RANK-ONE CONVEX ENERGY UNDER CERTAIN GEOMETRIC FLOWS

BAISHENG YAN

Abstract. In this paper, we generalize some of the interesting properties of energy $E(w) = \int_{\Omega} f(\nabla w) \, dx$ on maps $w$ generated by equilibrium or gradient-like flows to the geometric flows $u_t \cdot \text{div}(Df(\nabla u)) = 0$ on $\Omega \times I$, where $\Omega$ is a bounded domain in $\mathbb{R}^n$ and $I$ is some interval containing 0 in $\mathbb{R}$. We show that for suitably defined Lipschitz weak flows $u$ the map $w(x) = u(x, \theta(x))$ for any Lipschitz function $\theta$ vanishing on $\partial \Omega$ will have no lower energy than $u_0 = u(\cdot, 0)$ if $f$ is rank-one convex.

1. Introduction

Given a function $f: M^{m \times n} \to \mathbb{R}$ on the space of $m \times n$ matrices, Morrey’s quasiconvexity condition for $f$ at $0 \in M^{m \times n}$ requires that the following inequality

$$(1.1) \quad E(w) = \int_{\Omega} f(\nabla w(x)) \, dx \geq |\Omega| f(0) = E(0)$$

hold for all Lipschitz functions $w: \Omega \to \mathbb{R}^m$ with $w|_{\partial \Omega} = 0$. Here $\Omega$ is a bounded Lipschitz domain in $\mathbb{R}^n$ and $\nabla w = (w^i_{x,a})$ is the Jacobi matrix of $w$ defined pointwise on $\Omega$.

There has been a considerable amount of work on Morrey’s quasiconvexity in the calculus of variations and nonlinear elasticity [3, 5, 8, 9, 10]. Integral estimates like (1.1) are also important for other problems in geometric mapping theory [2, 7], where some convexity (such as rank-one convexity) property of $f$ may be known or easy to check. Recall that $f$ is rank-one convex if, for any given $\xi \in M^{m \times n}$, $p \in \mathbb{R}^m$ and $a \in \mathbb{R}^n$, the function $h(s) = f(\xi + sp \otimes a)$ is a convex function of $s$ on $\mathbb{R}$; here $p \otimes a$ stands for the matrix $(p'a_\alpha)$ in $M^{m \times n}$.

An important result of Sivaloganathan [11], following an earlier work of Ball [4], showed that if $f$ is $C^2$ and rank-one convex then the inequality (1.1) holds for all maps $w$ in the form of

$$(1.2) \quad w(x) = u(x, \theta(x)),$$

where $\theta \in C^1(\Omega)$ with $\theta(x) \in I$, some given interval containing 0, and $u(x, t)$, with $u(x, 0) = 0$, is a family of $C^2$-equilibria (also smooth in $t$) of energy $E$; that is,

$$(1.3) \quad \text{div} Df(\nabla u(x, t)) = 0 \quad \text{in} \ \Omega \ \text{for all} \ t \in I.$$
In particular, under the rank-one convexity of $f$, inequality (1.1) holds for all maps $w(x) = Q(\theta(x))x$ with any $C^2$ function $Q: I \to \mathbb{M}^{m \times n}$ such that $Q(0) = 0$ and $\theta \in C^1(I)$ with $\theta(x) \in I$.

Using similar computations to [4, 11], Evans [6] has observed some interesting properties of $E(w)$ for maps $w$ of the form (1.2) generated by the smooth gradient flows $u(x,t)\theta$ of rank-one convex energy $E$; that is, (1.4) 

$$u_t(x,t) = \text{div}(Df(\nabla u(x,t))).$$

The main steps in [4, 6, 11] are to consider the Lagrangian

$$L(x,t,a) = f(\nabla u(x,t)) + Df(\nabla u(x,t)) : (u_t(x,t) \otimes a)$$

and the action functional

$$A(\theta) = \int_\Omega L(x,\theta(x),\nabla \theta(x)) \, dx.$$

Note that $L_a(x,t,a) = L_a(x,t)$ is independent of $a$ and $L = f(\nabla u) + L_a \cdot a$. If $f$, $u$ are $C^2$, one can easily verify the following point-wise identity:

(1.5) $$L_t(x,\eta,\nabla \eta) - \text{div}(L_a(x,\eta)) = -u_t(x,\eta) \cdot [\text{div}(Df(\nabla u))](x,\eta)$$

for all $\eta \in C^1(\Omega)$. Therefore, if $u$ satisfies the equilibrium flow (1.3) then it can be shown that the action $A(\theta)$ is a null-Lagrangian [11]: if $u$ satisfies the gradient flow (1.4) then $L_t(x,\eta,\nabla \eta) - \text{div}(L_a(x,\eta)) \leq 0$ for all $\eta \in C^1(\Omega)$, which has been useful in [6].

In this paper, we try to extend the similar calculations to maps $w$ of form (1.2) generated by certain geometric flows $u(x,t)$ that may not satisfy smoothness requirements. Assume here $f$ is only $C^1$ and rank-one convex and, motivated by identity (1.5), we focus on the following geometric flow:

(1.6) $$u_t(x,t) \cdot \text{div} Df(\nabla u(x,t)) = 0 \quad \text{on } \Omega \times I.$$ 

This generalizes the equilibrium flow (1.3). Note that (1.6) can be written as a divergence form in the $(x,t)$-space as follows:

(1.7) $u_t \cdot \text{div} Df(\nabla u) = \text{div}(u_t \cdot Df(\nabla u)) - (f(\nabla u))_t = 0.$

Also, in order for the action functional $A(\theta)$ to be well-defined, it is necessary that the Lagrangian $L(x,t,a)$ be at least Carathéodory, that is, measurable in $x$ and continuous in $(t,a)$. Therefore, we will define weak solutions of flow equation (1.6) as follows.

**Definition 1.1.** Let $\Omega_I = \Omega \times I$ and $u \in W^{1,\infty}(\Omega_I;\mathbb{R}^m)$. We say $u$ is a Lipschitz weak solution to (1.6) or simply a weak flow provided that $\nabla u(x,t)$ and $u_t(x,t)$ are continuous in $t \in I$ for almost every $x \in \Omega$ and that, for any bounded Lipschitz domain $U \subseteq \Omega_I$ and for any $\zeta \in W^{1,\infty}_0(\Omega_I)$, it follows

(1.8) $$\int_U [Df(\nabla u) : (u_t \otimes \nabla \zeta) - f(\nabla u)\zeta] \, dxdt = 0.$$

We shall see that under some mild regularity assumption any weak solution to the equilibrium flow (1.3) will be a weak flow for (1.6).

The main result of this paper is the following theorem whose proof will be given in Section 3.
**Theorem 1.1.** Let $f$ be $C^1$ and rank-one convex and $u \in W^{1,\infty}(\Omega;\mathbb{R}^m)$ be a weak flow of (1.6) with $u(x,0) = u_0(x)$. Given any Lipschitz function $\theta \in W^{1,\infty}_0(\Omega)$ with $\theta(x) \in I$ for all $x \in \Omega$, let $w(x) = u(x,\theta(x))$. Then $w - u_0 \in W^{1,\infty}_0(\Omega;\mathbb{R}^m)$ and $E(w) \geq E(u_0)$.

**Remark 1.1.** (a) The flow equation (1.6) is under-determined in the sense that there are $m$ unknown functions $u'$ in $u$ and, in general, existence of weak flows for (1.6) is not known. One can try to study many simple flows that satisfy (1.6). For example, let $P$ be any constant matrix such that $P + P^T = 0$; then the flow

$$u_t = \text{div}(PDf(\nabla u)).$$

will satisfy (1.6). In the case when $m = 2$, $f(\xi) = \frac{1}{2}||\xi||^2$ and constant matrix function $P = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, the equation (1.9) for $u = (v, w)$ becomes $v_t = \Delta w$, $w_t = -\Delta v$, which reduces to the beam equation $v_{tt} + \Delta^2 v = 0$ in linear elasticity.

(b) Certain special weak flows of (1.6) for general rotation-invariant functions $f$ will be studied later. Note that the functions studied in [1] are rotation-invariant. It will be shown that any Lipschitz critical points of the functional

$$\int_{\Omega} \left( f(\nabla v(x)) + \frac{\lambda(x)}{2} |v(x)|^2 \right) \, dx,$$

where $\lambda \in L^1(\Omega)$ is any given function, will generate a weak flow of (1.6); see Section 4.

(c) This result may provide a new way of establishing quasiconvexity for certain rank-one convex functions. Given any $w \in W^{1,\infty}_0(\Omega;\mathbb{R}^m)$, let $u(x,t) = w(x) + (t - w^m(x))v(x,t)$, where $v(x,t)$ with $v^m \equiv 1$ is to be determined. Then $w(x) = u(x, w^m(x))$ and $u_0 = u(x,0) = (\tilde{w}^1, \ldots, \tilde{w}^{m-1}, 0)$, where $\tilde{w}^i(x) = w^i(x) - w^m(x)v^i(x,0)$. Therefore, if one could solve (1.6) for such a $u$ to determine $v^i(x,t)$ ($i = 1, \ldots, m-1$) then $E(w) \geq E(u_0)$; repeating in this way, one would obtain $E(w) \geq \cdots \geq E(u_{w,m-1}) = E(0)$; this would prove the quasiconvexity of $f$ at 0. However, in view of Šverák’s well-known example [12], this procedure will certainly fail in general.

2. **LAGRANGIANS GENERATED BY WEAK FLOWS**

Let $f: \mathbb{M}^m \to \mathbb{R}$ be a $C^1$ function. We define its derivative map $Df: \mathbb{M}^{m \times n} \to \mathbb{M}^{m \times n}$ by $Df(\xi) = (\frac{\partial f}{\partial \xi})$ for all $\xi \in \mathbb{M}^{m \times n}$. For $\xi, \eta \in \mathbb{M}^{m \times n}$, write $\xi: \eta = \text{tr}(\xi \eta)$.

If $p \in \mathbb{R}^m$, $\xi \in \mathbb{M}^{m \times n}$ then denote $b = p \cdot \xi = \xi^T p$ to be the vector in $\mathbb{R}^n$ with $b_\alpha = \sum_{i=1}^m p^i \xi^i_\alpha$ for $\alpha = 1, \ldots, n$. So we have $\xi: (p \otimes a) = (p \cdot \xi) \cdot a$.

Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^n$ and $I$ be an interval containing 0. Let $\Omega_I = \Omega \times I$ and $u \in W^{1,\infty}(\Omega_I;\mathbb{R}^m)$. Assume that $\nabla u(x,t)$ and $u_t(x,t)$ are continuous in $t \in I$ for almost every $x \in \Omega$. Define the Lagrangian

$$L(x,t,a) = f(\nabla u(x,t)) + Df(\nabla u(x,t)) : (u_t(x,t) \otimes a)$$

for $x \in \Omega$, $t \in I$ and $a \in \mathbb{R}^n$, which is a Carathéodory function in the sense that $L(x,t,a)$ is measurable in $x$ for all $(t,a)$ and is continuous in $(t,a)$ for almost every $x$. Define then the total action functional

$$A(\theta) = \int_{\Omega} L(x,\theta(x),\nabla \theta(x)) \, dx.$$
Given any \( \theta \in W^{1,\infty}(\Omega; I) \), define a function \( g : [0,1] \to \mathbb{R} \) by
\[
g(s) = A(s\theta) = \int_{\Omega} L(x, s\theta(x), s\nabla \theta(x)) \, dx.
\]
Let \( \Omega_0 = \{ x \in \Omega \mid \theta(x) \neq 0 \} \) and
\[
U_0 = \{ (x,t) \in \Omega_I \mid x \in \Omega_0, \ 0 < t < \theta(x) \ or \ 0 > t > \theta(x) \}.
\]
Then \( U_0 \) is a bounded Lipschitz domain in \( \Omega_I \). For any \( \epsilon > 0 \), let \( \rho_\epsilon : \mathbb{R} \to \mathbb{R} \) be defined by
\[
\rho_\epsilon(t) = \begin{cases} 1 & \text{if } |t| \geq 2\epsilon \\ \frac{|t|}{\epsilon} - 1 & \text{if } \epsilon \leq |t| \leq 2\epsilon \\ 0 & \text{if } |t| \leq \epsilon. \end{cases}
\]
Then we have the following result.

**Proposition 2.1.** Function \( g \) is continuous on \([0,1]\). Moreover, for all \( \phi \in C^\infty_0([0,1]) \),
\[
\int_0^1 g(s)\phi'(s) \, ds = \lim_{\epsilon \to 0^+} \int_{U_0} L \left( x,t, \frac{t\nabla \theta(x)}{\theta(x)} \right) \phi' \left( \frac{t}{\theta(x)} \right) \rho_\epsilon(\theta(x)) \, dx 
\]

**Proof.** The continuity of \( g \) follows from the Carathéodory property of Lagrangian \( L \). We now prove the limit identity. Let \( N = \{ x \in \Omega \mid \theta(x) = 0 \} \). Then \( g(s) = \int_{\Omega_0} L(x, s\theta, s\nabla \theta) \, dx + c \), where \( c = \int_N L(x,0,0) \, dx \). Let
\[
g_\epsilon(s) = \int_{\Omega_0} \rho_\epsilon(\theta(x)) L(x, s\theta(x), s\nabla \theta(x)) \, dx.
\]
Then
\[
\int_0^1 g(s)\phi'(s) \, ds = \lim_{\epsilon \to 0^+} \int_0^1 g_\epsilon(s)\phi'(s) \, ds.
\]
By Fubini’s theorem,
\[
\int_0^1 g_\epsilon(s)\phi'(s) \, ds = \int_{\Omega_0} \left( \int_0^1 L(x, s\theta, s\nabla \theta)\phi'(s) \, ds \right) \rho_\epsilon(\theta(x)) \, dx
\]
\[
= \int_{\Omega_0} \left( \int_0^{\theta(x)} L \left( x,t, \frac{t\nabla \theta(x)}{\theta(x)} \right) \phi' \left( \frac{t}{\theta(x)} \right) \, dt \right) \rho_\epsilon(\theta(x)) \, dx
\]
\[
= \int_{\Omega_0} L \left( x,t, \frac{t\nabla \theta(x)}{\theta(x)} \right) \phi' \left( \frac{t}{\theta(x)} \right) \rho_\epsilon(\theta(x)) \, dx \, dt.
\]
This completes the proof. \( \square \)

3. PROOF OF THE MAIN THEOREM

Let \( f \) be a \( C^1 \) rank-one convex function and \( u \) be a weak flow for (1.6) defined as above. The rank-one convexity of \( f \) is equivalent to that the inequality
\[
(3.1) \quad f(\xi + p \otimes a) \geq f(\xi) + Df(\xi) : (p \otimes a)
\]
holds for all \( \xi \in M^{m \times n}, \ p \in \mathbb{R}^m, \ a \in \mathbb{R}^n \).

Assume \( \theta \in W^{1,\infty}_0(\Omega; I) \) and let \( w(x) = u(x, \theta(x)) \). Then it can be shown that \( w \in W^{1,\infty}(\Omega; \mathbb{R}^m) \) and
\[
\nabla w(x) = (\nabla u)(x, \theta(x)) + u_t(x, \theta(x)) \otimes \nabla \theta(x), \quad \text{a.e. } x \in \Omega.
\]
From the rank-one convexity of \( f \), it follows that
\[
f(\nabla w(x)) \geq f(\nabla u(x, \theta(x))) + Df(\nabla u(x, \theta(x))): [u_t(x, \theta(x)) \otimes \nabla \theta(x)]
\]
\[
= L(x, \theta(x), \nabla \theta(x))
\]
for almost every \( x \in \Omega \), where \( L(x, t, a) \) is the Lagrangian defined above. Hence it follows that
\[
E(w) = \int_{\Omega} f(\nabla w(x)) \, dx \geq A(\theta),
\]
where \( A(\theta) \) is the total action of \( L \) defined above. Note that \( A(0) = E(u_0) \), where \( u_0(x) = u(x, 0) \). Therefore, in view of (3.2), Theorem 1.1 is proved once we prove the following result.

**Proposition 3.1.** As before, let \( g(s) = A(s\theta) \) for \( s \in [0, 1] \). Then \( g \) is a constant on \([0, 1]\).

**Proof.** It is enough to show \( \int_0^1 g(s) \phi'(s) \, ds = 0 \) for all \( \phi \in C^1_0([0, 1]) \). We use the same notation as above. By Proposition 2.1, it suffices to show
\[
\lim_{\epsilon \to 0^+} \int_{U_0} L(x, t, \frac{t\nabla \theta(x)}{\theta(x)}) \phi'(\frac{t}{\theta(x)}) \rho_\epsilon(\theta(x)) \, dx \, dt = 0,
\]
Let
\[
\zeta(x, t) = \rho_\epsilon(\theta(x)) \phi\left(\frac{t}{\theta(x)}\right).
\]
Since \( \theta \in W^{1, \infty}(\Omega) \), we have \( \zeta \in W^{1, \infty}(U_0) \) and
\[
\zeta_t = \phi'\left(\frac{t}{\theta(x)}\right) \rho_\epsilon(\theta(x)) \frac{\rho_\epsilon'(\theta(x)) \nabla \theta(x) - \rho_\epsilon(\theta(x)) \phi'\left(\frac{t}{\theta(x)}\right) \frac{t\nabla \theta(x)}{\theta^2(x)}}{\theta(x)}.
\]
Therefore,
\[
Df(\nabla u): (u_t \otimes \nabla \zeta) - f(\nabla u) \zeta_t = \phi\left(\frac{t}{\theta}\right) \rho_\epsilon'(\theta) Df(\nabla u): (u_t \otimes \nabla \theta) - L(x, t, \frac{t\nabla \theta}{\theta}) \phi'\left(\frac{t}{\theta}\right) \rho_\epsilon(\theta).
\]
Since \( u(x, t) \) is a weak flow of (1.6), by (1.8), it follows that
\[
\int_{U_0} L(x, t, \frac{t\nabla \theta}{\theta}) \phi'\left(\frac{t}{\theta}\right) \rho_\epsilon(\theta) \, dx \, dt = \int_{U_0} \phi\left(\frac{x}{\theta}\right) \rho_\epsilon'(\theta) Df(\nabla u): (u_t \otimes \nabla \theta) \, dx \, dt.
\]
Note that \( \rho_\epsilon'(t) = 0 \) when \(|t| > 2\epsilon\) or \(|t| < \epsilon\) and \( |\rho_\epsilon'(t)| = \frac{1}{\epsilon} \) when \( \epsilon < |t| < 2\epsilon\). Also note that all \( \phi, \nabla u, u_t \) and \( \nabla \theta \) are bounded. Hence
\[
\left| \int_{U_0} \phi\left(\frac{x}{\theta}\right) \rho_\epsilon'(\theta) Df(\nabla u): (u_t \otimes \nabla \theta) \, dx \, dt \right| \leq C |U'|/\epsilon,
\]
where \( U' \subset U_0 \) is the open set defined by
\[
U' = \{(x, t) \in U_0 | \epsilon < |\theta(x)| < 2\epsilon, 0 < t < \theta(x) \text{ or } 0 > t > \theta(x)\}.
\]
Let \( \Omega^\epsilon = \{ x \in \Omega | \epsilon < |\theta(x)| < 2\epsilon \} \). Obviously, \( U^\epsilon \subset \Omega^\epsilon \times (-2\epsilon, 2\epsilon) \), and hence \( |U^\epsilon| \leq 4\epsilon|\Omega^\epsilon| \). Clearly, \( |\Omega^\epsilon| \to 0 \) as \( \epsilon \to 0^+ \). Therefore, (3.3) follows from (3.5) and (3.6).

Finally, we remark that under some mild regularity assumption any weak solution to the equilibrium flow (1.3) will be a weak flow for (1.6). Here by a weak solution to (1.3) we mean a map \( u \in W^{1,\infty}(\Omega_I; \mathbb{R}^m) \) such that

\[
\int_U Df(\nabla u(x,t)) : \nabla \varphi(x,t) \, dx dt = 0
\]

for all bounded domains \( U \subseteq \Omega_I \) and all \( \varphi \in W^{1,1}_0(U; \mathbb{R}^m) \).

**Proposition 3.2.** Let \( u \in W^{1,\infty}(\Omega_I; \mathbb{R}^m) \) be a weak solution to (1.3). Assume the weak derivatives \( \nabla u_t \) exist and belong to \( L^p(U) \) for some \( p > 1 \). Then, for any bounded Lipschitz domain \( U \subseteq \Omega_I \) and any \( \zeta \in W^{1,\infty}_0(U) \), it follows

\[
\int_U [Df(\nabla u) : (u_t \otimes \nabla \zeta) - f(\nabla u)\zeta] \, dx dt = 0.
\]

**Proof.** The proof follows from a standard approximation argument. Let \( u^\epsilon \) be the standard mollifier of \( u \) on \( \Omega_I \). Use \( \varphi = \zeta u^\epsilon \) as test function in (3.7) above to obtain

\[
\int_U Df(\nabla u(x,t)) : \nabla \zeta(x,t)u^\epsilon_t(x,t) \, dx dt = 0.
\]

Since \( \nabla (\zeta u^\epsilon) \) is bounded in \( L^p(U) \), it follows from the above equation that

\[
\lim_{\epsilon \to 0} \int_U Df(\nabla u^\epsilon(x,t)) : \nabla (\zeta(x,t)u^\epsilon_t(x,t)) \, dx dt = 0.
\]

However, it is easy to see that

\[
\int_U Df(\nabla u^\epsilon(x,t)) : \nabla (\zeta(x,t)u^\epsilon_t(x,t)) \, dx dt = \int_U \int_U [Df(\nabla u) : (u_t \otimes \nabla \zeta) \times \nabla \zeta] \, dx dt = \int_U \int_U [Df(\nabla u) : (u^\epsilon_t \otimes \nabla \zeta)] \, dx dt = \int_U \int_U [-\zeta_t f(\nabla u^\epsilon) + Df(\nabla u^\epsilon) : (u^\epsilon_t \otimes \nabla \zeta)] \, dx dt.
\]

Letting \( \epsilon \to 0 \) in (3.9)-(3.12) yields (3.8). This completes the proof. \( \square \)

**Remark 3.1.** (a) The result implies that the regular equilibrium flows \( u \) always generate maps \( w \) via (1.2) that have higher energy than \( u_0 \). For example, if \( Q : I \to M^{m \times n} \) is \( C^1 \), then \( u(x,t) = Q(t)x \) will be a weak flow for (1.6).

(b) A singular equilibrium flow may instead generate some map \( w \) that violates the inequality (1.1) even for a quasiconvex function \( f \). As an example, consider \( u(x,t) = Q(t)x \) and \( Q(t) = ((1 - t)^{-2} - 1)J \), where \( J \) is any given \( n \times n \) matrix such that \( |J|^n = -n^{n/2} \det J > 0 \). Then \( u(x,t) \) is an equilibrium flow (of any \( f \)) for all \( t < 1 \) (but blowing up at \( t = 1 \)). Let \( f(\xi) = |J + \xi|^n - n^{n/2} \det(J + \xi) \) and \( \theta(x) = 1 - |x| \) for the unit ball in \( \mathbb{R}^n \). In this case, \( w(x) = u(x, \theta(x)) = ((|x|^{-2} - 1)Jx \) satisfies \( f(\nabla w(x)) = 0 \) for all \( x \neq 0 \) and \( f \) is rank-one convex (in fact, polyconvex), but \( f(0) > 0 \). Such a result has also been observed earlier in [13].
4. Rotation-invariant functions

In this final section, we study a special class of functions $f$ and some special weak flows. We assume $f : \mathbb{M}^{m \times n} \to \mathbb{R}$ is rotation-invariant; that is,

$$(4.1) \quad f(R \xi) = f(\xi), \quad \forall \ R \in SO(m), \ \xi \in \mathbb{M}^{m \times n},$$

where $SO(m)$ is the set of all rotations in $\mathbb{R}^m$ defined by

$$SO(m) = \{ R \in \mathbb{M}^{m \times m} \mid R^T R = I, \det R = 1 \}.$$

**Lemma 4.1.** If $f$ is $C^1$ and satisfies (4.1), then

$$\begin{align*}
(4.2) & \quad Df(R \xi) = R Df(\xi), \quad \forall \ R \in SO(m), \ \xi \in \mathbb{M}^{m \times n}, \\
(4.3) & \quad Df(\xi) \xi^T = \xi (Df(\xi))^T, \quad \forall \ \xi \in \mathbb{M}^{m \times n}.
\end{align*}$$

**Proof.** (4.2) follows easily from (4.1); so we prove (4.3) only. Let $P$ be any matrix in $\mathbb{R}^{m \times m}$ such that $P + P^T = 0$. Define $R(t) = e^{tP}$. Then $R(t)$ is $C^1$ in $t \in \mathbb{R}$ with $R(t) \in SO(m)$ and $R(0) = I, R'(0) = P$. Differentiate the equation $f(R(t) \xi) = f(\xi)$ with respect to $t$ and we have

$$Df(R(t) \xi) : [R'(t) \xi] = R'(t) : [\xi (Df(R(t) \xi))^T] = 0.$$

Letting $t = 0$ yields $P : [\xi (Df(\xi))^T] = 0$, which holds for all $P$ such that $P + P^T = 0$. This in turn implies $\xi (Df(\xi))^T$ is symmetric: $\xi (Df(\xi))^T = Df(\xi) \xi^T$, proving (4.3). \qed

We look for solutions of flow (1.6) in the form of $u(x, t) = R(t) v(x)$, where $R : I \to SO(m)$ is $C^1$ on $I$. We search for $v$ so that all such $u$’s are weak flows of (1.6).

We have the following result.

**Theorem 4.2.** Let $f : \mathbb{M}^{m \times n} \to \mathbb{R}$ be $C^1$ and satisfy (4.1). Suppose $v \in W^{1, \infty}(\Omega ; \mathbb{R}^m)$ is a weak solution to the equation

$$\begin{align*}
(4.4) & \quad \text{div}(Df(\nabla v)) = \lambda(x) v,
\end{align*}$$

where $\lambda \in L^1(\Omega)$ is a scalar function. Then $u(x, t) = R(t) v(x)$ is a weak flow of (1.6) for all $C^1$ functions $R : I \to SO(m)$.

**Proof.** Clearly, we have $u \in W^{1, \infty}(\Omega ; \mathbb{R}^m)$ and that $u_t(x, t) = R'(t) v(x)$, $\nabla u_t(x, t) = R(t) \nabla v(x)$ are Carathéodory functions. We need to prove (1.8). To this end, let $U$ be any bounded Lipschitz domain in $\Omega_I$ and $\zeta \in W_0^{1, \infty}(U)$. By (4.1) and (4.2),

$$f(\nabla u(x, t)) = f(\nabla v(x)), \quad Df(\nabla u(x, t)) = R(t) Df(\nabla v (x)).$$

Since $f(\nabla u)$ is independent of $t$, it is easy to see that (1.8) is equivalent to

$$\begin{align*}
(4.5) & \quad \int_U R(t) Df(\nabla v(x)) : [R'(t) v(x) \otimes \nabla \zeta(x, t)] \, dx \, dt \\
(4.6) & \quad = \int_U R(t)^T R'(t) : [Df(\nabla v(x)) \nabla \zeta(x, t) \otimes v(x)] \, dx \, dt = 0.
\end{align*}$$

Extending $\zeta$ to all $\Omega_I$ by zero outside $U$, one can write the integral in (4.6) as

$$\int_U R(t)^T R'(t) : [Df(\nabla v(x)) \nabla \zeta(x, t) \otimes v(x)] \, dx \, dt = \int_I P(t) : K(t) \, dt,$$
Therefore, examples of nontrivial critical points should come with the cases when
\[ (4.7) \]
implies that \( \lambda \) can then simplify the final result in the matrix form as follows:

\[ \text{Since } R(t) \in SO(m) \text{ for all } t, \text{ it follows that } P(t) + P(t)T = 0. \text{ We will show that } K(t) = K(t)T \text{ for almost every } t \in I. \text{ To see this, we first note that equation (4.4) implies that} \]

\[ \sum_{\alpha=1}^{n} \int_{\Omega} \frac{\partial f(\nabla \psi)}{\partial \psi_{\alpha}} \frac{\partial \phi}{\partial x_{\alpha}} \, dx = -\int_{\Omega} \lambda \psi \phi \, dx, \quad \forall \, i = 1, \ldots, m \]

for all \( \phi \in W^{1, \infty}_{0}(\Omega) \). For each \( j = 1, 2, \ldots, m \) and any fixed \( t \in I \), use \( \psi(x) = \psi^{j}(x)\zeta(x, t) \) as a test function in (4.7), and note that

\[ \frac{\partial (\psi \zeta)}{\partial x_{\alpha}} = \zeta \frac{\partial \psi}{\partial x_{\alpha}} + \psi \frac{\partial \zeta}{\partial x_{\alpha}}. \]

We can then simplify the final result in the matrix form as follows:

\[ \int_{\Omega} \xi Df(\nabla \psi)(\nabla \psi)^{T} \, dx + \int_{\Omega} Df(\nabla \psi) \nabla \zeta \otimes \psi \, dx 
\]

\[ = -\int_{\Omega} \lambda(x) \zeta(x, t) \psi(x) \otimes \psi(x) \, dx. \]

By (4.3), the first term on the left-hand side is a symmetric matrix, and so is the term on the right-hand side. This proves that \( K(t) = \int_{\Omega} Df(\nabla \psi) \nabla \zeta \otimes \psi \, dx \) is also a symmetric matrix. Therefore, \( P(t) : K(t) = 0 \) for almost every \( t \in I \). This proves (4.6); hence the theorem is proved. \( \Box \)

**Remark 4.1.** Any weak solution of (4.4) is a critical point of the functional (1.10) defined above. Rank-one convexity of \( f \) plays an important role in studying the existence of many critical points of this functional in a Sobolev space under certain growth conditions on \( f \). For instance, given any \( p \in \mathbb{R}^{m} \), the map \( \psi(x) = \psi(x)p \) will be a critical point of (1.10) if \( \psi(x) \) is a critical point of the functional

\[ J(\psi) = \int_{\Omega} \left( f(p \otimes \nabla \psi(x)) + \frac{\lambda(x)}{2} |\psi(x)|^{2} \right) \, dx. \]

Note that since \( f \) is rank-one convex the Euler-Lagrange equation for functional \( J(\psi) \) is semi-linear elliptic. Also, if \( \lambda(x) \geq 0 \), then the functional \( J(\psi) \) is convex and thus the only critical point is \( \psi(x) = 0 \) in a suitable Sobolev space \( W^{1, p}_{0}(\Omega) \). Therefore, examples of nontrivial critical points should come with the cases when \( \lambda(x) < 0 \).

**References**


Department of Mathematics, Michigan State University, East Lansing, MI 48824

E-mail address: yan@math.msu.edu