(1) Show that a real number \( x \) satisfies \( x^2 - 2x - 3 < 0 \) if and only if \(-1 < x < 3\).

**Proof.** (a) We first prove the “if” part; that is, if \(-1 < x < 3\) then \( x^2 - 2x - 3 < 0 \).

So, assume \(-1 < x < 3\). Then \( x + 1 > 0 \) and \( x - 3 < 0 \), and hence

\[
x^2 - 2x - 3 = (x + 1)(x - 3) < 0.
\]

(b) We now prove the “only if” part; that is, if \( x^2 - 2x - 3 < 0 \) then \(-1 < x < 3\).

So assume \( x^2 - 2x - 3 < 0 \). This implies that the product \((x + 1)(x - 3)\) is negative, which is possible only in the following two cases:

(1) \( x + 1 > 0 \) and \( x - 3 < 0 \);

(2) \( x + 1 < 0 \) and \( x - 3 > 0 \).

Case (1) gives that \(-1 < x < 3\), while Case (2) gives that \( x < -1 \) and \( x > 3 \), which is not possible. Therefore, we must have the first case; that is, \(-1 < x < 3\).

This completes the proof.

(2) For each \( p \in \mathbb{P}_2 \), define \( T(p)(x) = p(x+1) - 5 \) for all \( x \in \mathbb{R} \). Show that \( T \) is a function from \( \mathbb{P}_2 \) to \( \mathbb{P}_2 \), and show that \( T : \mathbb{P}_2 \to \mathbb{P}_2 \) is a bijection.

**Proof.** We first show that \( T \) is a function from \( \mathbb{P}_2 \) to \( \mathbb{P}_2 \). Given each input \( p = a_0 + a_1x + a_2x^2 \in \mathbb{P}_2 \), its output under \( T \) is

\[
T(p) = p(x+1) - 5 = a_0 + a_1(x+1) + a_2(x+1)^2 - 5,
\]

which, after expansion, is a polynomial of degree \( \leq 2 \); that is, \( T(p) \in \mathbb{P}_2 \). So, the formula \( T \) defines a function from \( \mathbb{P}_2 \) to \( \mathbb{P}_2 \).

We now show that this function \( T : \mathbb{P}_2 \to \mathbb{P}_2 \) is a bijection; that is, \( T \) is both injective and surjective.

**Step 1:** We show that \( T : \mathbb{P}_2 \to \mathbb{P}_2 \) is **injective**. Assume \( T(p) = T(q) \), where \( p, q \in \mathbb{P}_2 \). This implies

\[
p(x+1) - 5 = q(x+1) - 5 \quad \text{for all} \quad x \in \mathbb{R}.
\]

Hence \( p(x+1) = q(x+1) \) for all \( x \), and so \( p(y) = q(y) \) for all \( y \in \mathbb{R} \); that is, \( p = q \), as polynomials in \( \mathbb{P}_2 \). This proves that \( T \) is injective.

**Step 2:** We show that \( T : \mathbb{P}_2 \to \mathbb{P}_2 \) is **surjective**. Assume \( q \in \mathbb{P}_2 \) and we need to show that there exists an element \( p \in \mathbb{P}_2 \) such that \( T(p) = q \).

**[The Plan:] Suppose such a \( p \) exists. Then \( q(x) = T(p)(x) = p(x+1) - 5 \) for all \( x \in \mathbb{R} \). So \( p(x+1) = q(x) + 5 \) and, by letting \( y = x + 1 \), we have that \( p(y) = q(y - 1) + 5 \). This is how we define \( p \). The rest is to reverse this procedure to write the proof.]**

Define \( p(x) = q(x - 1) + 5 \). Since \( q \in \mathbb{P}_2 \), as in Step 1, we have that \( p \in \mathbb{P}_2 \). For this \( p \in \mathbb{P}_2 \), we have

\[
T(p)(x) = p(x+1) - 5 = [q(x+1-1) + 5] - 5 = q(x) \quad \text{for all} \quad x \in \mathbb{R};
\]

that is, \( T(p) = q \) in \( \mathbb{P}_2 \). This proves that every element \( q \in \mathbb{P}_2 \) is an image under \( T \); thus, by definition, \( T : \mathbb{P}_2 \to \mathbb{P}_2 \) is surjective.

This completes the proof.