(1) Let $x_1 = 0$, $x_2 = 1$ and $x_n = \frac{1}{2}(x_{n-1} + x_{n-2})$ for all $n \geq 3$. Show that

$$x_n = \frac{2^{n-1} + (-1)^n}{3 \times 2^{n-2}}$$

for all $n \in \mathbb{N}$.

**Proof.** Call the formula to be proved statement $A(n)$. We use the Mathematical Induction to show that $A(n)$ is true for all $n \in \mathbb{N}$.

1. We show that $A(1)$ is true. If $n = 1$, then $\frac{2^{n-1} + (-1)^n}{3 \times 2^{n-2}} = \frac{2^{-1} + (-1)^1}{3 \times 2^{-2}} = 0 = x_1$.

2. We show that $A(2)$ is true. If $n = 2$, then $\frac{2^{n-1} + (-1)^n}{3 \times 2^{n-2}} = \frac{2^1 + (-1)^2}{3 \times 2^0} = \frac{3}{3} = 1 = x_2$.

3. Assume that $A(1), A(2), \ldots, A(k)$ are true for some $k \geq 2$. We show that $A(k+1)$ is true. Since $A(k)$ and $A(k-1)$ are true, we have

$$x_k = \frac{2^{k-1} + (-1)^k}{3 \times 2^{k-2}}, \quad x_{k-1} = \frac{2^{k-2} + (-1)^{k-1}}{3 \times 2^{k-3}},$$

and thus, by the definition of $x_n$, we have

$$x_{k+1} = \frac{1}{2}(x_k + x_{k-1}) = \frac{1}{2} \left[ \frac{2^{k-1} + (-1)^k}{3 \times 2^{k-2}} + \frac{2^{k-2} + (-1)^{k-1}}{3 \times 2^{k-3}} \right]$$

$$= \frac{2^{k-1} + (-1)^k}{3 \times 2^{k-1}} + \frac{2^{k-2} + (-1)^{k-1}}{3 \times 2^{k-2}} = \frac{2^{k-1} + (-1)^k}{3 \times 2^{k-1}} + \frac{2^{k-1} + 2(-1)^{k-1}}{3 \times 2^{k-1}}$$

$$= \frac{2^k + (-1)^{k+1}}{3 \times 2^{k-1}} \quad \text{(Note that } (-1)^k + 2(-1)^{k-1} = (-1)^{k+1}. \text{)}$$

Hence $A(k+1)$ is true.

By induction, $A(n)$ is true for all $n \in \mathbb{N}$.

(2) Let $L : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be defined by $L(x) = (x_1 + x_2, x_1)$ for all $x = (x_1, x_2) \in \mathbb{R}^2$.

Prove that $L$ is injective and linear.

**Proof.**

**Step 1: $L$ is injective.** Let $x = (x_1, x_2), y = (y_1, y_2)$ be such that $L(x) = L(y)$. Then

$$(x_1 + x_2, x_1) = (y_1 + y_2, y_1).$$

Hence $x_1 + x_2 = y_1 + y_2$ and $x_1 = y_1$. So, plugging $x_1 = y_1$ into the first equation, $x_2 = y_2$. Hence, $x_1 = y_1$ and $x_2 = y_2$, so $x = y$; this proves that $L$ is injective.

**Step 2: $L$ is linear.** Given any $a, b \in \mathbb{R}$, and $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$, we have that $ax + by = (ax_1 + by_1, ax_2 + by_2)$ and hence, by the definition of $L$,

$$L(ax + by) = L(ax_1 + by_1, ax_2 + by_2)$$

$$= (ax_1 + by_1 + ax_2 + by_2, ax_1 + by_1)$$

$$= (ax_1 + ax_2 + by_1 + by_2, ax_1 + by_1)$$

$$= a(x_1 + x_2, x_1) + b(y_1 + y_2, y_1) = aL(x) + bL(y).$$

Therefore, $L(ax + by) = aL(x) + bL(y)$ for all $a, b \in \mathbb{R}$ and $x, y \in \mathbb{R}^2$. So, by definition, $L : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is linear.