(1) Prove that the following statement is true:
\[ \forall n \in \mathbb{N} \ (5 \mid n^2 \Rightarrow 5 \mid n). \]
Here the symbol “\( q \mid p \)” means that \( q \) is a factor of \( p \).

**Proof.** We prove the truth of the contrapositive statement:
\[ \forall n \in \mathbb{N} \ (5 \nmid n \Rightarrow 5 \nmid n^2). \]
Given \( n \in \mathbb{N} \) and assume \( 5 \nmid n \). We show that \( 5 \nmid n^2 \). Since \( 5 \nmid n \), we write \( n = 5k + r \) for some integers \( k \) and \( r \) with \( 1 \leq r \leq 4 \). We proceed with all cases of \( r \).

(a) When \( r = 1 \), we have
\[ n^2 = (5k + 1)^2 = 25k^2 + 10k + 1 = 5(5k^2 + 2k) + 1 \]
and hence \( 5 \nmid n^2 \).

(b) When \( r = 2 \), we have
\[ n^2 = (5k + 2)^2 = 25k^2 + 20k + 4 = 5(5k^2 + 4k) + 4 \]
and hence \( 5 \nmid n^2 \).

(c) When \( r = 3 \), we have
\[ n^2 = (5k + 3)^2 = 25k^2 + 30k + 9 = 5(5k^2 + 6k + 1) + 4 \]
and hence \( 5 \nmid n^2 \).

(d) When \( r = 4 \), we have
\[ n^2 = (5k + 4)^2 = 25k^2 + 40k + 16 = 5(5k^2 + 8k + 3) + 1 \]
and hence \( 5 \nmid n^2 \).

Therefore, in all cases, \( 5 \nmid n \Rightarrow 5 \nmid n^2 \).

(Note that we have actually proved that the remainder of \( n^2 \) divided by 5 is either 0, 1 or 4 and thus cannot be 2 or 3.)

(2) Prove that \( \sqrt{5} \) is irrational.

**Proof.** We use the contradiction method. Suppose \( \sqrt{5} \) is rational. Then
\[ \sqrt{5} = \frac{m}{n}, \] where \( m, n \in \mathbb{N} \) and have no common factors except 1.

Hence, squaring and rearranging,
\[ 5n^2 = m^2. \]
So \( 5 \mid m^2 \). By problem (1), we have \( 5 \mid m \). Let \( m = 5k \), where \( k \in \mathbb{N} \). Hence
\[ 5n^2 = (5k)^2 = 25k^2 \]
and thus \( n^2 = 5k^2 \) and \( 5 \mid n^2 \); therefore, by Problem (1) again, \( 5 \mid n \). We have thus deduced that 5 is a factor of both \( m \) and \( n \), which is a contradiction to the assumption that \( m \) and \( n \) have no common factors except 1.
Therefore, \( \sqrt{5} \) is not rational.