As defined, $\lim_{x \to 2^-} f(x) = 12$ and $\lim_{x \to 2^+} f(x) = a^2(2) - 2a = 2a^2 - 2a$. For f(x) to be continuous we must have $12 = 2a^2 - 2a \Rightarrow a = 3$ or $a = -2$.

45. As defined, $\lim_{x \to -1^-} f(x) = -2$ and $\lim_{x \to -1^+} f(x) = a(-1) + b = -a + b$, and $\lim_{x \to 1^-} f(x) = a(1) + b = a + b$ and $\lim_{x \to 1^+} f(x) = 3$. For f(x) to be continuous we must have $-2 = -a + b$ and $a + b = 3 \Rightarrow a = \frac{5}{2}$ and $b = \frac{1}{2}$.

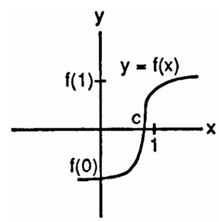
47. The function can be extended: $f(0) \approx 2.3$.



49. The function cannot be extended to be continuous at $x = 0$. If $f(0) = 1$, it will be continuous from the right. Or if $f(0) = -1$, it will be continuous from the left.



51. $f(x)$ is continuous on $[0, 1]$ and $f(0) < 0$, $f(1) > 0$ $\Rightarrow$ by the Intermediate Value Theorem $f(x)$ takes on every value between $f(0)$ and $f(1)$ $\Rightarrow$ the equation $f(x) = 0$ has at least one solution between $x = 0$ and $x = 1$.



53. Let $f(x) = x^3 - 15x + 1$, which is continuous on $[-4, 4]$. Then $f(-4) = -3$, $f(-1) = 15$, $f(1) = -13$, and $f(4) = 5$. By the Intermediate Value Theorem, $f(x) = 0$ for some x in each of the intervals $-4 < x < -1$, $-1 < x < 1$, and $1 < x < 4$. That is, $x^3 - 15x + 1 = 0$ has three solutions in $[-4, 4]$. Since a polynomial of degree 3 can have at most 3 solutions, these are the only solutions.

55. Answers may vary. Note that f is continuous for every value of x.
   (a) $f(0) = 10$, $f(1) = 1^3 - 8(1) + 10 = 3$. Since $3 < \pi < 10$, by the Intermediate Value Theorem, there exists a c so that $0 < c < 1$ and $f(c) = \pi$.
   (b) $f(0) = 10$, $f(-4) = (-4)^3 - 8(-4) + 10 = -22$. Since $-22 < -\sqrt{3} < 10$, by the Intermediate Value Theorem, there exists a c so that $-4 < c < 0$ and $f(c) = -\sqrt{3}$.
   (c) $f(0) = 10$, $f(1000) = (1000)^3 - 8(1000) + 10 = 999{,}992{,}010$. Since $10 < 5{,}000{,}000 < 999{,}992{,}010$, by the Intermediate Value Theorem, there exists a c so that $0 < c < 1000$ and $f(c) = 5{,}000{,}000$.

57. Answers may vary. For example, $f(x) = \frac{\sin(x-2)}{x-2}$ is discontinuous at $x = 2$ because it is not defined there. However, the discontinuity can be removed because f has a limit (namely 1) as $x \to 2$.

59. (a) Suppose x_0 is rational $\Rightarrow$ $f(x_0) = 1$. Choose $\epsilon = \frac{1}{2}$. For any $\delta > 0$ there is an irrational number x (actually infinitely many) in the interval $(x_0 - \delta, x_0 + \delta)$ $\Rightarrow$ $f(x) = 0$. Then $0 < |x - x_0| < \delta$ but $|f(x) - f(x_0)| = 1 > \frac{1}{2} = \epsilon$, so $\lim_{x \to x_0} f(x)$ fails to exist $\Rightarrow$ f is discontinuous at x_0 rational.

   On the other hand, x_0 irrational $\Rightarrow$ $f(x_0) = 0$ and there is a rational number x in $(x_0 - \delta, x_0 + \delta)$ $\Rightarrow$ $f(x) = 1$. Again $\lim_{x \to x_0} f(x)$ fails to exist $\Rightarrow$ f is discontinuous at x_0 irrational. That is, f is discontinuous at every point.

   (b) f is neither right-continuous nor left-continuous at any point x_0 because in every interval $(x_0 - \delta, x_0)$ or $(x_0, x_0 + \delta)$ there exist both rational and irrational real numbers. Thus neither limits $\lim_{x \to x_0^-} f(x)$ and $\lim_{x \to x_0^+} f(x)$ exist by the same arguments used in part (a).

61. No. For instance, if $f(x) = 0$, $g(x) = \lceil x \rceil$, then $h(x) = 0\left(\lceil x \rceil\right) = 0$ is continuous at $x = 0$ and $g(x)$ is not.

63. Yes, because of the Intermediate Value Theorem. If f(a) and f(b) did have different signs then f would have to equal zero at some point between a and b since f is continuous on $[a, b]$.

65. If $f(0) = 0$ or $f(1) = 1$, we are done (i.e., $c = 0$ or $c = 1$ in those cases). Then let $f(0) = a > 0$ and $f(1) = b < 1$ because $0 \le f(x) \le 1$. Define $g(x) = f(x) - x \Rightarrow$ g is continuous on $[0, 1]$. Moreover, $g(0) = f(0) - 0 = a > 0$ and $g(1) = f(1) - 1 = b - 1 < 0 \Rightarrow$ by the Intermediate Value Theorem there is a number c in $(0, 1)$ such that $g(c) = 0 \Rightarrow f(c) - c = 0$ or $f(c) = c$.

67. By Exercises 52 in Section 2.3, we have $\lim_{x \to c} f(x) = L \Leftrightarrow \lim_{h \to 0} f(c + h) = L$.
Thus, $f(x)$ is continuous at $x = c \Leftrightarrow \lim_{x \to c} f(x) = f(c) \Leftrightarrow \lim_{h \to 0} f(c + h) = f(c)$.

69. $x \approx 1.8794, -1.5321, -0.3473$

71. $x \approx 1.7549$

73. $x \approx 3.5156$

75. $x \approx 0.7391$

2.6 LIMITS INVOLVING INFINITY; ASMYPTOTES OF GRAPHS

1. (a) $\lim_{x \to 2} f(x) = 0$
(b) $\lim_{x \to -3^+} f(x) = -2$
(c) $\lim_{x \to -3^-} f(x) = 2$
(d) $\lim_{x \to -3} f(x) =$ does not exist
(e) $\lim_{x \to 0^+} f(x) = -1$
(f) $\lim_{x \to 0^-} f(x) = +\infty$
(g) $\lim_{x \to 0} f(x) =$ does not exist
(h) $\lim_{x \to \infty} f(x) = 1$
(i) $\lim_{x \to -\infty} f(x) = 0$

Note: In these exercises we use the result $\lim_{x \to \pm\infty} \frac{1}{x^{m/n}} = 0$ whenever $\frac{m}{n} > 0$. This result follows immediately from Theorem 8 and the power rule in Theorem 1: $\lim_{x \to \pm\infty} \left(\frac{1}{x^{m/n}}\right) = \lim_{x \to \pm\infty} \left(\frac{1}{x}\right)^{m/n} = \left(\lim_{x \to \pm\infty} \frac{1}{x}\right)^{m/n} = 0^{m/n} = 0$.

3. (a) -3
(b) -3

5. (a) $\frac{1}{2}$
(b) $\frac{1}{2}$

7. (a) $-\frac{5}{3}$
(b) $-\frac{5}{3}$

9. $-\frac{1}{x} \le \frac{\sin 2x}{x} \le \frac{1}{x} \Rightarrow \lim_{x \to \infty} \frac{\sin 2x}{x} = 0$ by the Sandwich Theorem

11. $\lim_{t \to \infty} \frac{2 - t + \sin t}{t + \cos t} = \lim_{t \to \infty} \frac{\frac{2}{t} - 1 + \left(\frac{\sin t}{t}\right)}{1 + \left(\frac{\cos t}{t}\right)} = \frac{0 - 1 + 0}{1 + 0} = -1$

13. (a) $\lim_{x \to \infty} \frac{2x + 3}{5x + 7} = \lim_{x \to \infty} \frac{2 + \frac{3}{x}}{5 + \frac{7}{x}} = \frac{2}{5}$
(b) $\frac{2}{5}$ (same process as part (a))

15. (a) $\lim_{x \to \infty} \frac{x + 1}{x^2 + 3} = \lim_{x \to \infty} \frac{\frac{1}{x} + \frac{1}{x^2}}{1 + \frac{3}{x^2}} = 0$
(b) 0 (same process as part (a))

17. (a) $\lim_{x \to \infty} \frac{7x^3}{x^3 - 3x^2 + 6x} = \lim_{x \to \infty} \frac{7}{1 - \frac{3}{x} + \frac{6}{x^2}} = 7$
(b) 7 (same process as part (a))

19. (a) $\lim_{x \to \infty} \frac{10x^5 + x^4 + 31}{x^6} = \lim_{x \to \infty} \frac{\frac{10}{x} + \frac{1}{x^2} + \frac{31}{x^6}}{1} = 0$ (b) 0 (same process as part (a))

21. (a) $\lim_{x \to \infty} \frac{-2x^3 - 2x + 3}{3x^3 + 3x^2 - 5x} = \lim_{x \to \infty} \frac{-2 - \frac{2}{x^2} + \frac{3}{x^3}}{3 + \frac{3}{x} - \frac{5}{x^2}} = -\frac{2}{3}$
(b) $-\frac{2}{3}$ (same process as part (a))

23. $\lim_{x \to \infty} \sqrt{\frac{8x^2 - 3}{2x^2 + x}} = \lim_{x \to \infty} \sqrt{\frac{8 - \frac{3}{x^2}}{2 + \frac{1}{x}}} = \sqrt{\lim_{x \to \infty} \frac{8 - \frac{3}{x^2}}{2 + \frac{1}{x}}} = \sqrt{\frac{8-0}{2+0}} = \sqrt{4} = 2$

25. $\lim_{x \to -\infty} \left(\frac{1 - x^3}{x^2 - 7x}\right)^5 = \lim_{x \to -\infty} \left(\frac{\frac{1}{x^2} - x}{1 - \frac{7}{x}}\right)^5 = \left(\lim_{x \to -\infty} \frac{\frac{1}{x^2} - x}{1 - \frac{7}{x}}\right)^5 = \left(\frac{0+\infty}{1-0}\right)^5 = \infty$

27. $\lim_{x \to \infty} \frac{2\sqrt{x} + x^{-1}}{3x - 7} = \lim_{x \to \infty} \frac{\left(\frac{2}{x^{1/2}}\right) + \left(\frac{1}{x^2}\right)}{3 - \frac{7}{x}} = 0$

29. $\lim_{x \to -\infty} \frac{\sqrt[3]{x} - \sqrt[5]{x}}{\sqrt[3]{x} + \sqrt[5]{x}} = \lim_{x \to -\infty} \frac{1 - x^{(1/5)-(1/3)}}{1 + x^{(1/5)-(1/3)}} = \lim_{x \to -\infty} \frac{1 - \left(\frac{1}{x^{2/15}}\right)}{1 + \left(\frac{1}{x^{2/15}}\right)} = 1$

31. $\lim_{x \to \infty} \frac{2x^{5/3} - x^{1/3} + 7}{x^{8/5} + 3x + \sqrt{x}} = \lim_{x \to \infty} \frac{2x^{1/15} - \frac{1}{x^{19/15}} + \frac{7}{x^{8/5}}}{1 + \frac{3}{x^{3/5}} + \frac{1}{x^{11/10}}} = \infty$

33. $\lim_{x \to \infty} \frac{\sqrt{x^2+1}}{x+1} = \lim_{x \to \infty} \frac{\sqrt{x^2+1}/\sqrt{x^2}}{(x+1)/\sqrt{x^2}} = \lim_{x \to \infty} \frac{\sqrt{(x^2+1)/x^2}}{(x+1)/x} = \lim_{x \to \infty} \frac{\sqrt{1+1/x^2}}{(1+1/x)} = \frac{\sqrt{1+0}}{(1+0)} = 1$

35. $\lim_{x \to \infty} \frac{x-3}{\sqrt{4x^2+25}} = \lim_{x \to \infty} \frac{(x-3)/\sqrt{x^2}}{\sqrt{4x^2+25}/\sqrt{x^2}} = \lim_{x \to \infty} \frac{(x-3)/x}{\sqrt{(4x^2+25)/x^2}} = \lim_{x \to \infty} \frac{(1-3/x)}{\sqrt{4+25/x^2}} = \frac{(1-0)}{\sqrt{4+0}} = \frac{1}{2}$

37. $\lim_{x \to 0^+} \frac{1}{3x} = \infty$ $\left(\frac{\text{positive}}{\text{positive}}\right)$

39. $\lim_{x \to 2^-} \frac{3}{x-2} = -\infty$ $\left(\frac{\text{positive}}{\text{negative}}\right)$

41. $\lim_{x \to -8^+} \frac{2x}{x+8} = -\infty$ $\left(\frac{\text{negative}}{\text{positive}}\right)$

43. $\lim_{x \to 7} \frac{4}{(x-7)^2} = \infty$ $\left(\frac{\text{positive}}{\text{positive}}\right)$

45. (a) $\lim_{x \to 0^+} \frac{2}{3x^{1/3}} = \infty$ (b) $\lim_{x \to 0^-} \frac{2}{3x^{1/3}} = -\infty$

47. $\lim_{x \to 0} \frac{4}{x^{2/5}} = \lim_{x \to 0} \frac{4}{(x^{1/5})^2} = \infty$

49. $\lim_{x \to \left(\frac{\pi}{2}\right)^-} \tan x = \infty$

51. $\lim_{\theta \to 0^-} (1 + \csc\theta) = -\infty$

53. (a) $\lim_{x \to 2^+} \frac{1}{x^2-4} = \lim_{x \to 2^+} \frac{1}{(x+2)(x-2)} = \infty$ $\left(\frac{1}{\text{positive}\cdot\text{positive}}\right)$
(b) $\lim_{x \to 2^-} \frac{1}{x^2-4} = \lim_{x \to 2^-} \frac{1}{(x+2)(x-2)} = -\infty$ $\left(\frac{1}{\text{positive}\cdot\text{negative}}\right)$
(c) $\lim_{x \to -2^+} \frac{1}{x^2-4} = \lim_{x \to -2^+} \frac{1}{(x+2)(x-2)} = -\infty$ $\left(\frac{1}{\text{positive}\cdot\text{negative}}\right)$
(d) $\lim_{x \to -2^-} \frac{1}{x^2-4} = \lim_{x \to -2^-} \frac{1}{(x+2)(x-2)} = \infty$ $\left(\frac{1}{\text{negative}\cdot\text{negative}}\right)$

55. (a) $\lim_{x \to 0^+} \frac{x^2}{2} - \frac{1}{x} = 0 + \lim_{x \to 0^+} \frac{1}{-x} = -\infty$ $\left(\frac{1}{\text{negative}}\right)$
(b) $\lim_{x \to 0^-} \frac{x^2}{2} - \frac{1}{x} = 0 + \lim_{x \to 0^-} \frac{1}{-x} = \infty$ $\left(\frac{1}{\text{positive}}\right)$

(c) $\lim_{x \to \sqrt[3]{2}} \frac{x^2}{2} - \frac{1}{x} = \frac{2^{2/3}}{2} - \frac{1}{2^{1/3}} = 2^{-1/3} - 2^{-1/3} = 0$

(d) $\lim_{x \to -1} \frac{x^2}{2} - \frac{1}{x} = \frac{1}{2} - \left(\frac{1}{-1}\right) = \frac{3}{2}$

57. (a) $\lim_{x \to 0^+} \frac{x^2 - 3x + 2}{x^3 - 2x^2} = \lim_{x \to 0^+} \frac{(x-2)(x-1)}{x^2(x-2)} = -\infty$ $\left(\frac{\text{negative}\cdot\text{negative}}{\text{positive}\cdot\text{negative}}\right)$

(b) $\lim_{x \to 2^+} \frac{x^2 - 3x + 2}{x^3 - 2x^2} = \lim_{x \to 2^+} \frac{(x-2)(x-1)}{x^2(x-2)} = \lim_{x \to 2^+} \frac{x-1}{x^2} = \frac{1}{4}, x \neq 2$

(c) $\lim_{x \to 2^-} \frac{x^2 - 3x + 2}{x^3 - 2x^2} = \lim_{x \to 2^-} \frac{(x-2)(x-1)}{x^2(x-2)} = \lim_{x \to 2^-} \frac{x-1}{x^2} = \frac{1}{4}, x \neq 2$

(d) $\lim_{x \to 2} \frac{x^2 - 3x + 2}{x^3 - 2x^2} = \lim_{x \to 2} \frac{(x-2)(x-1)}{x^2(x-2)} = \lim_{x \to 2} \frac{x-1}{x^2} = \frac{1}{4}, x \neq 2$

(e) $\lim_{x \to 0} \frac{x^2 - 3x + 2}{x^3 - 2x^2} = \lim_{x \to 0} \frac{(x-2)(x-1)}{x^2(x-2)} = -\infty$ $\left(\frac{\text{negative}\cdot\text{negative}}{\text{positive}\cdot\text{negative}}\right)$

59. (a) $\lim_{t \to 0^+} \left[2 - \frac{3}{t^{1/3}}\right] = -\infty$

(b) $\lim_{t \to 0^-} \left[2 - \frac{3}{t^{1/3}}\right] = \infty$

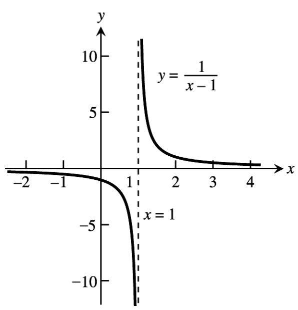
61. (a) $\lim_{x \to 0^+} \left[\frac{1}{x^{2/3}} + \frac{2}{(x-1)^{2/3}}\right] = \infty$

(b) $\lim_{x \to 0^-} \left[\frac{1}{x^{2/3}} + \frac{2}{(x-1)^{2/3}}\right] = \infty$

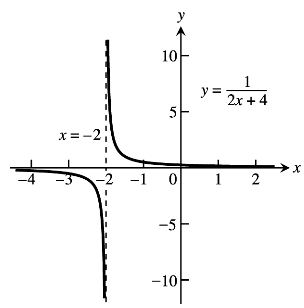
(c) $\lim_{x \to 1^+} \left[\frac{1}{x^{2/3}} + \frac{2}{(x-1)^{2/3}}\right] = \infty$

(d) $\lim_{x \to 1^-} \left[\frac{1}{x^{2/3}} + \frac{2}{(x-1)^{2/3}}\right] = \infty$

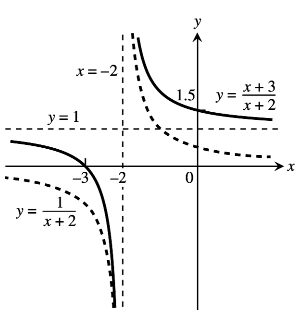
63. $y = \frac{1}{x-1}$



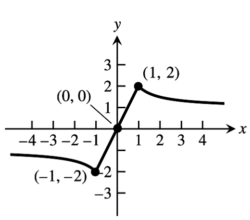
65. $y = \frac{1}{2x+4}$



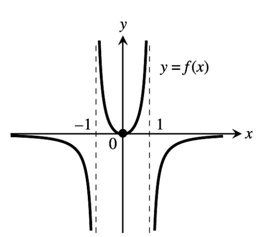
67. $y = \frac{x+3}{x+2} = 1 + \frac{1}{x+2}$



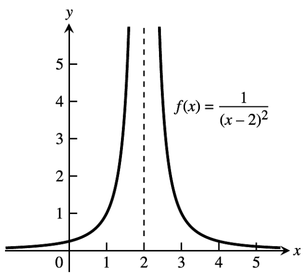
69. Here is one possibility.



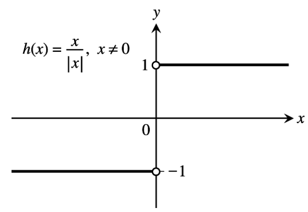
71. Here is one possibility.



73. Here is one possibility.



75. Here is one possibility.



77. Yes. If $\lim\limits_{x \to \infty} \frac{f(x)}{g(x)} = 2$ then the ratio of the polynomials' leading coefficients is 2, so $\lim\limits_{x \to -\infty} \frac{f(x)}{g(x)} = 2$ as well.

79. At most 1 horizontal asymptote: If $\lim\limits_{x \to \infty} \frac{f(x)}{g(x)} = L$, then the ratio of the polynomials' leading coefficients is L, so $\lim\limits_{x \to -\infty} \frac{f(x)}{g(x)} = L$ as well.

81. $\lim\limits_{x \to \infty} \left(\sqrt{x^2+25} - \sqrt{x^2-1}\right) = \lim\limits_{x \to \infty} \left[\sqrt{x^2+25} - \sqrt{x^2-1}\right] \cdot \left[\frac{\sqrt{x^2+25}+\sqrt{x^2-1}}{\sqrt{x^2+25}+\sqrt{x^2-1}}\right] = \lim\limits_{x \to \infty} \frac{(x^2+25)-(x^2-1)}{\sqrt{x^2+25}+\sqrt{x^2-1}}$
$= \lim\limits_{x \to \infty} \frac{26}{\sqrt{x^2+25}+\sqrt{x^2-1}} = \lim\limits_{x \to \infty} \frac{\frac{26}{x}}{\sqrt{1+\frac{25}{x^2}}+\sqrt{1-\frac{1}{x^2}}} = \frac{0}{1+1} = 0$

83. $\lim\limits_{x \to -\infty} \left(2x + \sqrt{4x^2+3x-2}\right) = \lim\limits_{x \to -\infty} \left[2x + \sqrt{4x^2+3x-2}\right] \cdot \left[\frac{2x-\sqrt{4x^2+3x-2}}{2x-\sqrt{4x^2+3x-2}}\right] = \lim\limits_{x \to -\infty} \frac{(4x^2)-(4x^2+3x-2)}{2x-\sqrt{4x^2+3x-2}}$
$= \lim\limits_{x \to -\infty} \frac{-3x+2}{2x-\sqrt{4x^2+3x-2}} = \lim\limits_{x \to -\infty} \frac{\frac{-3x+2}{\sqrt{x^2}}}{\frac{2x}{\sqrt{x^2}}-\sqrt{4+\frac{3}{x}-\frac{2}{x^2}}} = \lim\limits_{x \to -\infty} \frac{\frac{-3x+2}{-x}}{\frac{2x}{-x}-\sqrt{4+\frac{3}{x}-\frac{2}{x^2}}} = \lim\limits_{x \to -\infty} \frac{3-\frac{2}{x}}{-2-\sqrt{4+\frac{3}{x}-\frac{2}{x^2}}}$
$= \frac{3-0}{-2-2} = -\frac{3}{4}$

85. $\lim\limits_{x \to \infty} \left(\sqrt{x^2+3x} - \sqrt{x^2-2x}\right) = \lim\limits_{x \to \infty} \left[\sqrt{x^2+3x} - \sqrt{x^2-2x}\right] \cdot \left[\frac{\sqrt{x^2+3x}+\sqrt{x^2-2x}}{\sqrt{x^2+3x}+\sqrt{x^2-2x}}\right] = \lim\limits_{x \to \infty} \frac{(x^2+3x)-(x^2-2x)}{\sqrt{x^2+3x}+\sqrt{x^2-2x}}$
$= \lim\limits_{x \to \infty} \frac{5x}{\sqrt{x^2+3x}+\sqrt{x^2-2x}} = \lim\limits_{x \to \infty} \frac{5}{\sqrt{1+\frac{3}{x}}+\sqrt{1-\frac{2}{x}}} = \frac{5}{1+1} = \frac{5}{2}$

87. For any $\epsilon > 0$, take $N = 1$. Then for all $x > N$ we have that $|f(x) - k| = |k - k| = 0 < \epsilon$.

89. For every real number $-B < 0$, we must find a $\delta > 0$ such that for all x, $0 < |x - 0| < \delta \Rightarrow \frac{-1}{x^2} < -B$. Now, $-\frac{1}{x^2} < -B < 0 \Leftrightarrow \frac{1}{x^2} > B > 0 \Leftrightarrow x^2 < \frac{1}{B} \Leftrightarrow |x| < \frac{1}{\sqrt{B}}$. Choose $\delta = \frac{1}{\sqrt{B}}$, then $0 < |x| < \delta \Rightarrow |x| < \frac{1}{\sqrt{B}}$ $\Rightarrow \frac{-1}{x^2} < -B$ so that $\lim\limits_{x \to 0} -\frac{1}{x^2} = -\infty$.

91. For every real number $-B < 0$, we must find a $\delta > 0$ such that for all x, $0 < |x - 3| < \delta \Rightarrow \frac{-2}{(x-3)^2} < -B$. Now, $\frac{-2}{(x-3)^2} < -B < 0 \Leftrightarrow \frac{2}{(x-3)^2} > B > 0 \Leftrightarrow \frac{(x-3)^2}{2} < \frac{1}{B} \Leftrightarrow (x-3)^2 < \frac{2}{B} \Leftrightarrow 0 < |x-3| < \sqrt{\frac{2}{B}}$. Choose $\delta = \sqrt{\frac{2}{B}}$, then $0 < |x - 3| < \delta \Rightarrow \frac{-2}{(x-3)^2} < -B < 0$ so that $\lim\limits_{x \to 3} \frac{-2}{(x-3)^2} = -\infty$.

93. (a) We say that f(x) approaches infinity as x approaches x_0 from the left, and write $\lim\limits_{x \to x_0^-} f(x) = \infty$, if for every positive number B, there exists a corresponding number $\delta > 0$ such that for all x, $x_0 - \delta < x < x_0 \Rightarrow f(x) > B$.

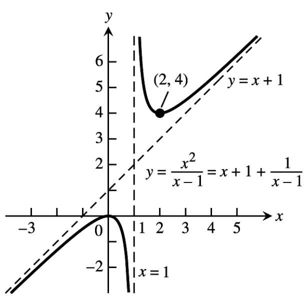
(b) We say that f(x) approaches minus infinity as x approaches x_0 from the right, and write $\lim\limits_{x \to x_0^+} f(x) = -\infty$, if for every positive number B (or negative number $-B$) there exists a corresponding number $\delta > 0$ such that for all x, $x_0 < x < x_0 + \delta \Rightarrow f(x) < -B$.

(c) We say that f(x) approaches minus infinity as x approaches x_0 from the left, and write $\lim_{x \to x_0^-} f(x) = -\infty$, if for every positive number B (or negative number $-B$) there exists a corresponding number $\delta > 0$ such that for all x, $x_0 - \delta < x < x_0 \Rightarrow f(x) < -B$.

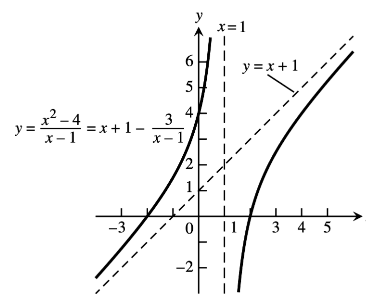
95. For $B > 0$, $\frac{1}{x} < -B < 0 \Leftrightarrow -\frac{1}{x} > B > 0 \Leftrightarrow -x < \frac{1}{B} \Leftrightarrow -\frac{1}{B} < x$. Choose $\delta = \frac{1}{B}$. Then $-\delta < x < 0$ $\Rightarrow -\frac{1}{B} < x \Rightarrow \frac{1}{x} < -B$ so that $\lim_{x \to 0^-} \frac{1}{x} = -\infty$.

97. For $B > 0$, $\frac{1}{x-2} > B \Leftrightarrow 0 < x - 2 < \frac{1}{B}$. Choose $\delta = \frac{1}{B}$. Then $2 < x < 2 + \delta \Rightarrow 0 < x - 2 < \delta \Rightarrow 0 < x - 2 < \frac{1}{B}$ $\Rightarrow \frac{1}{x-2} > B > 0$ so that $\lim_{x \to 2^+} \frac{1}{x-2} = \infty$.

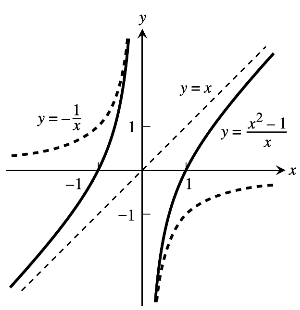
99. $y = \frac{x^2}{x-1} = x + 1 + \frac{1}{x-1}$



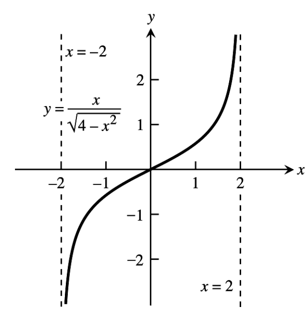
101. $y = \frac{x^2-4}{x-1} = x + 1 - \frac{3}{x-1}$



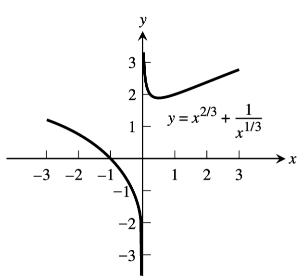
103. $y = \frac{x^2-1}{x} = x - \frac{1}{x}$



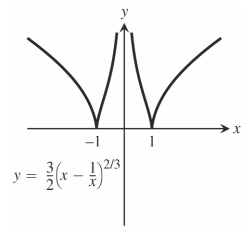
105. $y = \frac{x}{\sqrt{4 - x^2}}$



107. $y = x^{2/3} + \frac{1}{x^{1/3}}$

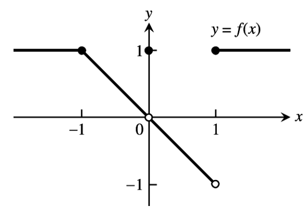


109. (a) $y \to \infty$ (see accompanying graph)
(b) $y \to \infty$ (see accompanying graph)
(c) cusps at $x = \pm 1$ (see accompanying graph)



CHAPTER 2 PRACTICE EXERCISES

1. At $x = -1$: $\lim_{x \to -1^-} f(x) = \lim_{x \to -1^+} f(x) = 1$
$\Rightarrow \lim_{x \to -1} f(x) = 1 = f(-1)$
$\Rightarrow$ f is continuous at $x = -1$.
At $x = 0$: $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = 0 \Rightarrow \lim_{x \to 0} f(x) = 0$.
But $f(0) = 1 \neq \lim_{x \to 0} f(x)$
$\Rightarrow$ f is discontinuous at $x = 0$.
If we define $f(0) = 0$, then the discontinuity at $x = 0$ is removable.
At $x = 1$: $\lim_{x \to 1^-} f(x) = -1$ and $\lim_{x \to 1^+} f(x) = 1$
$\Rightarrow \lim_{x \to 1} f(x)$ does not exist
$\Rightarrow$ f is discontinuous at $x = 1$.



3. (a) $\lim_{t \to t_0} (3f(t)) = 3 \lim_{t \to t_0} f(t) = 3(-7) = -21$
(b) $\lim_{t \to t_0} (f(t))^2 = \left(\lim_{t \to t_0} f(t)\right)^2 = (-7)^2 = 49$
(c) $\lim_{t \to t_0} (f(t) \cdot g(t)) = \lim_{t \to t_0} f(t) \cdot \lim_{t \to t_0} g(t) = (-7)(0) = 0$
(d) $\lim_{t \to t_0} \frac{f(t)}{g(t)-7} = \frac{\lim_{t \to t_0} f(t)}{\lim_{t \to t_0} (g(t) - 7)} = \frac{\lim_{t \to t_0} f(t)}{\lim_{t \to t_0} g(t) - \lim_{t \to t_0} 7} = \frac{-7}{0-7} = 1$
(e) $\lim_{t \to t_0} \cos(g(t)) = \cos\left(\lim_{t \to t_0} g(t)\right) = \cos 0 = 1$
(f) $\lim_{t \to t_0} |f(t)| = \left|\lim_{t \to t_0} f(t)\right| = |-7| = 7$
(g) $\lim_{t \to t_0} (f(t) + g(t)) = \lim_{t \to t_0} f(t) + \lim_{t \to t_0} g(t) = -7 + 0 = -7$
(h) $\lim_{t \to t_0} \left(\frac{1}{f(t)}\right) = \frac{1}{\lim_{t \to t_0} f(t)} = \frac{1}{-7} = -\frac{1}{7}$

5. Since $\lim_{x \to 0} x = 0$ we must have that $\lim_{x \to 0} (4 - g(x)) = 0$. Otherwise, if $\lim_{x \to 0} (4 - g(x))$ is a finite positive number, we would have $\lim_{x \to 0^-} \left[\frac{4-g(x)}{x}\right] = -\infty$ and $\lim_{x \to 0^+} \left[\frac{4-g(x)}{x}\right] = \infty$ so the limit could not equal 1 as $x \to 0$. Similar reasoning holds if $\lim_{x \to 0} (4 - g(x))$ is a finite negative number. We conclude that $\lim_{x \to 0} g(x) = 4$.

7. (a) $\lim_{x \to c} f(x) = \lim_{x \to c} x^{1/3} = c^{1/3} = f(c)$ for every real number c $\Rightarrow$ f is continuous on $(-\infty, \infty)$.
(b) $\lim_{x \to c} g(x) = \lim_{x \to c} x^{3/4} = c^{3/4} = g(c)$ for every nonnegative real number c $\Rightarrow$ g is continuous on $[0, \infty)$.
(c) $\lim_{x \to c} h(x) = \lim_{x \to c} x^{-2/3} = \frac{1}{c^{2/3}} = h(c)$ for every nonzero real number c $\Rightarrow$ h is continuous on $(-\infty, 0)$ and $(-\infty, \infty)$.
(d) $\lim_{x \to c} k(x) = \lim_{x \to c} x^{-1/6} = \frac{1}{c^{1/6}} = k(c)$ for every positive real number c $\Rightarrow$ k is continuous on $(0, \infty)$

9. (a) $\lim_{x\to 0} \frac{x^2-4x+4}{x^3+5x^2-14x} = \lim_{x\to 0} \frac{(x-2)(x-2)}{x(x+7)(x-2)} = \lim_{x\to 0} \frac{x-2}{x(x+7)}$, $x \neq 2$; the limit does not exist because $\lim_{x\to 0^-} \frac{x-2}{x(x+7)} = \infty$ and $\lim_{x\to 0^+} \frac{x-2}{x(x+7)} = -\infty$

(b) $\lim_{x\to 2} \frac{x^2-4x+4}{x^3+5x^2-14x} = \lim_{x\to 2} \frac{(x-2)(x-2)}{x(x+7)(x-2)} = \lim_{x\to 2} \frac{x-2}{x(x+7)}$, $x \neq 2$, and $\lim_{x\to 2} \frac{x-2}{x(x+7)} = \frac{0}{2(9)} = 0$

11. $\lim_{x\to 1} \frac{1-\sqrt{x}}{1-x} = \lim_{x\to 1} \frac{1-\sqrt{x}}{(1-\sqrt{x})(1+\sqrt{x})} = \lim_{x\to 1} \frac{1}{1+\sqrt{x}} = \frac{1}{2}$

13. $\lim_{h\to 0} \frac{(x+h)^2-x^2}{h} = \lim_{h\to 0} \frac{(x^2+2hx+h^2)-x^2}{h} = \lim_{h\to 0} (2x+h) = 2x$

15. $\lim_{x\to 0} \frac{\frac{1}{2+x}-\frac{1}{2}}{x} = \lim_{x\to 0} \frac{2-(2+x)}{2x(2+x)} = \lim_{x\to 0} \frac{-1}{4+2x} = -\frac{1}{4}$

17. $\lim_{x\to 1} \frac{x^{1/3}-1}{\sqrt{x}-1} = \lim_{x\to 1} \frac{(x^{1/3}-1)}{(\sqrt{x}-1)} \cdot \frac{(x^{2/3}+x^{1/3}+1)(\sqrt{x}+1)}{(\sqrt{x}+1)(x^{2/3}+x^{1/3}+1)} = \lim_{x\to 1} \frac{(x-1)(\sqrt{x}+1)}{(x-1)(x^{2/3}+x^{1/3}+1)} = \lim_{x\to 1} \frac{\sqrt{x}+1}{x^{2/3}+x^{1/3}+1}$
$= \frac{1+1}{1+1+1} = \frac{2}{3}$

19. $\lim_{x\to 0} \frac{\tan 2x}{\tan \pi x} = \lim_{x\to 0} \frac{\sin 2x}{\cos 2x} \cdot \frac{\cos \pi x}{\sin \pi x} = \lim_{x\to 0} \left(\frac{\sin 2x}{2x}\right)\left(\frac{\cos \pi x}{\cos 2x}\right)\left(\frac{\pi x}{\sin \pi x}\right)\left(\frac{2x}{\pi x}\right) = 1 \cdot 1 \cdot 1 \cdot \frac{2}{\pi} = \frac{2}{\pi}$

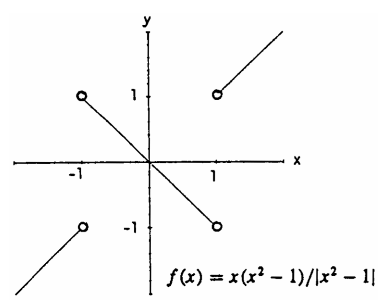
21. $\lim_{x\to \pi} \sin\left(\frac{x}{2} + \sin x\right) = \sin\left(\frac{\pi}{2} + \sin \pi\right) = \sin\left(\frac{\pi}{2}\right) = 1$

23. $\lim_{x\to 0} \frac{8x}{3\sin x - x} = \lim_{x\to 0} \frac{8}{3\frac{\sin x}{x} - 1} = \frac{8}{3(1)-1} = 4$

25. $\lim_{x\to 0^+} [4\,g(x)]^{1/3} = 2 \Rightarrow \left[\lim_{x\to 0^+} 4\,g(x)\right]^{1/3} = 2 \Rightarrow \lim_{x\to 0^+} 4\,g(x) = 8$, since $2^3 = 8$. Then $\lim_{x\to 0^+} g(x) = 2$.

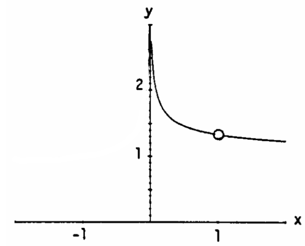
27. $\lim_{x\to 1} \frac{3x^2+1}{g(x)} = \infty \Rightarrow \lim_{x\to 1} g(x) = 0$ since $\lim_{x\to 1} (3x^2+1) = 4$

29. At $x = -1$: $\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} \frac{x(x^2-1)}{|x^2-1|}$
$= \lim_{x\to -1^-} \frac{x(x^2-1)}{x^2-1} = \lim_{x\to -1^-} x = -1$, and
$\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} \frac{x(x^2-1)}{|x^2-1|} = \lim_{x\to -1^+} \frac{x(x^2-1)}{-(x^2-1)}$
$= \lim_{x\to -1} (-x) = -(-1) = 1$. Since
$\lim_{x\to -1^-} f(x) \neq \lim_{x\to -1^+} f(x)$
$\Rightarrow \lim_{x\to -1} f(x)$ does not exist, the function f cannot be extended to a continuous function at $x = -1$.



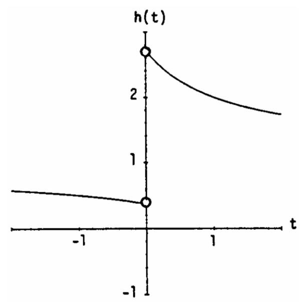
At $x = 1$: $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{x(x^2-1)}{|x^2-1|} = \lim_{x\to 1^-} \frac{x(x^2-1)}{-(x^2-1)} = \lim_{x\to 1^-} (-x) = -1$, and
$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{x(x^2-1)}{|x^2-1|} = \lim_{x\to 1^+} \frac{x(x^2-1)}{x^2-1} = \lim_{x\to 1^+} x = 1$. Again $\lim_{x\to 1} f(x)$ does not exist so f cannot be extended to a continuous function at $x = 1$ either.

31. Yes, f does have a continuous extension to a = 1: define $f(1) = \lim\limits_{x \to 1} \frac{x-1}{x-\sqrt[4]{x}} = \frac{4}{3}$.



$f(x) = \frac{x-1}{x-\sqrt[4]{x}}, \quad a = 1$

33. From the graph we see that $\lim\limits_{t \to 0^-} h(t) \neq \lim\limits_{t \to 0^+} h(t)$ so h <u>cannot</u> be extended to a continuous function at a = 0.



$h(t) = (1+|t|)^{1/t}, \quad a = 0$

35. (a) $f(-1) = -1$ and $f(2) = 5 \Rightarrow$ f has a root between -1 and 2 by the Intermediate Value Theorem.
(b), (c) root is 1.32471795724

37. $\lim\limits_{x \to \infty} \frac{2x+3}{5x+7} = \lim\limits_{x \to \infty} \frac{2+\frac{3}{x}}{5+\frac{7}{x}} = \frac{2+0}{5+0} = \frac{2}{5}$

39. $\lim\limits_{x \to -\infty} \frac{x^2-4x+8}{3x^3} = \lim\limits_{x \to -\infty} \left(\frac{1}{3x} - \frac{4}{3x^2} + \frac{8}{3x^3}\right) = 0 - 0 + 0 = 0$

41. $\lim\limits_{x \to -\infty} \frac{x^2-7x}{x+1} = \lim\limits_{x \to -\infty} \frac{x-7}{1+\frac{1}{x}} = -\infty$

43. $\lim\limits_{x \to \infty} \frac{\sin x}{\lfloor x \rfloor} \le \lim\limits_{x \to \infty} \frac{1}{\lfloor x \rfloor} = 0$ since int $x \to \infty$ as $x \to \infty \Rightarrow \lim\limits_{x \to \infty} \frac{\sin x}{\lfloor x \rfloor} = 0$.

45. $\lim\limits_{x \to \infty} \frac{x + \sin x + 2\sqrt{x}}{x + \sin x} = \lim\limits_{x \to \infty} \frac{1 + \frac{\sin x}{x} + \frac{2}{\sqrt{x}}}{1 + \frac{\sin x}{x}} = \frac{1+0+0}{1+0} = 1$

47. (a) $y = \frac{x^2+4}{x-3}$ is undefined at x = 3: $\lim\limits_{x \to 3^-} \frac{x^2+4}{x-3} = -\infty$ and $\lim\limits_{x \to 3^+} \frac{x^2+4}{x-3} = +\infty$, thus x = 3 is a vertical asymptote.
(b) $y = \frac{x^2-x-2}{x^2-2x+1}$ is undefined at x = 1: $\lim\limits_{x \to 1^-} \frac{x^2-x-2}{x^2-2x+1} = -\infty$ and $\lim\limits_{x \to 1^+} \frac{x^2-x-2}{x^2-2x+1} = -\infty$, thus x = 1 is a vertical asymptote.
(c) $y = \frac{x^2+x-6}{x^2+2x-8}$ is undefined at x = 2 and -4: $\lim\limits_{x \to 2} \frac{x^2+x-6}{x^2+2x-8} = \lim\limits_{x \to 2} \frac{x+3}{x+4} = \frac{5}{6}$; $\lim\limits_{x \to -4^-} \frac{x^2+x-6}{x^2+2x-8} = \lim\limits_{x \to -4^-} \frac{x+3}{x+4} = \infty$
$\lim\limits_{x \to -4^+} \frac{x^2+x-6}{x^2+2x-8} = \lim\limits_{x \to -4^+} \frac{x+3}{x+4} = -\infty$. Thus x = -4 is a vertical asymptote.

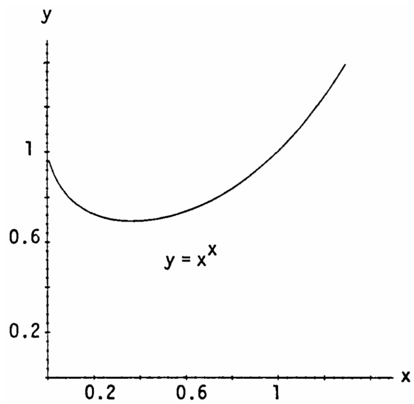
CHAPTER 2 ADDITIONAL AND ADVANCED EXERCISES

1. (a)

x	0.1	0.01	0.001	0.0001	0.00001
x^x	0.7943	0.9550	0.9931	0.9991	0.9999

Apparently, $\lim\limits_{x \to 0^+} x^x = 1$

(b)



3. $\lim\limits_{v \to c^-} L = \lim\limits_{v \to c^-} L_0 \sqrt{1 - \frac{v^2}{c^2}} = L_0 \sqrt{1 - \frac{\lim\limits_{v \to c^-} v^2}{c^2}} = L_0 \sqrt{1 - \frac{c^2}{c^2}} = 0$
The left-hand limit was needed because the function L is undefined if $v > c$ (the rocket cannot move faster than the speed of light).

5. $|10 + (t - 70) \times 10^{-4} - 10| < 0.0005 \Rightarrow |(t - 70) \times 10^{-4}| < 0.0005 \Rightarrow -0.0005 < (t - 70) \times 10^{-4} < 0.0005$
$\Rightarrow -5 < t - 70 < 5 \Rightarrow 65° < t < 75° \Rightarrow$ Within 5° F.

7. Show $\lim\limits_{x \to 1} f(x) = \lim\limits_{x \to 1} (x^2 - 7) = -6 = f(1)$.
Step 1: $|(x^2 - 7) + 6| < \epsilon \Rightarrow -\epsilon < x^2 - 1 < \epsilon \Rightarrow 1 - \epsilon < x^2 < 1 + \epsilon \Rightarrow \sqrt{1 - \epsilon} < x < \sqrt{1 + \epsilon}$.
Step 2: $|x - 1| < \delta \Rightarrow -\delta < x - 1 < \delta \Rightarrow -\delta + 1 < x < \delta + 1$.
Then $-\delta + 1 = \sqrt{1 - \epsilon}$ or $\delta + 1 = \sqrt{1 + \epsilon}$. Choose $\delta = \min\left\{1 - \sqrt{1 - \epsilon}, \sqrt{1 + \epsilon} - 1\right\}$, then
$0 < |x - 1| < \delta \Rightarrow |(x^2 - 7) - 6| < \epsilon$ and $\lim\limits_{x \to 1} f(x) = -6$. By the continuity test, f(x) is continuous at $x = 1$.

9. Show $\lim\limits_{x \to 2} h(x) = \lim\limits_{x \to 2} \sqrt{2x - 3} = 1 = h(2)$.
Step 1: $\left|\sqrt{2x - 3} - 1\right| < \epsilon \Rightarrow -\epsilon < \sqrt{2x - 3} - 1 < \epsilon \Rightarrow 1 - \epsilon < \sqrt{2x - 3} < 1 + \epsilon \Rightarrow \frac{(1 - \epsilon)^2 + 3}{2} < x < \frac{(1 + \epsilon)^2 + 3}{2}$.
Step 2: $|x - 2| < \delta \Rightarrow -\delta < x - 2 < \delta$ or $-\delta + 2 < x < \delta + 2$.
Then $-\delta + 2 = \frac{(1 - \epsilon)^2 + 3}{2} \Rightarrow \delta = 2 - \frac{(1 - \epsilon)^2 + 3}{2} = \frac{1 - (1 - \epsilon)^2}{2} = \epsilon - \frac{\epsilon^2}{2}$, or $\delta + 2 = \frac{(1 + \epsilon)^2 + 3}{2}$
$\Rightarrow \delta = \frac{(1 + \epsilon)^2 + 3}{2} - 2 = \frac{(1 + \epsilon)^2 - 1}{2} = \epsilon + \frac{\epsilon^2}{2}$. Choose $\delta = \epsilon - \frac{\epsilon^2}{2}$, the smaller of the two values. Then,
$0 < |x - 2| < \delta \Rightarrow \left|\sqrt{2x - 3} - 1\right| < \epsilon$, so $\lim\limits_{x \to 2} \sqrt{2x - 3} = 1$. By the continuity test, h(x) is continuous at $x = 2$.

11. Suppose L_1 and L_2 are two different limits. Without loss of generality assume $L_2 > L_1$. Let $\epsilon = \frac{1}{3}(L_2 - L_1)$.
Since $\lim\limits_{x \to x_0} f(x) = L_1$ there is a $\delta_1 > 0$ such that $0 < |x - x_0| < \delta_1 \Rightarrow |f(x) - L_1| < \epsilon \Rightarrow -\epsilon < f(x) - L_1 < \epsilon$
$\Rightarrow -\frac{1}{3}(L_2 - L_1) + L_1 < f(x) < \frac{1}{3}(L_2 - L_1) + L_1 \Rightarrow 4L_1 - L_2 < 3f(x) < 2L_1 + L_2$. Likewise, $\lim\limits_{x \to x_0} f(x) = L_2$
so there is a δ_2 such that $0 < |x - x_0| < \delta_2 \Rightarrow |f(x) - L_2| < \epsilon \Rightarrow -\epsilon < f(x) - L_2 < \epsilon$
$\Rightarrow -\frac{1}{3}(L_2 - L_1) + L_2 < f(x) < \frac{1}{3}(L_2 - L_1) + L_2 \Rightarrow 2L_2 + L_1 < 3f(x) < 4L_2 - L_1$
$\Rightarrow L_1 - 4L_2 < -3f(x) < -2L_2 - L_1$. If $\delta = \min\{\delta_1, \delta_2\}$ both inequalities must hold for $0 < |x - x_0| < \delta$:
$\left.\begin{array}{l} 4L_1 - L_2 < 3f(x) < 2L_1 + L_2 \\ L_1 - 4L_2 < -3f(x) < -2L_2 - L_1 \end{array}\right\} \Rightarrow 5(L_1 - L_2) < 0 < L_1 - L_2$. That is, $L_1 - L_2 < 0$ and $L_1 - L_2 > 0$,
a contradiction.

13. (a) Since $x \to 0^+, 0 < x^3 < x < 1 \Rightarrow (x^3 - x) \to 0^- \Rightarrow \lim_{x \to 0^+} f(x^3 - x) = \lim_{y \to 0^-} f(y) = B$ where $y = x^3 - x$.

(b) Since $x \to 0^-, -1 < x < x^3 < 0 \Rightarrow (x^3 - x) \to 0^+ \Rightarrow \lim_{x \to 0^-} f(x^3 - x) = \lim_{y \to 0^+} f(y) = A$ where $y = x^3 - x$.

(c) Since $x \to 0^+, 0 < x^4 < x^2 < 1 \Rightarrow (x^2 - x^4) \to 0^+ \Rightarrow \lim_{x \to 0^+} f(x^2 - x^4) = \lim_{y \to 0^+} f(y) = A$ where $y = x^2 - x^4$.

(d) Since $x \to 0^-, -1 < x < 0 \Rightarrow 0 < x^4 < x^2 < 1 \Rightarrow (x^2 - x^4) \to 0^+ \Rightarrow \lim_{x \to 0^+} f(x^2 - x^4) = A$ as in part (c).

15. Show $\lim_{x \to -1} f(x) = \lim_{x \to -1} \frac{x^2 - 1}{x + 1} = \lim_{x \to -1} \frac{(x+1)(x-1)}{(x+1)} = -2, x \neq -1$.

Define the continuous extension of f(x) as $F(x) = \begin{cases} \frac{x^2-1}{x+1}, & x \neq -1 \\ -2 & , x = -1 \end{cases}$. We now prove the limit of f(x) as $x \to -1$ exists and has the correct value.

Step 1: $\left| \frac{x^2-1}{x+1} - (-2) \right| < \epsilon \Rightarrow -\epsilon < \frac{(x+1)(x-1)}{(x+1)} + 2 < \epsilon \Rightarrow -\epsilon < (x-1) + 2 < \epsilon, x \neq -1 \Rightarrow -\epsilon - 1 < x < \epsilon - 1$.

Step 2: $|x - (-1)| < \delta \Rightarrow -\delta < x + 1 < \delta \Rightarrow -\delta - 1 < x < \delta - 1$.

Then $-\delta - 1 = -\epsilon - 1 \Rightarrow \delta = \epsilon$, or $\delta - 1 = \epsilon - 1 \Rightarrow \delta = \epsilon$. Choose $\delta = \epsilon$. Then $0 < |x - (-1)| < \delta$ $\Rightarrow \left| \frac{x^2-1}{x+1} - (-2) \right| < \epsilon \Rightarrow \lim_{x \to -1} F(x) = -2$. Since the conditions of the continuity test are met by F(x), then f(x) has a continuous extension to F(x) at $x = -1$.

17. (a) Let $\epsilon > 0$ be given. If x is rational, then $f(x) = x \Rightarrow |f(x) - 0| = |x - 0| < \epsilon \Leftrightarrow |x - 0| < \epsilon$; i.e., choose $\delta = \epsilon$. Then $|x - 0| < \delta \Rightarrow |f(x) - 0| < \epsilon$ for x rational. If x is irrational, then $f(x) = 0 \Rightarrow |f(x) - 0| < \epsilon$ $\Leftrightarrow 0 < \epsilon$ which is true no matter how close irrational x is to 0, so again we can choose $\delta = \epsilon$. In either case, given $\epsilon > 0$ there is a $\delta = \epsilon > 0$ such that $0 < |x - 0| < \delta \Rightarrow |f(x) - 0| < \epsilon$. Therefore, f is continuous at $x = 0$.

(b) Choose $x = c > 0$. Then within any interval $(c - \delta, c + \delta)$ there are both rational and irrational numbers. If c is rational, pick $\epsilon = \frac{c}{2}$. No matter how small we choose $\delta > 0$ there is an irrational number x in $(c - \delta, c + \delta) \Rightarrow |f(x) - f(c)| = |0 - c| = c > \frac{c}{2} = \epsilon$. That is, f is not continuous at any rational $c > 0$. On the other hand, suppose c is irrational $\Rightarrow f(c) = 0$. Again pick $\epsilon = \frac{c}{2}$. No matter how small we choose $\delta > 0$ there is a rational number x in $(c - \delta, c + \delta)$ with $|x - c| < \frac{c}{2} = \epsilon \Leftrightarrow \frac{c}{2} < x < \frac{3c}{2}$. Then $|f(x) - f(c)| = |x - 0|$ $= |x| > \frac{c}{2} = \epsilon \Rightarrow$ f is not continuous at any irrational $c > 0$.

If $x = c < 0$, repeat the argument picking $\epsilon = \frac{|c|}{2} = \frac{-c}{2}$. Therefore f fails to be continuous at any nonzero value $x = c$.

19. Yes. Let R be the radius of the equator (earth) and suppose at a fixed instant of time we label noon as the zero point, 0, on the equator $\Rightarrow 0 + \pi R$ represents the midnight point (at the same exact time). Suppose x_1 is a point on the equator “just after" noon $\Rightarrow x_1 + \pi R$ is simultaneously “just after" midnight. It seems reasonable that the temperature T at a point just after noon is hotter than it would be at the diametrically opposite point just after midnight: That is, $T(x_1) - T(x_1 + \pi R) > 0$. At exactly the same moment in time pick x_2 to be a point just before midnight $\Rightarrow x_2 + \pi R$ is just before noon. Then $T(x_2) - T(x_2 + \pi R) < 0$. Assuming the temperature function T is continuous along the equator (which is reasonable), the Intermediate Value Theorem says there is a point c between 0 (noon) and πR (simultaneously midnight) such that $T(c) - T(c + \pi R) = 0$; i.e., there is always a pair of antipodal points on the earth's equator where the temperatures are the same.

21. (a) At $x = 0$: $\lim_{a \to 0} r_+(a) = \lim_{a \to 0} \frac{-1 + \sqrt{1+a}}{a} = \lim_{a \to 0} \left(\frac{-1 + \sqrt{1+a}}{a} \right) \left(\frac{-1 - \sqrt{1+a}}{-1 - \sqrt{1+a}} \right)$

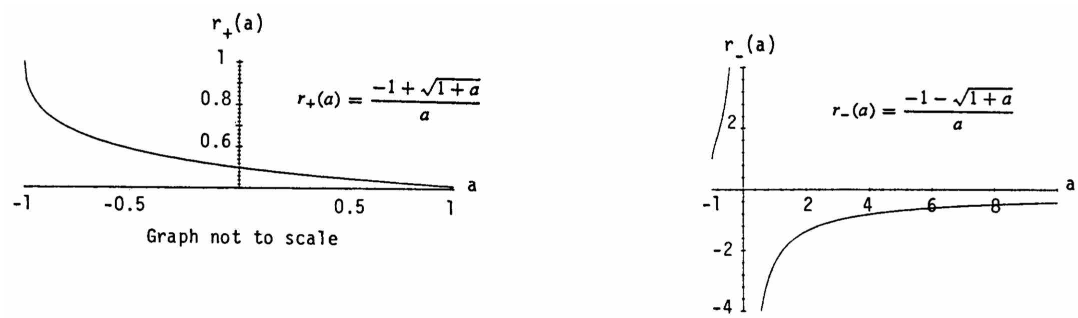
$= \lim_{a \to 0} \frac{1 - (1+a)}{a(-1 - \sqrt{1+a})} = \frac{-1}{-1 - \sqrt{1+0}} = \frac{1}{2}$

At $x = -1$: $\lim_{a \to -1^+} r_+(a) = \lim_{a \to -1^+} \frac{1 - (1+a)}{a(-1 - \sqrt{1+a})} = \lim_{a \to -1} \frac{-a}{a(-1 - \sqrt{1+a})} = \frac{-1}{-1 - \sqrt{0}} = 1$

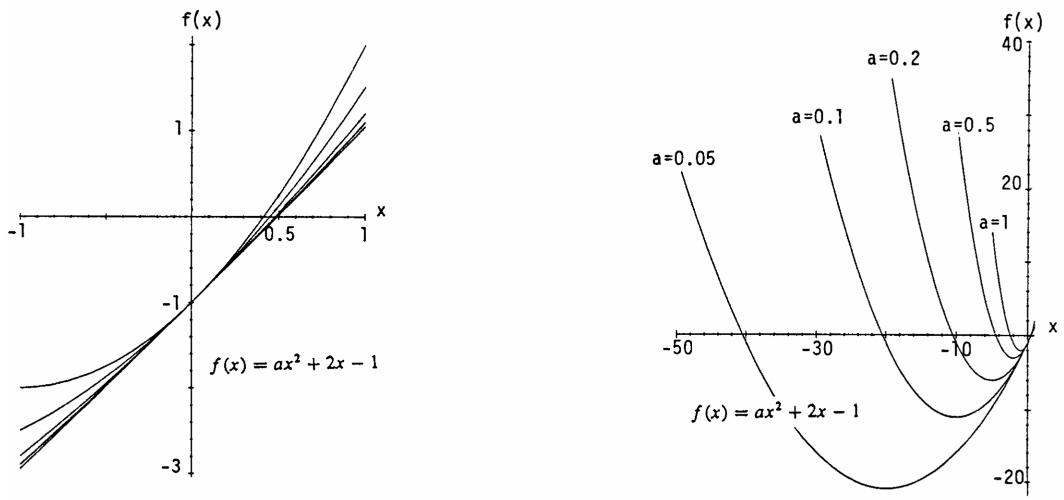
(b) At $x = 0$: $\lim_{a \to 0^-} r_-(a) = \lim_{a \to 0^-} \frac{-1-\sqrt{1+a}}{a} = \lim_{a \to 0^-} \left(\frac{-1-\sqrt{1+a}}{a}\right)\left(\frac{-1+\sqrt{1+a}}{-1+\sqrt{1+a}}\right)$
$= \lim_{a \to 0^-} \frac{1-(1+a)}{a\left(-1+\sqrt{1+a}\right)} = \lim_{a \to 0^-} \frac{-a}{a\left(-1+\sqrt{1+a}\right)} = \lim_{a \to 0^-} \frac{-1}{-1+\sqrt{1+a}} = \infty$ (because the denominator is always negative); $\lim_{a \to 0^+} r_-(a) = \lim_{a \to 0^+} \frac{-1}{-1+\sqrt{1+a}} = -\infty$ (because the denominator is always positive). Therefore, $\lim_{a \to 0} r_-(a)$ does not exist.

At $x = -1$: $\lim_{a \to -1^+} r_-(a) = \lim_{a \to -1^+} \frac{-1-\sqrt{1+a}}{a} = \lim_{a \to -1^+} \frac{-1}{-1+\sqrt{1+a}} = 1$

(c)



(d)



23. (a) The function f is bounded on D if $f(x) \ge M$ and $f(x) \le N$ for all x in D. This means $M \le f(x) \le N$ for all x in D. Choose B to be $\max\{|M|, |N|\}$. Then $|f(x)| \le B$. On the other hand, if $|f(x)| \le B$, then $-B \le f(x) \le B \Rightarrow f(x) \ge -B$ and $f(x) \le B \Rightarrow f(x)$ is bounded on D with $N = B$ an upper bound and $M = -B$ a lower bound.

(b) Assume $f(x) \le N$ for all x and that $L > N$. Let $\epsilon = \frac{L-N}{2}$. Since $\lim_{x \to x_0} f(x) = L$ there is a $\delta > 0$ such that $0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \epsilon \Leftrightarrow L - \epsilon < f(x) < L + \epsilon \Leftrightarrow L - \frac{L-N}{2} < f(x) < L + \frac{L-N}{2}$ $\Leftrightarrow \frac{L+N}{2} < f(x) < \frac{3L-N}{2}$. But $L > N \Rightarrow \frac{L+N}{2} > N \Rightarrow N < f(x)$ contrary to the boundedness assumption $f(x) \le N$. This contradiction proves $L \le N$.

(c) Assume $M \le f(x)$ for all x and that $L < M$. Let $\epsilon = \frac{M-L}{2}$. As in part (b), $0 < |x - x_0| < \delta$ $\Rightarrow L - \frac{M-L}{2} < f(x) < L + \frac{M-L}{2} \Leftrightarrow \frac{3L-M}{2} < f(x) < \frac{M+L}{2} < M$, a contradiction.

25. $\lim_{x \to 0} = \frac{\sin(1-\cos x)}{x} = \lim_{x \to 0} \frac{\sin(1-\cos x)}{1-\cos x} \cdot \frac{1-\cos x}{x} \cdot \frac{1+\cos x}{1+\cos x} = \lim_{x \to 0} \frac{\sin(1-\cos x)}{1-\cos x} \cdot \lim_{x \to 0} \frac{1-\cos^2 x}{x(1+\cos x)} = 1 \cdot \lim_{x \to 0} \frac{\sin^2 x}{x(1+\cos x)}$
$= \lim_{x \to 0} \frac{\sin x}{x} \cdot \frac{\sin x}{1+\cos x} = 1 \cdot \left(\frac{0}{2}\right) = 0.$

27. $\lim\limits_{x \to 0} \frac{\sin(\sin x)}{x} = \lim\limits_{x \to 0} \frac{\sin(\sin x)}{\sin x} \cdot \frac{\sin x}{x} = \lim\limits_{x \to 0} \frac{\sin(\sin x)}{\sin x} \cdot \lim\limits_{x \to 0} \frac{\sin x}{x} = 1 \cdot 1 = 1.$

29. $\lim\limits_{x \to 2} \frac{\sin(x^2-4)}{x-2} = \lim\limits_{x \to 2} \frac{\sin(x^2-4)}{x^2-4} \cdot (x+2) = \lim\limits_{x \to 2} \frac{\sin(x^2-4)}{x^2-4} \cdot \lim\limits_{x \to 2}(x+2) = 1 \cdot 4 = 4$

31. Since the highest power of x in the numerator is 1 more than the highest power of x in the denominator, there is an oblique asymptote. $y = \frac{2x^{3/2}+2x-3}{\sqrt{x}+1} = 2x - \frac{3}{\sqrt{x}+1}$, thus the oblique asymptote is $y = 2x$.

33. As $x \to \pm\infty$, $x^2+1 \to x^2 \Rightarrow \sqrt{x^2+1} \to \sqrt{x^2}$; as $x \to -\infty$, $\sqrt{x^2} = -x$, and as $x \to +\infty$, $\sqrt{x^2} = x$; thus the oblique asymptotes are $y = x$ and $y = -x$.