CONVEX INTEGRATION AND INFINITELY MANY WEAK SOLUTIONS TO THE PERONA–MALIK EQUATION IN ALL DIMENSIONS

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Abstract. We prove that for all smooth nonconstant initial data the initial-Neumann boundary value problem for the Perona–Malik equation in image processing possesses infinitely many Lipschitz weak solutions on smooth bounded convex domains in all dimensions. Such existence results have not been known except for the one-dimensional problems. Our approach is motivated by reformulating the Perona–Malik equation as a nonhomogeneous partial differential inclusion with linear constraint but some uncontrollable components of gradient. We establish a general existence result by a suitable Baire category method under a pivotal density hypothesis. We finally fulfill this density hypothesis by convex integration based on certain approximations from an explicit formula of lamination convex hull of some matrix set involved.

Key words. Perona–Malik equation, differential inclusion, convex integration, Baire’s category method, infinitely many weak solutions on convex domains

AMS subject classifications. 35M13, 35K20, 35D30, 35F60, 49K20

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1. Introduction. In this paper, we study the initial and Neumann boundary value problem

\[
\begin{align*}
\frac{\partial u}{\partial t} &= \text{div} \left( \frac{D u}{1 + |D u|^2} \right) \quad \text{in } \Omega \times (0, T), \\
\frac{\partial u}{\partial \mathbf{n}} &= 0 \quad \text{on } \partial \Omega \times (0, T), \\
u(x, 0) &= u_0(x), \quad x \in \Omega,
\end{align*}
\]

where \( \Omega \subset \mathbb{R}^n \) is a smooth bounded convex domain, \( T > 0 \) is a given number, \( u = u(x, t) \) is the unknown function with \( u_t \) denoting its time-derivative and \( D u = (u_{x_1}, \ldots, u_{x_n}) \) its spatial gradient, \( \mathbf{n} \) is outer unit normal on \( \partial \Omega \), and \( u_0(x) \) is a given smooth function satisfying

\[
Du_0 \neq 0 \quad \text{in } \Omega, \quad \partial u_0 / \partial \mathbf{n} = 0 \quad \text{on } \partial \Omega.
\]

Problem (1.1), especially when \( n = 2 \), is a well-known model in image processing introduced by Perona and Malik in [27] for denoising and edge enhancement of a computer vision. In this model, \( u(x, t) \) represents an improved version of the initial gray level \( u_0(x) \) of a noisy picture. The anisotropic diffusion \( \text{div} \left( \frac{D u}{1 + |D u|^2} \right) \) is forward parabolic in the subcritical region where \( |D u| < 1 \) and backward parabolic in the supercritical region where \( |D u| > 1 \).

The expectation of the Perona–Malik model is that disturbances with small gradient in the subcritical region will be smoothed out by the forward parabolic diffusion,
while sharp edges corresponding to large gradient in the supercritical region will be enhanced by the backward parabolic equation. Such expected phenomenology has been implemented and observed in some numerical experiments, showing the stability and effectiveness of the model. On the other hand, many analytical works have shown that the model is highly ill-posed when the initial datum $u_0$ is transcritical in $\Omega$; namely, there are subregions in $\Omega$ where $|Du_0| < 1$ and where $|Du_0| > 1$, respectively. For transcritical initial data, due to the backward parabolicity, even a proper notion and the existence of well-posed solutions to (1.1) have remained largely unsettled. Most analytical works have focused on the study of singular perturbations, Young measure solutions, numerical scheme analyses, and examples and properties of certain classical solutions; see, e.g., [3, 6, 12, 13, 14, 17, 18].

The present paper addresses the analytical issue concerning the existence of certain exact weak solutions to problem (1.1). Let $\Omega_T = \Omega \times (0,T)$. We say that a Lipschitz function $u \in W^{1,\infty}(\Omega_T)$ is a weak solution to (1.1) provided that for all $\zeta \in C^\infty(\bar{\Omega}_T)$ and $s \in [0,T]$,

\begin{equation}
\int_{\Omega} u(x,s)\zeta(x,s)dx + \int_0^s \int_{\Omega} (-u\zeta_t + \sigma(Du) \cdot D\zeta) \, dxdt = \int_{\Omega} u_0(x)\zeta(x,0)dx,
\end{equation}

where $\sigma(p) = \frac{p}{1+|p|^2}$ ($p \in \mathbb{R}^n$) is the Perona–Malik function. The first existence result on such weak solutions was established by Zhang [32] for the one-dimensional problem, whose pivotal idea is to reformulate the one-dimensional Perona–Malik equation as a differential inclusion with linear constraint and then prove the existence using a modified method of convex integration following the ideas of [20, 24]. Based on a similar approach of differential inclusion, we have recently proved in [19] that for all dimensions $n$, if the domain $\Omega$ is a ball and the nonconstant initial function $u_0$ is smooth and radially symmetric, then (1.1) admits infinitely many radially symmetric Lipschitz weak solutions.

The main purpose of this paper is to extend the results of [19, 32] to problem (1.1) on all $n$-dimensional smooth convex domains for all nonconstant smooth initial data.

Our main result of the paper is the following theorem.

**Theorem 1.1.** Let $\Omega \subset \mathbb{R}^n$ be a bounded convex domain with $\partial \Omega$ of $C^{2+\alpha}$, and let $u_0 \in C^{2+\alpha}(\Omega)$ satisfy (1.2) for some constant $0 < \alpha < 1$. Then (1.1) possesses infinitely many weak solutions. Moreover, if $\|Du_0\|_{L^\infty(\Omega)} \geq 1$ and $\lambda > 0$, then these weak solutions $u$ will satisfy the almost gradient maximum principle:

$$\|Du\|_{L^\infty(\Omega_T)} \leq \|Du_0\|_{L^\infty(\Omega)} + \lambda.$$
solutions, there exists a unique subcritical classical solution; we have the following
result (see also [17, Theorem 6.1]).

**Theorem 1.2.** In addition to the assumptions above, suppose \( \| Du_0 \|_{L^\infty(\Omega)} < 1 \). Then (1.1) has a unique solution \( u \in C^{2+\alpha,2+\alpha/2}(\bar{\Omega}_T) \) satisfying

\[
\| Du \|_{L^\infty(\Omega_T)} = \| Du_0 \|_{L^\infty(\Omega)} < 1.
\]

We remark that the convexity of the domain is needed to guarantee a gradient maximum principle for the classical solution to an initial-Neumann boundary value problem of a class of quasi-linear uniformly parabolic equations (see Theorem 2.1 below). This gradient maximum principle turns out to be crucial for the proof of the main theorems, and an example in [1, Theorem 4.1] showed that such a gradient maximum principle may fail even for the heat equation without the convexity of the domain; however, such a convexity condition seemed to be overlooked in [17, Theorem 6.1].

The proof of Theorem 1.2 is given in Section 2 after some preliminary results. The majority of this paper is devoted to the proof of Theorem 1.1. For this purpose, in what follows, we assume the initial function \( u_0 \) satisfies

(1.4) \[
\int_\Omega u_0(x) dx = 0
\]
since otherwise one can solve solution \( \tilde{u} \) of (1.1) with new initial datum \( \tilde{u}_0 = u_0 - \frac{1}{|\Omega|} \int_\Omega u_0 dx \); then \( u = \tilde{u} + \frac{1}{|\Omega|} \int_\Omega u_0 dx \) will solve (1.1).

Our proof of Theorem 1.1 is based on a crucial generalization of the ideas of [19, 32, 33]. Let us discuss this generalization in some details because it exhibits several different features from the one-dimensional setup.

Assume \( u \in W^{1,\infty}(\Omega_T) \) is a weak solution to (1.1), and suppose there exists a vector function \( v \in W^{1,\infty}(\Omega_T; \mathbb{R}^n) \) such that \( \text{div} v = u \) and \( v_t = \sigma(Du) \) a.e. in \( \Omega_T \). Let \( w = (u, v): \Omega_T \to \mathbb{R}^{1+n} \), with a space-time Jacobian matrix denoted by

\[
\nabla w = \begin{pmatrix} Du & u_t \\ Dv & v_t \end{pmatrix}
\]
as an element in the matrix space \( \mathbf{M}^{(1+n) \times (n+1)} \). Given \( s \in \mathbb{R} \), define the set \( K(s) \) in \( \mathbf{M}^{(1+n) \times (n+1)} \) by

(1.5) \[
K(s) = \left\{ \begin{pmatrix} p & c \\ B & \sigma(p) \end{pmatrix} \ \bigg| \ p \in \mathbb{R}^n, c \in \mathbb{R}, B \in \mathbf{M}^{n \times n}, \text{tr} B = s \right\}.
\]

Then \( w = (u, v) \) solves the nonhomogeneous partial differential inclusion:

\[
\nabla w(x, t) \in K(u(x, t)) \quad \text{a.e. } (x, t) \in \Omega_T.
\]

Conversely, suppose we have found a function \( \Phi = (u^*, v^*) \), where \( u^* \in W^{1,\infty}(\Omega_T) \) and \( v^* \in W^{1,\infty}(\Omega_T; \mathbb{R}^n) \), such that

(1.6) \[
\begin{cases}
  u^*(x, 0) = u_0(x) \ (x \in \Omega), \\
  \text{div } v^* = u^* \text{ a.e. in } \Omega_T, \\
  v^*(\cdot, t) \cdot n|_{\partial \Omega} = 0 \ \forall \ t \in [0, T].
\end{cases}
\]
Assume \( w = (u, v) \in W^{1,\infty}(\Omega_T; \mathbb{R}^{1+n}) \) solves the Dirichlet problem of nonhomogeneous differential inclusion:

\[
\begin{aligned}
\nabla w(x,t) &\in K(u(x,t)) \quad \text{a.e. } (x,t) \in \Omega_T, \\
w(x,t) &= \Phi(x,t), \quad \text{a.e. } (x,t) \in \partial\Omega_T.
\end{aligned}
\]  

Then it can be verified that \( u \) is a weak solution to (1.1) (see Lemma 3.2).

The Dirichlet problem (1.7) falls into the framework of general nonhomogeneous partial differential inclusions studied by Dacorogna and Marcellini [10] using Baire’s category method and by Müller and Sychev [26] using the convex integration method; see also [20]. The study of such differential inclusions has stemmed from the successful understanding of homogeneous differential inclusions of the form \( Du(x) \in K \) first encountered in the study of crystal microstructure by Ball and James [2], Chipot and Kinderlehrer [7], and Müller and Sverák [24]. Recently, the method of differential inclusions has been successfully applied to other important problems; see, e.g., [8, 11, 23, 25, 30, 31].

We point out that the existence result of [26] is not applicable to problem (1.7) even in dimension \( n = 1 \), as has already been noticed in [32, 33]. A key condition in the main existence theorem of [26], when applied to (1.7), would require that the boundary function \( \Phi \) satisfy

\[
\nabla \Phi(x,t) \in U(u^*(x,t)) \cup K(u^*(x,t)) \quad \text{a.e. } (x,t) \in \Omega_T,
\]

where \( U(s) \subset M^{(1+n) \times (n+1)}(s \in \mathbb{R}) \) are bounded sets that are reducible to \( K(s) \) in the sense that, for every \( s_0 \in \mathbb{R} \), \( \xi_0 \in U(s_0), \epsilon > 0 \), and bounded Lipschitz domain \( G \subset \mathbb{R}^{n+1} \), there exist a piecewise affine function \( w \in W^{1,\infty}_0(G; \mathbb{R}^{1+n}) \) and a \( \delta > 0 \) satisfying, for a.e. \( z = (x,t) \in G \),

\[
\xi_0 + \nabla w(z) \in \bigcap_{|s-s_0|<\delta} U(s), \quad \int_G \text{dist}(\xi_0 + \nabla w(z), K(s_0)) \, dz < \epsilon|G|.
\]

The second condition would imply \( \text{tr} B_0 = s_0 \) for each \( \xi_0 = \begin{pmatrix} p & c \\ B & \beta \end{pmatrix} \in U(s_0) \) and \( s_0 \in \mathbb{R} \); but then \( \bigcap_{|s-s_0|<\delta} U(s) = \emptyset \), which makes the first condition impossible.

However, certain geometric structures of the set \( K(0) \) still turn out useful, especially when it comes to the relaxation of homogeneous differential inclusion \( \nabla \omega(z) \in K(0) \) with \( \omega = (\varphi, \psi) \). We explicitly compute the first-order lamination set \( L(K(0)) \) of \( K(0) \) consisting of all \( \xi \in M^{(1+n) \times (n+1)} \setminus K(0) \) such that \( \xi = \lambda \xi_1 + (1-\lambda) \xi_2 \) for some \( \lambda \in (0,1) \) and \( \xi_1, \xi_2 \in K(0) \) with \( \text{rank}(\xi_1 - \xi_2) = 1 \). We obtain the explicit formula (see Theorem 4.1)

\[
L(K(0)) = \left\{ \begin{pmatrix} p & c \\ B & \beta \end{pmatrix} \mid \text{tr} B = 0, \ |\beta|^2 + (p \cdot \beta)^2 - p \cdot \beta < 0 \right\},
\]

which enables us to extract enough information on the diagonal components of differential inclusion \( \nabla \omega(z) \in K(0) \) and establish a relaxation result on \( (D\varphi, \psi) \) (see Theorem 4.6). Although for such relaxation we must have \( \text{div} \psi = 0 \), the resulting \( \varphi \) can be arbitrarily small; this is important for the subsequent handling of the linear constraint \( \text{div} v = u \) in problem (1.7).

Another difficulty concerning problem (1.7) is that when \( n = 1 \), one can control \( \|v_x\|_{L^\infty(\Omega_T)} \) in terms of \( u = v_x \); however, for \( n \geq 2 \), it is impossible to control \( \|Dv\|_{L^\infty(\Omega_T)} \) in terms of \( u = \text{div} v \). So, if \( n \geq 2 \), the space \( W^{1,\infty}(\Omega_T; \mathbb{R}^n) \) is not
suitable for the function $v$. It turns out that a suitable space for $v$ is the space $W^{1,2}(0,T;L^2(\Omega;\mathbb{R}^n))$ of abstract functions (see Lemma 3.1); in this setting, the linear constraint $\text{div}v = u$ must be understood in the sense of distributions.

We design a new approach to overcome the lack of control on $Dv$: instead of defining an admissible class for $w = (u,v)$, we define a suitable admissible class for only the functions $u \in W^{1,\infty}(\Omega T)$, treating $v$ as auxiliary functions. Of course, during all the relevant constructions, the linear constraint $\text{div}v = u$ must be satisfied. In this regard, we need a linear operator $\mathcal{R}$ that serves as a (distributional) right inverse of the divergence operator: $\text{div}\mathcal{R} = \text{Id}$. By the results of Bourgain and Brézis [4], such an operator may not exist as a bounded operator on certain spaces, but for our purpose, it suffices to construct such an operator $\mathcal{R}$ that is bounded from $L^\infty(Q \times I)$ to $L^\infty(Q \times I;\mathbb{R}^n)$ for the box domains $Q \times I$ in $\mathbb{R}^{n+1}$; this is achieved by following some constructions in [4].

Finally, we remark that although the result of this paper relies heavily on the explicit formula of $L(K(0))$, the method can handle some general forward-backward parabolic equations. In fact, the method can be adapted to show that for all Perona–Malik-type equations the infinitely many Lipschitz weak solutions $u(x,t) \in \Omega_T$, exhibiting staircase-like behaviors in these weak solutions; such staircase microscopic behaviors in the true weak solutions are in much agreement with the numerical simulations shown in [27, Figure 12], which had only been previously ascertained in terms of the similar macroscopic behaviors of Young measure solutions. We intend to leave these further generalizations and results and the relevant discussions to our future work.

The rest of the paper is organized as follows. In section 2, we collect several necessary preliminary results, some of which cannot be found in the standard references; the proof of Theorem 1.2 is also given in this section. In section 3, we set up a new general procedure for proving Theorem 1.1 under a pivotal density hypothesis of an admissible class $\mathcal{U}$; this setup is suitable for a Baire category method and simplifies some of the arguments even for the one-dimensional problem. In section 4, as the heart of the matter for fulfilling the density hypothesis and thus proving Theorem 1.1, we present the essential geometric considerations, including an explicit computation of the set $L(K(0))$ above and establishing a critical relaxation property (Theorem 4.6) by convex integration with linear constraint. In section 5, we construct the suitable admissible class $\mathcal{U}$ after defining a specific boundary function $\Phi = (u^*, v^*)$. In section 6, we fulfill the key density hypothesis for admissible class $\mathcal{U}$ (Theorem 6.1) and finally complete the proof of Theorem 1.1 according to the setup of section 3.

2. Some preliminary results.

2.1. Uniformly parabolic quasi-linear equations. We refer to the standard references (e.g., [21, 22]) for general theory of parabolic equations, including some notation concerning functions and domains of class $C^{k+\alpha}$ for integer $k \geq 0$ and number $0 < \alpha < 1$.

Assume $f \in C^3([0,\infty))$ is a function satisfying

$$(2.1) \quad \theta \leq f(s) + 2sf'(s) \leq \Theta \quad \forall s \geq 0,$$

where $\Theta \geq \theta > 0$ are constants. This condition is equivalent to $\theta \leq (sf(s^2))' \leq \Theta$ for all $s \in \mathbb{R}$; hence, $\theta \leq f(s) \leq \Theta$ for all $s \geq 0$. Let

$$A(p) = f(|p|^2)p \quad (p \in \mathbb{R}^n).$$
Then we have

\[ A_p^i_p(p) = f(|p|^2)\delta_{ij} + 2f'(|p|^2)p_ip_j \quad (i, j = 1, 2, \ldots, n; \ p \in \mathbb{R}^n) \]

and hence the uniform ellipticity condition:

\[ \theta|q|^2 \leq \sum_{i,j=1}^n A_p^i_p(p)q_iq_j \leq \Theta|q|^2 \quad \forall \ p, \ q \in \mathbb{R}^n. \]

**Theorem 2.1.** Let \( \Omega \subset \mathbb{R}^n \) be a bounded convex domain with \( \partial \Omega \) of \( C^{2+\alpha} \), and let \( u_0 \in C^{2+\alpha}(\Omega) \) satisfy \( Du_0 \cdot n = 0 \) on \( \partial \Omega \). Then the initial-Neumann boundary value problem

\[
\begin{cases}
  u_t = \text{div}(A(Du)) & \text{in } \Omega_T, \\
  \partial u / \partial n = 0 & \text{on } \partial \Omega \times (0, T), \\
  u(x, 0) = u_0(x) & \text{for } x \in \Omega
\end{cases}
\]

has a unique solution \( u \in C^{2+\alpha, \frac{2+\alpha}{2}}(\Omega_T) \). Moreover, the gradient maximum principle holds:

\[ \|Du\|_{L^\infty(\Omega_T)} = \|Du_0\|_{L^\infty(\Omega)}. \]

**Proof.** 1. As problem (2.3) is uniformly parabolic by (2.2), the existence of unique classical solution \( u \) in \( C^{2+\alpha, \frac{2+\alpha}{2}}(\Omega_T) \) follows from the standard theory; see [22, Theorem 13.24]. Moreover, since \( A \in C^3(\mathbb{R}^n) \), standard bootstrap arguments based on the regularity theory of linear parabolic equations [21, 22] show that the solution \( u \) has all continuous partial derivatives \( u_{x,x,x} \) and \( u_{x,t} \) within \( \Omega_T \) for \( 1 \leq i, j, k \leq n \).

2. We now prove the gradient maximum principle (2.4), which, by [1, Theorem 4.1], in general requires the convexity of \( \Omega \). Let \( v = |Du|^2 \). Then, inside \( \Omega_T \), we compute

\[
\Delta v = 2Du \cdot D(\Delta u) + 2|D^2 u|^2,
\]

\[
u_t = \text{div}(A(Du)) = \text{div}(f(v)Du) = f'(v)Du \cdot Du + f(v)\Delta u,
\]

\[
Du_t = f''(v)(Du \cdot Du)Du + f'(v)(D^2 u)Du + f'(v)(\Delta u)Du + f(v)D(\Delta u).
\]

Plugging these equations into \( \nu_t = 2Du \cdot Du_t \), we obtain

\[ \nu_t - \mathcal{L}(v) - B \cdot Dv = -2f(|Du|^2)D^2 u|2 \leq 0 \quad \text{in } \Omega_T, \]

where operator \( \mathcal{L}(v) \) and coefficient \( B \) are defined by

\[
\mathcal{L}(v) = f(|Du|^2)\Delta v + 2f'(|Du|^2)Du \cdot (D^2 v)Du,
\]

\[
\]

We write \( \mathcal{L}(v) = \sum_{i,j=1}^n a_{ij}v_{x,x} \), with coefficients \( a_{ij} = a_{ij}(x, t) \) given by

\[ a_{ij} = A_p^i_p(Du) = f(|Du|^2)\delta_{ij} + 2f'(|Du|^2)u_{x}u_{x,j} \quad (i, j = 1, \ldots, n). \]

Note that on \( \Omega_T \) all eigenvalues of the matrix \( (a_{ij}) \) lie in \([0, \Theta]\).
3. We show that
\[
\max_{(x,t) \in \Omega_T} v(x,t) = \max_{x \in \Omega} v(x,0),
\]
which proves (2.4). We prove this by contradiction. Suppose
\[
(2.6) \quad M := \max_{(x,t) \in \Omega_T} v(x,t) > \max_{x \in \Omega} v(x,0).
\]
Let \((x_0, t_0) \in \Omega_T\) be such that \(v(x_0, t_0) = M\); then \(t_0 > 0\). If \(x_0 \in \Omega\), then the strong maximum principle applied to (2.5) would imply that \(v\) is constant on \(\Omega_{t_0}\), which yields \(v(x, 0) \equiv M\) on \(\Omega\), a contradiction to (2.6). Consequently, \(x_0 \in \partial \Omega\), and thus \(v(x_0, t_0) = M > v(x, t)\) for all \((x, t) \in \Omega_T\). We can then apply Hopf’s lemma for parabolic equations [29] to (2.5) to deduce that \(\partial v(x_0, t_0)/\partial n > 0\). However, since \(\Omega\) is convex and \(\partial u/\partial n = 0\) on \(\bar{\Omega} \times [0, T]\), a result of [1, Lemma 2.1] (see also [16, Theorem 2]) asserts that \(\partial v/\partial n \leq 0\) on \(\partial \Omega \times [0, T]\), which gives a desired contradiction. \(\square\)

2.2. Modification of the Perona–Malik function. We need to modify the Perona–Malik function \(\sigma(p) = \frac{4}{1+|p|^2}\) to obtain a uniformly parabolic problem of type (2.3). For this purpose, let
\[
\rho(s) = \frac{s}{1 + s^2} \quad (s \geq 0),
\]
and, for \(0 < \delta < 1/2\), let \(m = m_\pm(\delta)\) be the solutions of \(\rho(m) = \delta\); that is,
\[
(2.7) \quad m_\pm(\delta) = \frac{1 \pm \sqrt{1 - 4\delta^2}}{2\delta}.
\]

The following result can be proved similarly as in [6, 32]; we omit the proof (see Figure 1).

Lemma 2.2. Let \(0 < \delta < 1/2\) and \(1 < \Lambda < m_+(\delta)\). Then there exists a function \(\rho^* \in C^3([0, \infty))\) satisfying
\[
\begin{cases}
\rho^*(s) = \rho(s) & \forall 0 \leq s \leq m_-(\delta), \\
\rho^*(s) < \rho(s) & \forall m_-(\delta) < s \leq \Lambda, \\
\theta \leq (\rho^*)'(s) \leq \Theta & \forall 0 \leq s < \infty,
\end{cases}
\]
where \(\Theta > \theta > 0\) are some constants. Moreover, define \(f(0) = 1\) and \(f(s) = \rho^*(\sqrt{s})/\sqrt{s}\) for \(s > 0\); then \(f \in C^3([0, \infty))\) and (2.1) is fulfilled.

2.3. Proof of Theorem 1.2. Let \(M = \|Du_0\|_{L^\infty(\Omega)}\) assume \(0 < M < 1\). Let \(\delta = \frac{M}{1 + M^2}\), and select arbitrary \(\Lambda\) with \(1 < \Lambda < m_+(\delta) = \frac{1}{\delta}\). Note that \(0 < \delta < 1/2\) and \(m_-(\delta) = M\). Use this choice of \(\delta\) and \(\Lambda\) in Lemma 2.2 to obtain a \(C^3\) function \(f\) as stated in the lemma. Let \(A(p) = f(|p|^2)p\). For this \(A(p)\), problem (2.3) is uniformly parabolic; hence, by Theorem 2.1, problem (2.3) has a unique solution \(u \in C^{2+\alpha,1+\gamma}(\bar{\Omega}_T)\) satisfying
\[
\|Du\|_{L^\infty(\Omega_T)} = \|Du_0\|_{L^\infty(\Omega)} < 1.
\]
Since \(A(p) = \sigma(p)\) on \(|p| \leq m_-(\delta) = M\), it follows that \(A(Du(x,t)) = \sigma(Du(x,t))\) on \(\Omega_T\); this proves that \(u\) is a classical solution of (1.1). On the other hand, we easily see that any classical solution of (1.1) satisfying \(\|Du\|_{L^\infty(\Omega_T)} < 1\) is also a classical solution of (2.3) and hence must be unique. This completes the proof of Theorem 1.2.
2.4. Right inverse of the divergence operator. To deal with the linear constraint \( \text{div} \, v = u \), we follow an argument of [4, Lemma 4] to construct a right inverse \( \mathcal{R} \) of the divergence operator: \( \text{div} \mathcal{R} = \text{Id} \) (in the sense of distributions in \( \Omega_T \)). For the purpose of this paper, the construction of \( \mathcal{R} \) is restricted to the box domains, by which we mean domains given by \( Q = J_1 \times J_2 \times \cdots \times J_n \), where \( J_i = (a_i, b_i) \subset \mathbb{R} \) is a finite open interval.

Given such a box \( Q \), we define a linear operator \( \mathcal{R}_n : L^\infty(Q) \to L^\infty(Q; \mathbb{R}^n) \) inductively on dimension \( n \). If \( n = 1 \), for \( u \in L^\infty(J_1) \), we define \( v = \mathcal{R}_1 u \) by

\[
v(x_1) = \int_{a_1}^{b_1} u(s) \, ds \quad (x_1 \in J_1).
\]

Assume \( n = 2 \). Let \( u \in L^\infty(J_1 \times J_2) \). Set \( \tilde{u}(x_1) = \int_{a_2}^{b_2} u(x_1, s) \, ds \) for \( x_1 \in J_1 \). Then \( \tilde{u} \in L^\infty(J_1) \). Let \( \tilde{v} = \mathcal{R}_1 \tilde{u} \); that is,

\[
\tilde{v}(x_1) = \int_{a_1}^{b_1} \tilde{u}(s) \, ds = \int_{a_1}^{b_1} \int_{a_2}^{b_2} u(s, \tau) \, d\tau \, ds \quad (x_1 \in J_1).
\]

Let \( \rho_2 \in C^c_c(a_2, b_2) \) be such that \( 0 \leq \rho_2(s) \leq \frac{c_0}{b_2 - a_2} \) and \( \int_{a_2}^{b_2} \rho_2(s) \, ds = 1 \). Define \( v = \mathcal{R}_2 u \in L^\infty(J_1 \times J_2; \mathbb{R}^2) \) by \( v = (v^1, v^2) \) with \( v^1(x_1, x_2) = \rho_2(x_2) \tilde{v}(x_1) \) and

\[
v^2(x_1, x_2) = \int_{a_2}^{b_2} u(x_1, s) \, ds - \tilde{u}(x_1) \int_{a_2}^{b_2} \rho_2(s) \, ds.
\]

Note that if \( u \in W^{1, \infty}(J_1 \times J_2) \), then \( \tilde{u} \in W^{1, \infty}(J_1) \); hence \( v = \mathcal{R}_2 u \in W^{1, \infty}(J_1 \times J_2; \mathbb{R}^2) \) and \( \text{div} \, v = u \) a.e. in \( J_1 \times J_2 \). Moreover, if \( u \in C^1(J_1 \times J_2; \mathbb{R}^2) \), then \( v \) is in \( C^1(J_1 \times J_2; \mathbb{R}^2) \).

Assume that we have defined the operator \( \mathcal{R}_{n-1} \). Let \( u \in L^\infty(Q) \) with \( Q = J_1 \times J_2 \times \cdots \times J_n \) and \( x = (x', x_n) \in Q \), where \( x' \in Q' = J_1 \times \cdots \times J_{n-1} \) and \( x_n \in J_n \). Set \( \tilde{u}(x') = \int_{a_n}^{b_n} u(x', s) \, ds \) for \( x' \in Q' \). Then \( \tilde{u} \in L^\infty(Q') \). By the assumption, \( v = \mathcal{R}_{n-1} \tilde{u} \in L^\infty(Q'; \mathbb{R}^{n-1}) \) is defined. Write \( \tilde{v}(x') = (Z^1(x'), \ldots, Z^{n-1}(x')) \), and let \( \rho_n \in C^c_c(a_n, b_n) \) be a function satisfying \( 0 \leq \rho_n(s) \leq \frac{c_0}{b_n - a_n} \) and \( \int_{a_n}^{b_n} \rho_n(s) \, ds = 1 \).
Define \( v = R_n u \in L^\infty(Q; \mathbb{R}^n) \) as follows. Let \( v(x) = (v^1(x), v^2(x), \ldots, v^n(x)) \), where, with \( x = (x', x_n) \in Q \),
\[
\begin{align*}
v^k(x', x_n) &= \rho_n(x_n)Z^k(x') \quad (k = 1, 2, \ldots, n - 1), \\
v^n(x', x_n) &= \int_{a_n}^{x_n} u(x', s)ds - \hat{u}(x') \int_{a_n}^{x_n} \rho_n(s)ds.
\end{align*}
\]

Then \( R_n : L^\infty(Q) \to L^\infty(Q; \mathbb{R}^n) \) is a well-defined linear operator; moreover,
\begin{equation}
(2.8) \quad \|R_n u\|_{L^\infty(Q)} \leq C_n (|J_1| + \cdots + |J_n|) u\|_{L^\infty(Q)},
\end{equation}
where \( C_n > 0 \) is a constant depending only on \( n \).

As in the case \( n = 2 \), we see that if \( u \in W^{1,\infty}(Q) \), then \( v = R_n u \in W^{1,\infty}(Q; \mathbb{R}^n) \) and \( \text{div}v = u \) a.e. in \( Q \). Also, if \( u \in C^1(\Omega) \), then \( v = R_n u \) is in \( C^1(Q; \mathbb{R}^n) \). Moreover, if \( u \in W^{1,\infty}_0(Q) \) satisfies \( \int_Q u(x)dx = 0 \), then one can easily show that \( v = R_n u \in W^{1,\infty}_0(Q; \mathbb{R}^n) \).

Let \( I \) be a finite open interval in \( \mathbb{R} \). We now extend the operator \( R_n \) to an operator \( R \) on \( L^\infty(Q \times I) \) by defining, for a.e. \((x, t) \in Q \times I\),
\begin{equation}
(2.9) \quad (Ru)(x, t) = (R_n u(\cdot, t))(x) \quad \forall u \in L^\infty(Q \times I).
\end{equation}

Then \( R : L^\infty(Q \times I) \to L^\infty(Q \times I; \mathbb{R}^n) \) is a bounded linear operator.

We have the following result.

**Theorem 3.1.** Let \( u \in W^{1,\infty}_0(Q \times I) \) satisfy \( \int_Q u(x, t)dx = 0 \) for all \( t \in I \). Then \( v = Ru \in W^{1,\infty}_0(Q \times I; \mathbb{R}^n) \), \( \text{div} v = u \) a.e. in \( Q \times I \), and
\begin{equation}
(2.10) \quad \|v\|_{L^\infty(Q \times I)} \leq C_n (|J_1| + \cdots + |J_n|) u\|_{L^\infty(Q \times I)},
\end{equation}
where \( Q = J_1 \times \cdots \times J_n \) and \( C_n \) is the same constant as in (2.8). Moreover, if \( u \in C^1(\overline{Q \times I}) \), then \( v = Ru \in C^1(\overline{Q \times I}; \mathbb{R}^n) \).

**Proof.** Given \( u \in W^{1,\infty}_0(Q \times I) \), let \( v = Ru \). We easily verify that \( v \) is Lipschitz continuous in \( t \), and hence \( v_t \) exists. It also follows that \( v_t = R(u_t) \). Clearly, if \( \int_Q u(x, t)dx = 0 \), then \( v(x, t) = 0 \) whenever \( t \in \partial I \) or \( x \in \partial Q \). This proves that \( v \in W^{1,\infty}_0(Q \times I; \mathbb{R}^n) \), and the estimate (2.10) follows from (2.8). Finally, from the definition of \( Ru \), we see that if \( u \in C^1(\overline{Q \times I}) \), then \( v = Ru \in C^1(\overline{Q \times I}; \mathbb{R}^n) \).

**3. General setup for existence.** In this section we set up the general procedure for proving our main theorem, Theorem 1.1.

**3.1. Sufficient conditions for weak solutions.** Since our setup differs from the usual formulation of differential inclusions, we first prove the next two results to clarify some relevant issues, which are elementary but not too obvious.

**Lemma 3.1.** Suppose \( u \in W^{1,\infty}(\Omega_T) \) is such that \( u(x, 0) = u_0(x) \) \((x \in \Omega)\), there exists a vector function \( v \in W^{1,2}(0, T; L^2(\Omega; \mathbb{R}^n)) \) with weak time-derivative \( v_t \) satisfying \( v_t = \sigma(Du) \) a.e. in \( \Omega_T \), and for each \( \zeta \in C^\infty(\overline{\Omega_T}) \) and \( t \in [0, T] \),
\begin{equation}
(3.1) \quad \int_{\Omega} v(x, t) \cdot D\zeta(x, t)dx = -\int_{\Omega} u(x, t)\zeta(x, t)dx.
\end{equation}

Then \( u \) is a weak solution to (1.1).

**Proof.** To verify (1.3), given any \( \zeta \in C^\infty(\overline{\Omega_T}) \), let
\[
g(t) = \int_{\Omega} u(x, t)\zeta(x, t)dx, \quad h(t) = \int_{\Omega} u(x, t)\zeta_t(x, t)dx \quad (0 \leq t \leq T).
\]
Then for each \( \psi \in C^\infty_c(0,T) \), by (3.1),
\[
\int_0^T \psi(t)g(t)\,dt = -\int_0^T \int_\Omega \psi(t)v(x,t)\cdot D\zeta(x,t)\,dxdt,
\]
\[
\int_0^T \psi(t)h(t)\,dt = -\int_0^T \int_\Omega \psi(t)v(x,t)\cdot D\zeta_t(x,t)\,dxdt.
\]
Since \( v \in W^{1,2}(0,T); L^2(\Omega; \mathbb{R}^n) \) and \( v_t = \sigma(Du) \), one has
\[
\int_0^T \int_\Omega (\psi(t)D\zeta_t(x,t))_t \cdot v(x,t)\,dxdt = -\int_0^T \int_\Omega \psi(t)\sigma(Du(x,t)) \cdot D\zeta(x,t)\,dxdt.
\]
Now as \( (\psi D\zeta) = \psi_1 D\zeta + \psi_2 D\zeta_t \), combining previous equations, we have
\[
\int_0^T \psi_1(t)g(t)\,dt = \int_0^T \psi_1(t)\left( -h(t) + \int_\Omega \sigma(Du(x,t)) \cdot D\zeta(x,t)\,dx \right)\,dt;
\]
this proves that \( g \) is weakly differentiable in \((0,T)\) with weak derivative
\[
g'(t) = h(t) - \int_\Omega \sigma(Du(x,t)) \cdot D\zeta(x,t)\,dx \quad \text{a.e.} \ t \in (0,T).
\]
From this, upon integrating, (1.3) follows for all \( s \in [0,T] \).

In what follows, suppose \( \Phi = (u^*, v^*) \in W^{1,\infty}(\Omega_T; \mathbb{R}^{1+n}) \), with \( u^*, v^* \) satisfying (1.6). Let \( W^{1,\infty}_u(\Omega_T) \) and \( W^{1,\infty}_v(\Omega_T; \mathbb{R}^n) \) denote the usual Dirichlet classes with boundary traces \( u^*, v^* \), respectively.

Let \( \mathcal{U} \) be some nonempty and bounded subset of \( W^{1,\infty}_v(\Omega_T) \) such that for each \( u \in \mathcal{U} \), there exists a vector function \( v \in W^{1,\infty}_v(\Omega_T; \mathbb{R}^n) \) satisfying \( \text{div} v = u \) a.e. in \( \Omega_T \) and \( \|v_t\|_{L^\infty(\Omega_T)} \leq 1/2 \). Any such set \( \mathcal{U} \) is called an admissible class.

Given any \( \epsilon > 0 \), define \( \mathcal{U}_\epsilon \) to be the set of \( u \in \mathcal{U} \) such that there exists a vector function \( v \in W^{1,\infty}_v(\Omega_T; \mathbb{R}^n) \) satisfying \( \text{div} v = u \) a.e. in \( \Omega_T \), \( \|v_t\|_{L^\infty(\Omega_T)} \leq 1/2 \), and
\[
\int_{\Omega_T} |v_t(x,t) - \sigma(Du(x,t))| \,dxdt \leq \epsilon |\Omega_T|.
\]
(Note that \( \mathcal{U}_\epsilon = \mathcal{U} \) for all \( \epsilon \geq 1 \).)

**Lemma 3.2.** Let \( u \in \mathcal{U} \). Then any vector function \( v \in W^{1,\infty}_v(\Omega_T; \mathbb{R}^n) \) determined above satisfies the integral identity (3.1) for all \( \zeta \in C^\infty(\Omega_T) \) and \( t \in [0,T] \).

**Proof.** Let \( \zeta \in C^\infty(\Omega_T) \), and define
\[
h(t) = \int_\Omega (v(x,t) \cdot D\zeta(x,t) + u(x,t)\zeta(x,t))\,dx.
\]
Then \( h \) is continuous on \([0,T]\), and for each \( \psi \in C^1[0,T] \),
\[
\int_0^T h(t)\psi(t)\,dt = \int_0^T \int_\Omega \psi(t) (v(x,t) \cdot D\zeta(x,t) + u(x,t)\zeta(x,t))\,dxdt
\]
\[
= \int_0^T \int_\Omega \psi(t) (v(x,t) \cdot D\zeta(x,t) + \text{div}(v(x,t)\zeta(x,t)))\,dxdt
\]
\[
= \int_0^T \int_\Omega \text{div}(\zeta(x,t)\psi(t)v(x,t))\,dxdt
\]
\[
= \int_0^T \int_\Omega \text{div}(\zeta(x,t)(t)v_*(x,t))\,dxdt = 0,
\]
resulting from \( v|_{\partial\Omega_T} = v^*|_{\partial\Omega_T} \) and \( v^*(\cdot,t) \cdot \mathbf{n}|_{\partial\Omega} = 0 \) for all \( t \in [0,T] \). Hence \( h \equiv 0 \) on \([0,T]\). This completes the proof. \( \blacksquare \)
3.2. General existence theorem by Baire’s category method. We prove a general existence theorem under a density hypothesis.

**Theorem 3.3.** Let $\mathcal{U} \subset W^{1,\infty}_0(\Omega_T)$ be an admissible class. Assume, for each $\epsilon > 0$, that $\mathcal{U}$ is dense in $\mathcal{U}$ under the $L^\infty$-norm. Then, given any $\varphi \in \mathcal{U}$, for each $\eta > 0$, there exists a weak solution $u \in W^{1,\infty}_0(\Omega_T)$ to problem (1.1) satisfying $\|u - \varphi\|_{L^\infty(\Omega_T)} < \eta$. Furthermore, if $\mathcal{U}$ contains a function in $W^{1,\infty}_0(\Omega_T)$ that is not a weak solution to (1.1), then (1.1) admits infinitely many weak solutions.

**Proof.** 1. Let $\mathcal{X}$ be the closure of $\mathcal{U}$ in the metric space $L^\infty(\Omega_T)$. Then $(\mathcal{X}, L^\infty)$ is a complete metric space. By assumption, $\mathcal{U}$ is a dense subset of $\mathcal{X}$. Furthermore, since $\mathcal{U}$ is bounded in $W^{1,\infty}_0(\Omega_T)$, we have $\mathcal{X} \subset W^{1,\infty}_0(\Omega_T)$.

2. Let $\mathcal{Y} = L^1(\Omega_T; \mathbb{R}^n)$. For $h > 0$, define $T_h : \mathcal{X} \to \mathcal{Y}$ as follows. Given any $u \in \mathcal{X}$, write $u = u^* + w$ with $w \in W^{1,\infty}_0(\Omega_T)$

$$T_h(u) = Du^* + D(\rho_h \ast w),$$

where $\rho_h(z) = h^{-N}\rho(z/h)$, with $z = (x,t)$ and $N = n + 1$, is the standard mollifier on $\mathbb{R}^N$, and $\rho \ast w$ is the usual convolution on $\mathbb{R}^N$ with $w$ extended to be zero outside $\Omega_T$. Then, for each $h > 0$, the map $T_h : (\mathcal{X}, L^\infty) \to (\mathcal{Y}, L^1)$ is continuous, and for each $u \in \mathcal{X}$,

$$\lim_{h \to 0^+} \|T_h(u) - Du\|_{L^1(\Omega_T)} = \lim_{h \to 0^+} \|\rho_h \ast Dw - Dw\|_{L^1(\Omega_T)} = 0.$$

Therefore, the spatial gradient operator $D : \mathcal{X} \to \mathcal{Y}$ is the pointwise limit of a sequence of continuous functions $T_h : \mathcal{X} \to \mathcal{Y}$; hence $D : \mathcal{X} \to \mathcal{Y}$ is a Baire-one function. By Baire’s category theorem (see, e.g., [5, Theorem 10.13]), there exists a residual set $\mathcal{G} \subset \mathcal{X}$ such that the operator $D$ is continuous at each point of $\mathcal{G}$. Since $\mathcal{X} \setminus \mathcal{G}$ is of first category, the set $\mathcal{G}$ is dense in $\mathcal{X}$. Therefore, given any $\varphi \in \mathcal{X}$, for each $\eta > 0$, there exists a function $u \in \mathcal{G}$ such that $\|u - \varphi\|_{L^\infty(\Omega_T)} < \eta$.

3. We now prove that each function $u \in \mathcal{G}$ is a weak solution to (1.1). Let $u \in \mathcal{G}$ be given. By the density of $\mathcal{U}$ in $(\mathcal{X}, L^\infty)$, for each $j \in \mathbb{N}$, there exists a function $u_j \in \mathcal{U}_{1/j}$ such that $\|u_j - u\|_{L^\infty(\Omega_T)} < 1/j$. Since the operator $D : (\mathcal{X}, L^\infty) \to (\mathcal{Y}, L^1)$ is continuous at $u$, we have $D u_j \to D u$ in $L^1(\Omega_T; \mathbb{R}^n)$. Furthermore, from the definition of $\mathcal{U}_{1/j}$, there exists a vector function $v_j \in W^{1,\infty}_0(\Omega_T; \mathbb{R}^n)$ such that, for each $\zeta \in C^\infty(\Omega_T)$ and $t \in [0,T]$,

$$\int_\Omega v_j(x,t) \cdot D\zeta(x,t) dx = - \int_\Omega u_j(x,t)\zeta(x,t) dx,$$

$$\|(v_j)\|_{L^\infty(\Omega_T)} \leq \frac{1}{2}, \quad \int_0^T \int_\Omega |(v_j)_t - \sigma(Du_j)| \, dx \, dt \leq \frac{1}{2}|\Omega_T|.$$  

Since $v_j(x,0) = v^*(x,0) \in W^{1,\infty}(\Omega; \mathbb{R}^n)$, from $\|(v_j)\|_{L^\infty(\Omega_T)} \leq 1/2$, it follows that both sequences $\{v_j\}$ and $\{(v_j)_t\}$ are bounded in $L^2(\Omega_T; \mathbb{R}^n) \approx L^2((0,T); L^2(\Omega; \mathbb{R}^n))$. We may assume $v_j \rightharpoonup v$ and $(v_j)_t \rightharpoonup v_t$ weakly in $L^2((0,T); L^2(\Omega; \mathbb{R}^n))$ for some $v \in W^{1,2}((0,T); L^2(\Omega; \mathbb{R}^n))$. Upon taking the limit as $j \to \infty$ in (3.2) and noticing $v \in C([0,T]; L^2(\Omega; \mathbb{R}^n))$, we obtain that

$$\int_\Omega v(x,t) \cdot D\zeta(x,t) dx = - \int_\Omega u(x,t)\zeta(x,t) dx \quad (t \in [0,T]),$$

$$v_t(x,t) = \sigma(Du(x,t)) \quad \text{a.e.} \ (x,t) \in \Omega_T.$$  

Consequently, by Lemma 3.1, $u$ is a weak solution to (1.1).
4. Finally, assume $\mathcal{U}$ contains a function that is not a weak solution to (1.1); hence $G \neq \mathcal{U}$. Then $G$ cannot be a finite set since otherwise the $L^\infty$-closure $\mathcal{X} = \overline{G} = \overline{\mathcal{U}}$ would be a finite set, making $\mathcal{U} = G$; therefore, in this case, (1.1) admits infinitely many weak solutions. This completes the proof. $\square$

The rest of the paper is devoted to the construction of a suitable admissible class $\mathcal{U} \subset W^{1,\infty}(\Omega_T)$ fulfilling the density property:

$$U_\epsilon \text{ is dense in } U \text{ under the } L^\infty\text{-norm for each } \epsilon > 0.$$  

4. Geometric considerations: Relaxation of $\nabla \omega(z) \in K(0)$. Let $K(s)$ be the matrix set defined by (1.5) above. Since $K(s)$ is a translation of set $K(0)$, we focus on the set $K_0 = K(0)$; that is,

$$K_0 = \left\{ \begin{pmatrix} p & c \\ B & \sigma(p) \end{pmatrix} \bigg| p \in \mathbb{R}^n, c \in \mathbb{R}, B \in \mathbb{M}^{n \times n}, \text{tr} B = 0 \right\},$$

where $\sigma(p) = \frac{p}{1 + |p|}$ is the Perona–Malik function.

4.1. Rank-one lamination of $K_0$. We first compute certain rank-one structures of the set $K_0$.

Let $L(K_0)$ be the set of all matrices $\xi \in \mathbb{M}^{(1+n)\times(n+1)}$ that are not in $K_0$ but are representable by $\xi = \lambda \xi_1 + (1 - \lambda)\xi_2$ for some $\lambda \in (0,1)$ and $\xi_1, \xi_2 \in K_0$ with rank($\xi_1 - \xi_2$) = 1, or equivalently, let

$$L(K_0) = \{ \xi \notin K_0 \ | \ \xi + t_+ \eta \in K_0 \text{ for some } t_- < 0 < t_+ \text{ and rank } \eta = 1 \}.$$

Suppose $\xi = \begin{pmatrix} p & c \\ B & \beta \end{pmatrix} \in L(K_0)$, with $\xi + t_+ \eta \in K_0$, where $t_- < 0 < t_+$ and $\eta$ is a rank-one matrix given by

$$\eta = \begin{pmatrix} a \\ \alpha \end{pmatrix} \otimes (q, b) = \begin{pmatrix} aq & ab \\ \alpha \otimes q & b \alpha \end{pmatrix}, \quad a^2 + |a|^2 \neq 0, \ b^2 + |q|^2 \neq 0,$$

for some $a, b \in \mathbb{R}$ and $\alpha, q \in \mathbb{R}^n$; here $\alpha \otimes q$ denotes the rank-one or zero matrix $(\alpha^t q_j)$ in $\mathbb{M}^{n \times n}$.

Condition $\xi + t_+ \eta \in K_0$ with $t_- < 0 < t_+$ is equivalent to the following:

$$\text{tr } B = 0, \quad \alpha \cdot q = 0, \quad \sigma(p + t_+ aq) = \beta + t_+ b \alpha.$$

If $aq = 0$, then $\sigma(p) = \beta + t_+ b \alpha$ has two different solutions of $t$ only when $b \alpha = 0$, but then we would have $\sigma(p) = \beta$ and thus $\xi \in K_0$, a contradiction. Therefore, $aq \neq 0$.

By rescaling $\eta$ and $t_\pm$, we assume $a = 1$ and $|q| = 1$; namely,

$$\eta = \begin{pmatrix} q \\ \alpha \otimes q \end{pmatrix}, \quad |q| = 1, \quad \alpha \cdot q = 0.$$

Case 1. Assume $b \alpha = 0$. In this case, by (4.1), the equation $\sigma(p + t q) = \beta$ has two solutions of $t$ of opposite signs, and thus we must have $p = xq$ and $\beta = uq$, and $\sigma(xq + t q) = uq$ becomes a quadratic equation $x + t = u + u(x^2 + 2xt + t^2)$, which has two solutions $t = t_\pm$ of opposite signs if and only if $u \neq 0$ and $x^2 - \frac{4u}{u} + 1 < 0$; this condition can be written as

$$|\beta|^2 + (p \cdot \beta)^2 - p \cdot \beta = u^2 + (xu)^2 - xu < 0.$$
By (4.4) and (4.6), we have

Then the necessary and sufficient condition for (4.3) to have two solutions

\[ \eta = \begin{pmatrix} q & b \\ \frac{1}{q} \gamma & \gamma \end{pmatrix}, \quad |q| = 1, \gamma \cdot q = 0, \gamma \neq 0, b \neq 0. \]

Since the equation \( \sigma(p + tq) = \beta + t\gamma \) has two solutions \( t = t_{\pm} \) of opposite signs, we must have \( p = xq + y\gamma \) and \( \beta = uq + v\gamma \), and the equation \( \sigma(p + tq) = \beta + t\gamma \) becomes a system of two equations:

\[
\begin{align*}
\begin{cases}
x + t &= u(1 + (x + t)^2 + |\gamma|^2 y^2), \\
y + t &= (1 + (x + t)^2 + |\gamma|^2 y^2). 
\end{cases}
\end{align*}
\]

This system has two solutions \( t = t_{\pm} \) of opposite signs, and thus \( u \neq 0 \) and \( y \neq 0 \). So (4.2) is equivalent to a system of two quadratic equations:

\[
\begin{align*}
\begin{cases}
t^2 + (2x - \frac{1}{u})t + x^2 + |\gamma|^2 y^2 + 1 - \frac{x}{u} = 0, \\
t^2 + (v + x)t + xv - yu = 0.
\end{cases}
\end{align*}
\]

The necessary and sufficient condition for (4.3) to have two solutions \( t = t_{\pm} \) of opposite signs is that the two quadratic equations of \( t \) have the same coefficients and the constant terms are negative, which yields that

\[
x = \frac{1}{u} + v, \quad x^2 + |\gamma|^2 y^2 + 1 - \frac{x}{u} = xv - yu < 0.
\]

Here, if \( v = 0 \), then \( x = \frac{1}{u} \), and taking this into the inequality, we have \( 1 + |\gamma|^2 y^2 < 0 \), a contradiction. So \( v \neq 0 \). Therefore

\[
(4.4) \quad uv = xu - 1, \quad |\gamma|^2 = \frac{1 - xu}{yu} - \frac{1}{y^2}
\]

and

\[
(4.5) \quad xv - yu = (\frac{x}{u} - \frac{y}{v}) (xu - 1) < 0.
\]

We now solve for \( x, y, u, v \) from (4.4) in terms of \( p \) and \( \beta \). From \( p = xq + y\gamma \), \( \beta = uq + v\gamma \), it follows that

\[
q = \frac{1}{xv - yu} (vp - y\beta), \quad \gamma = \frac{1}{xv - yu} (-up + x\beta).
\]

By (4.4) and (4.6), we have

\[
p \cdot \beta = xu + yv|\gamma|^2 = 1 - \frac{v}{y}, \quad \frac{x}{u} = \frac{\frac{y}{p} |p|^2}{\frac{y}{p} \cdot \beta - |\beta|^2},
\]

where \( \frac{y}{p} \cdot \beta - |\beta|^2 \neq 0 \) by (4.5). Let \( k = x/u, \ l = y/v \). Then

\[
(4.7) \quad l = \frac{1}{1 - p \cdot \beta}, \quad k = \frac{(1 - p \cdot \beta)|p|^2 - p \cdot \beta}{(1 - p \cdot \beta)p \cdot \beta - |\beta|^2}.
\]
Moreover,
\[ k - l = \frac{x}{u} - \frac{y}{v} = \frac{|p|^2 - lp \cdot \beta}{p \cdot \beta - l|\beta|^2} - l = \frac{|p - l\beta|^2}{p \cdot \beta - l|\beta|^2}. \]

From \(|q| = 1\), we have \(xv - yu = (k - l)uv = -|vp - lv\beta|\), and hence
\[ u = -\text{sgn}(v) \frac{|p - l\beta|}{k - l} = -\text{sgn}(v) \frac{p \cdot \beta - l|\beta|^2}{|p - l\beta|}, \]
where \(\text{sgn}(v) = v/|v|\) is the sign of \(v \neq 0\). We then obtain \(x, v, y\) by
\[ x = ku, \quad v = x - \frac{1}{u} = ku - \frac{1}{u}, \quad y = lv = 1ku - \frac{l}{u}. \]

In this way, we have solved \(x, y, u, v\) in terms of \(p, \beta\) uniquely up to the sign change. We can check that both conditions in (4.4) are satisfied.

Let us consider inequality (4.5) for these solutions. The equation on \(|\gamma|^2\) in (4.4) implies \(\xi(1-xu) > 1\), and hence the inequality (4.5) yields \(\xi(1-xu) \geq \frac{\xi(1-xu)}{1-xu} > 1\). So \(0 < xu < 1\), and thus \(x/u > y/v > 1\), i.e., \(k > l > 1\). Then we deduce the inequality
\[ |\beta|^2 + (p \cdot \beta)^2 - p \cdot \beta < 0. \]

### 4.2. Exact formula of \(L(K_0)\).
In fact, inequality (4.10) exactly characterizes the set \(L(K_0)\). We have the following result.

**Theorem 4.1.**

\[ L(K_0) = \left\{ \begin{pmatrix} p \\ c \\ B \end{pmatrix} \mid \text{tr } B = 0, \ |\beta|^2 + (p \cdot \beta)^2 - p \cdot \beta < 0 \right\}. \]

Moreover, given any \(\xi \in L(K_0)\), there exist a rank-one matrix
\[ \eta = \begin{pmatrix} q \\ b \gamma \\ \gamma \end{pmatrix}, \]
with \(|q| = 1\), \(\gamma \cdot q = 0\), \(b \neq 0\) and two numbers \(t_- < 0 < t_+\) such that
\[ \xi + t_\pm \eta \in K_0, \]
where \(|b| > 0\) can be arbitrarily small.

**Proof.** Let \(S\) be the set defined on the right-hand side of (4.11). The previous calculations show that \(L(K_0) \subseteq S\). To verify the reverse inclusion \(S \subseteq L(K_0)\), let \(\xi = \begin{pmatrix} p \\ c \\ B \end{pmatrix} \in S\). Then \(|\beta|^2 + (p \cdot \beta)^2 - p \cdot \beta < 0\). So \(0 < p \cdot \beta < 1\) and \((1 - p \cdot \beta)p \cdot \beta - |\beta|^2 > 0\), and hence we can define \(l, k\) by (4.7), so that \(l > 0, k > 0\). To fix the sign, we define \(u\) by (4.8) with \(a + \text{sgn}\):\[ u = \frac{p \cdot \beta - l|\beta|^2}{|p - l\beta|} = \frac{(1 - p \cdot \beta)p \cdot \beta - |\beta|^2}{|(1 - p \cdot \beta)p \cdot \beta - |\beta|^2|} > 0. \]

We now define \(x, v, y\) by (4.9). Then \(x > 0, v < 0, y < 0\). After deducing that \(xv - yu < 0\), we finally define \(q, \gamma\) by (4.6). It is then straightforward to check the following:
\[ p = xq + y\gamma, \quad \beta = uq + v\gamma, \quad |q| = 1, \quad \gamma \cdot q = 0, \]
\[ x = \frac{1}{u} + v, \quad x^2 + |\gamma|^2 y^2 + 1 - \frac{x}{u} = xv - yu < 0. \]

In particular, equation \( \sigma(p + tq) = \beta + t\gamma \) has two solutions \( t = t_{\pm} \) with \( t_- < 0 < t_+ \). Now let \( \eta = (\frac{q}{\pm\sqrt{\gamma}}, b) \), where \( b \neq 0 \) is arbitrary. Then \( \xi + t_{\pm}\eta \in K_0 \), and so \( \xi \in L(K_0) \). The proof is now complete. \( \square \)

Remark 4.2. The quantities \( q \in S^{n-1}, \gamma \in \mathbb{R}^n \), and \( t_{\pm} \) defined in the proof depend continuously on \( (p, \beta) \in \mathbb{R}^{n+n} \) with \( |\beta|^2 + (p \cdot \beta)^2 - p \cdot \beta < 0 \). We may also take \( b > 0 \) (or \( b < 0 \)) to be a continuous function of all such \((p, \beta)\).

4.3. The approximating sets \( S_{\delta} \) and \( S_{\delta} \). Given any \( 0 \leq \delta \leq 1/2 \), let

\[ S_{\delta} = \{(p, \beta) \in \mathbb{R}^{n+n} \mid |(1-p \cdot \beta)p - \beta| + |\beta|^2 + (p \cdot \beta)^2 - p \cdot \beta < 0\}, \]

Then \( S_0 = L(K_0) \). Immediate properties of the open sets \( S_{\delta} \) are that

\[ S_{\delta_2} \subset S_{\delta_1} \quad \text{for} \quad 0 \leq \delta_1 < \delta_2 \leq 1/2, \quad S_{1/2} = \emptyset, \]

\[ S_{\delta} \neq \emptyset \quad \text{for} \quad 0 \leq \delta < 1/2, \quad S_0 = \bigcup_{0<\delta<1/2} S_{\delta}. \]

In what follows, we always assume \( 0 < \delta < 1/2 \), unless otherwise stated.

**Proposition 4.2.** Let \( \xi \in S_{\delta} \) and \( \xi_+ \in K_0 \) with \( \text{rank}(\xi_+ - \xi_-) = 1 \) satisfy that \( \xi \) lies in the open line segment \((\xi_-, \xi_+)\). Then \((\xi_-, \xi_+) \subset S_{\delta} \).

**Proof.** Consider functions

\[ F(\xi) = |(1-p \cdot \beta)p - \beta|, \quad G(\xi) = |\beta|^2 + (p \cdot \beta)^2 - p \cdot \beta \quad \forall \xi = \begin{pmatrix} p \\ B \\ \beta \end{pmatrix}. \]

Then both \( F \) and \( G \) vanish on set \( K_0 \). For the given \( \xi \) and \( \xi_\pm \), let \( f(\tau) = F(\tau\xi) \) and \( g(\tau) = G(\tau\xi) \), where \( \xi^\tau = \tau\xi_+ + (1-\tau)\xi_- \). The rank-one condition implies that the corresponding term \( p^\tau \cdot \beta^\tau \) is linear in \( \tau \); hence \( g(\tau) \) is a quadratic polynomial of \( \tau \), and \( f(\tau) \) is the length of a vector quadratic in \( \tau \). Since both \( f(\tau) \) and \( g(\tau) \) vanish when \( \tau = 0 \) and \( 1 \), we must have \( g(\tau) = C_1\tau(1-\tau) \) and \( f(\tau) = C_2\tau(1-\tau) \) for some constants \( C_1, C_2 \). Since \( \xi = \xi^\lambda \in S_{\delta} \) for some \( 0 < \lambda < 1 \), we have

\[ \delta f(\lambda) + g(\lambda) = \lambda(1-\lambda)(\delta C_2 + C_1) < 0. \]

This implies the constant \( \delta C_2 + C_1 \) is negative. Hence \( \delta f(\tau) + g(\tau) = \tau(1-\tau)(\delta C_2 + C_1) < 0 \) for all \( \tau \in (0,1) \), which proves exactly \( \xi^\tau \in S_{\delta} \) for all \( \tau \in (0,1) \). \( \square \)

Remark 4.3. The lamination convex hull \( K_{lc} \) of a set \( K \subset \mathbb{M}^{n \times n} \) is defined to be the smallest set \( S \subset \mathbb{M}^{n \times n} \) containing \( K \) with the property that if \( \xi_\pm \in S \) with \( \text{rank}(\xi_+ - \xi_-) = 1 \), then \((\xi_-, \xi_+) \subset S \) (see [9]). As in the proof of Proposition 4.2, one can see that \( K_0^{lc} = K_0 \cup L(K_0) \).

**Lemma 4.3.** Let \( m_+(\delta) \) be defined by (2.7) above. It follows that

\[ \sup_{(p,\beta) \in S_{\delta}} |p| = m_+(\delta), \quad \inf_{(p,\beta) \in S_{\delta}} |p| = m_-(\delta). \]
Proof. Let \((p, \beta) \in S_\delta\). Then \(0 < p \cdot \beta < 1\) and \(\beta \cdot ((1 - p \cdot \beta)p - \beta) > 0\). We write
\[
p \cdot \beta = |\beta| \cos \theta, \quad \beta \cdot ((1 - p \cdot \beta)p - \beta) = |\beta|((1 - p \cdot \beta)p - \beta) \cos \theta',
\]
for some \(0 \leq \theta, \theta' < \pi/2\). A simple geometry shows that \(\theta \leq \theta'\). As \(|\beta|^2 + (p \cdot \beta)^2 - p \cdot \beta < 0\), we have
\[
|\beta|^2 + |p|^2|\beta|^2 \cos^2 \theta - |p||\beta| \cos \theta < 0,
\]
and so
\[
|\beta| < \frac{1}{2}, \quad \frac{1 - \sqrt{1 - 4|\beta|^2}}{2|\beta| \cos \theta} < |p| < \frac{1 + \sqrt{1 - 4|\beta|^2}}{2|\beta| \cos \theta}.
\]
Condition \((p, \beta) \in S_\delta\) becomes \(\delta < |\beta| \cos \theta\); so \(|\beta| \geq |\beta| \cos \theta > \delta\) and
\[
\frac{1 - \sqrt{1 - 4\delta^2}}{2\delta} < \frac{1 - \sqrt{1 - 4|\beta|^2}}{2|\beta|} \leq \frac{1 - \sqrt{1 - 4\delta^2}}{2\delta} \leq \frac{1 - \sqrt{1 - 4|\beta|^2}}{2|\beta| \cos \theta} < \frac{1 + \sqrt{1 - 4\delta^2}}{2\delta}.
\]
This proves that
\[
(4.13) \quad \frac{1 - \sqrt{1 - 4\delta^2}}{2\delta} \leq \inf_{(p, \beta) \in S_\delta} |p| \leq \sup_{(p, \beta) \in S_\delta} |p| \leq \frac{1 + \sqrt{1 - 4\delta^2}}{2\delta}.
\]
Next, fix any \(\beta \in \mathbb{R}^n\) with \(\delta < |\beta| < 1/2\). Let \(l, l'\) be any numbers satisfying
\[
(4.14) \quad \frac{1 - \sqrt{1 - 4|\beta|^2}}{2|\beta|} < l < l' < \frac{1 + \sqrt{1 - 4|\beta|^2}}{2|\beta|},
\]
and let \(p = \frac{l}{|\beta|} \beta\), \(p' = \frac{l'}{|\beta|} \beta\). Then \((p, \beta), (p', \beta)\) are both in \(S_\delta\) with \(|p| = l, |p'| = l'\); so
\[
\inf_{(p, \beta) \in S_\delta} |p| \leq l < l' \leq \sup_{(p, \beta) \in S_\delta} |p|.
\]
As \(l, l'\) are arbitrary and satisfy (4.14), we have
\[
\inf_{(p, \beta) \in S_\delta} |p| \leq \frac{1 - \sqrt{1 - 4|\beta|^2}}{2|\beta|} < \frac{1 + \sqrt{1 - 4|\beta|^2}}{2|\beta|} \leq \sup_{(p, \beta) \in S_\delta} |p|.
\]
Finally, taking \(|\beta| \to \delta^+\) and combining with (4.13) completes the proof. □

As an immediate consequence of the previous lemma, we have the following result.

**Corollary 4.4.** \(S_\delta \subset \{(p, \beta) \mid m_-(\delta) < |p| < m_+ (\delta), \delta < |\beta| < 1/2\}\).

### 4.4. A useful convex integration lemma.

The following result is important for convex integration with linear constraint. For a more general case, see [28, Lemma 2.1].

**Lemma 4.5.** Let \(\lambda_1, \lambda_2 > 0\) and \(\eta_1 = -\lambda_1 \eta, \eta_2 = \lambda_2 \eta\) with
\[
\eta = \begin{pmatrix} q & b \\ \gamma \otimes q & \gamma \end{pmatrix}, \quad |q| = 1, \gamma \cdot q = 0, b \neq 0.
\]
Let \(G \subset \mathbb{R}^{n+1}\) be a bounded domain. Then for each \(\epsilon > 0\), there exists a function \(\omega = (\varphi, \psi) \in C^\infty_c(\mathbb{R}^{n+1}; \mathbb{R}^{1+n})\) with \(\text{supp}(\omega) \subset \subset G\) that satisfies the following properties:
(a) $\text{div} \psi = 0$ in $G$,
(b) $\{z \in G \mid \nabla \omega(z) \notin \{\eta_1, \eta_2\}\} < \epsilon$,
(c) $\text{dist}(\nabla \omega(z), \{\eta_1, \eta_2\}) < \epsilon$ for all $z \in G$,
(d) $\|\omega\|_{L^\infty(G)} < \epsilon$,
(e) $\int_{\mathbb{R}^n} \varphi(x,t) \, dx = 0$ for each $t \in \mathbb{R}$.

Proof. 1. The proof follows a simplified version of [28, Lemma 2.1]. Define a map $\mathcal{P} : C^1(\mathbb{R}^{n+1}) \to C^0(\mathbb{R}^{n+1}; \mathbb{R}^{1+n})$ by setting $\mathcal{P}(h) = (u,v)$, where, for $h(x,t) \in C^1(\mathbb{R}^{n+1})$,

$$u(x,t) = q \cdot Dh(x,t), \quad v(x,t) = \frac{1}{b} (\gamma \otimes q - q \otimes \gamma) Dh(x,t).$$

We easily see that $\mathcal{P}(h) = (u,v) \in C^\infty_c(\mathbb{R}^{n+1}; \mathbb{R}^{1+n})$, supp$(\mathcal{P}(h)) \subset$ supp$(h)$, div $v \equiv 0$, and $\int_{\mathbb{R}^n} u(x,t) \, dx = 0$ for all $t \in \mathbb{R}$, for all $h \in C^\infty_c(\mathbb{R}^{n+1})$. For $h(x,t) = f(q \cdot x + bt)$ with $f \in C^\infty(\mathbb{R})$, $w = (u,v) = \mathcal{P}(h)$ is given by $u(x,t) = f'(q \cdot x + bt)$ and $v(x,t) = f''(q \cdot x + bt)\frac{q}{b^2}$, and hence $\nabla w(x,t) = f''(q \cdot x + bt)q$. We also note that $\mathcal{P}(gh) = g\mathcal{P}(h) + h\mathcal{P}(g)$, and hence

$$(4.15) \quad \nabla \mathcal{P}(gh) = g \nabla \mathcal{P}(h) + h \nabla \mathcal{P}(g) + B(\nabla g, \nabla h) \quad \forall \, g, \, h \in C^\infty(\mathbb{R}^{n+1}),$$

where $B(\nabla g, \nabla h)$ is a bilinear map of $\nabla g$ and $\nabla h$; so $|B(\nabla h, \nabla g)| \leq C|\nabla h||\nabla g|$ for some constant $C > 0$.

2. Let $G_\epsilon \subset \subset G$ be a smooth subdomain such that $|G \setminus G_\epsilon| < \epsilon/2$, and let $\rho_\epsilon \in C^\infty_c(G)$ be a cut-off function satisfying $0 \leq \rho_\epsilon \leq 1$ in $G$, $\rho_\epsilon = 1$ on $G_\epsilon$. As $G$ is bounded, $G \subset \{x \in G \mid k < q \cdot x + bt < l\}$ for some numbers $k < l$. For each $\tau > 0$, we can find a function $f_\tau \in C^\infty_c(k,l)$ satisfying

$$-\lambda_1 \leq f''_\tau \leq \lambda_2, \quad |\{s \in (k,l) \mid f''(s) \notin [-\lambda_1, \lambda_2]\}| < \tau, \quad \|f_\tau\|_{L^\infty} + \|f''_\tau\|_{L^\infty} < \tau.$$

3. Define $\omega = (\phi, \psi) = \mathcal{P}(\rho_\epsilon(x,t)h_\tau(x,t))$, where $h_\tau(x,t) = f_\tau(q \cdot x + bt)$. Then $\|h_\tau\|_{C^1} \leq C\|f_\tau\|_{C^1} \leq C\tau$, $\omega \in C^\infty_c(\mathbb{R}^{n+1}; \mathbb{R}^{1+n})$, supp$(\omega) \subset$ supp$(\rho_\epsilon) \subset \subset G$, and (a) and (e) are satisfied. Note that

$$|\omega| \leq |\rho_\epsilon||\mathcal{P}(h_\tau)| + |h_\tau||\mathcal{P}(\rho_\epsilon)| \leq C_\epsilon \tau,$$

where $C_\epsilon > 0$ is a constant depending on $\|\rho_\epsilon\|_{C^2(G)}$. So we can choose a $\tau_1 > 0$ so small that (d) is satisfied for all $0 < \tau < \tau_1$. Note also that

$$\{z \in G \mid \nabla \omega(z) \notin \{\eta_1, \eta_2\}\} \subseteq (G \setminus G_\epsilon) \cup \{z \in G_\epsilon \mid f''(q \cdot x + bt) \notin [-\lambda_1, \lambda_2]\}.$$

Since $|\{z \in G_\epsilon \mid f''(q \cdot x + bt) \notin [-\lambda_1, \lambda_2]\}| \leq N|\{s \in (k,l) \mid f''(s) \notin [-\lambda_1, \lambda_2]\}|$ for some constant $N > 0$ depending only on set $G$, there exists a $\tau_2 > 0$ such that

$$|\{z \in G \mid \nabla \omega(z) \notin \{\eta_1, \eta_2\}\}| \leq \frac{\epsilon}{2} + N\tau < \epsilon$$

for all $0 < \tau < \tau_2$. Therefore, (b) is satisfied. Finally, note that

$$\rho_\epsilon \text{div} \mathcal{P}(h_\tau(x,t)) = \rho_\epsilon f''(q \cdot x + bt)q \in [\eta_1, \eta_2] \quad \text{in} \, G$$

and, by (4.15), for all $z = (x,t) \in G$,

$$|\nabla \omega(z) - \rho_\epsilon \text{div} \mathcal{P}(h_\tau(x,t))| \leq |h_\tau| |\nabla \mathcal{P}(\rho_\epsilon)| + |B(\nabla h_\tau, \nabla \rho_\epsilon)| \leq C'_\epsilon \tau < \epsilon$$

for all $0 < \tau < \tau_3$, where $C'_\epsilon > 0$ is a constant depending on $\|\rho_\epsilon\|_{C^2(G)}$, and $\tau_3 > 0$ is another constant. Hence (c) is satisfied. Taking $0 < \tau < \min\{\tau_1, \tau_2, \tau_3\}$, the proof is complete. \(\square\)
4.5. Relaxation of $\nabla \omega(z) \in K_0$. We now study the relaxation of homogeneous inclusion

$$\nabla \omega(z) \in K_0.$$ 

We state and prove the following result in a form suitable for later use.

**Theorem 4.6.** Let $K$ be a compact subset of $S_\delta$. Let $Q \times I$ be a box in $\mathbb{R}^{n+1}$. Then, given any $\epsilon > 0$, there exists a $\rho_0 > 0$ such that for each box $Q \times I \subset Q \times I$, point $(p, \beta) \in K$, and number $\rho > 0$ sufficiently small, there exists a function $\omega = (\varphi, \psi) \in C^\infty_c(Q \times I; \mathbb{R}^{1+n})$ satisfying the following properties:

(a) $\text{div} \psi = 0$ in $Q \times I$,

(b) $(p' + D_\varphi(z), \beta' + \psi_1(z)) \in S_\delta$ for all $z \in Q \times I$ and $|(p', \beta') - (p, \beta)| \leq \rho_0,$

(c) $\|\omega\|_{L^\infty(Q \times I)} < \rho,$

(d) $\int_{Q \times I} |\beta + \psi_1(z) - \sigma(p + D_\varphi(z))|dz < \epsilon|Q \times I|/|\bar{Q} \times I|,$

(e) $\int_Q \varphi(x, t) dx = 0$ for all $t \in I$,

(f) $\|\varphi\|_{L^\infty(Q \times I)} < \rho.$

**Proof.** Let $\xi = \xi(p, \beta) = (p \ 0 \ \beta)$ for $(p, \beta) \in K \subset S_\delta$, where $O$ is the $n \times n$ zero matrix. We omit the dependence on $(p, \beta)$ in the following whenever it is clear from the context. Since $\xi \in S_\delta \subset L(K_0)$ on $K$, it follows from Theorem 4.1 and its remark that given any $\rho > 0$, there exist continuous functions $q : K \to S^{n-1}, \gamma : K \to \mathbb{R}^n$, $t_\pm : K \to \mathbb{R}$, and $b : K \to (0, \infty)$ with $\gamma \cdot q = 0, t_- < 0 < t_+$ on $K$ such that letting $\eta = (\begin{smallmatrix} q \\ \frac{b}{\pm} \gamma \end{smallmatrix})$ on $K$, we have

$$\xi + t_\pm \eta \in K_0, \quad 0 < b < \frac{\rho}{t_+ - t_-} \text{ on } K.$$ 

Writing $\xi_\pm = (\begin{smallmatrix} p_\pm \\ B_\pm \end{smallmatrix}) = \xi + t_\pm \eta$ on $K$, we have $\xi = \lambda \xi_+ + (1 - \lambda) \xi_-, \lambda = \frac{t_-}{t_+ - t_-} \in (0, 1)$ on $K$.

Proposition 4.2 implies that on $K$, both $\xi_\tau^+ = \tau \xi_+ + (1 - \tau) \xi_-$ and $\xi_\tau^- = (1 - \tau) \xi_+ + \tau \xi_-$ belong to $S_\delta$ for each $\tau \in (0, 1)$. Let $0 < \tau < \min_K \min\{\lambda, 1 - \lambda\} \leq \frac{1}{2}$ be a small number to be selected later. Then $\lambda' = \frac{\lambda}{1 - \lambda} \in (0, 1), \xi = \lambda' \xi_+^\tau + (1 - \lambda') \xi_-^\tau$ on $K$. Moreover, on $K$, $\xi_\tau^- - \xi_\tau^+ = (1 - 2\tau)(\xi_+^\tau - \xi_-^\tau)$ is rank-one, $[\xi_\tau^+, \xi_\tau^-] \subset (\xi_-, \xi_+) \subset S_\delta$, and

$$c\tau \leq \xi_\tau^+ - \xi_-^\tau = |\xi_\tau^+ - \xi_-^\tau| = \tau|\xi_+ - \xi_-| = \tau(t_+ - t_-)|\eta| \leq C\tau,$$

where $C = \max_K (t_+ - t_-)|\eta| \geq \min_K (t_+ - t_-)|\eta| = c > 0$. By continuity, $H_\tau = \bigcup_{(p, \beta) \in K} [\xi_\tau^+(p, \beta), \xi_\tau^-(p, \beta)]$ is a compact subset of $S_\delta$, where $S_\delta$ is open in

$$\Sigma_0 = \left\{ \begin{pmatrix} p & c \\ B & \beta \end{pmatrix} \mid \text{tr} B = 0 \right\}.$$

So $d_\tau = \text{dist}(H_\tau, \partial|_{\Sigma_0} S_\delta) > 0$, where $\partial|_{\Sigma_0}$ is the relative boundary in the space $\Sigma_0$.

Let $\eta_1 = -\lambda_1 \eta = -\lambda'(1 - 2\tau)(t_+ - t_-)\eta, \eta_2 = \lambda_2 \eta = (1 - \lambda')(1 - 2\tau)(t_+ - t_-)\eta$ on $K$, where $\lambda_1 = \tau(t_+ + 1 - \tau)(1 - \tau)t_- > 0, \lambda_2 = (1 - \tau)t_+ + \tau t_- > 0$ on $K$ with $\tau > 0$ sufficiently small. Applying Lemma 4.5 to matrices $\eta_1, \eta_2$ and set $G = Q \times I$, we obtain that for each $\rho > 0$, there exists a function $\omega = (\varphi, \psi) \in C^\infty_c(Q \times I; \mathbb{R}^{1+n})$
and an open set \( G_\rho \subset \subset Q \times I \) satisfying the following conditions:

\[
\begin{aligned}
1. & \quad \text{div} \psi = 0 \text{ in } Q \times I, \\
2. & \quad |(Q \times I) \setminus G_\rho| < \rho; \quad \xi + \nabla \omega(z) \in \{\xi^-_\rho, \xi^+_\rho\} \text{ for all } z \in G_\rho, \\
3. & \quad \xi + \nabla \omega(z) \in [\xi^-_\rho, \xi^+_\rho]_\rho \text{ for all } z \in Q \times I, \\
4. & \quad |\omega|_{L^\infty(Q \times I)} < \rho, \\
5. & \quad \int_Q \varphi(x,t) \, dx = 0 \text{ for all } t \in I, \\
6. & \quad \|\varphi\|_{L^\infty(Q \times I)} < 2\rho,
\end{aligned}
\]

(4.16)

where \([\xi^-_\rho, \xi^+_\rho]_\rho\) denotes the \(\rho\)-neighborhood of closed line segment \([\xi^-_\rho, \xi^+_\rho]\). From (4.16.3), (4.16.6) follows as

\[
|\varphi| < |c_+ - c_-| + \rho = (t_+ - t_-)|b| + \rho < 2\rho \quad \text{in } Q \times I.
\]

By (4.16.3), \(|\beta + \psi(z)| \leq C + \rho\) for \(z \in Q \times I\); hence

\[
\begin{aligned}
\int_{Q \times I} |\beta + \psi_t - \sigma(p + D\varphi)| \, dz & \leq \int_{G_\rho} |\beta + \psi_t - \sigma(p + D\varphi)| \, dz + \left(C + \rho + \frac{1}{2}\right)\rho \\
& \leq |Q \times I| \max \{|\beta^+ - \sigma(p^+)|\} + \left(C + \rho + \frac{1}{2}\right)\rho \\
& \leq C|Q \times I|\tau + |Q \times I| \max \{|\sigma(p^+) - \sigma(p^+)\}| + \left(C + \rho + \frac{1}{2}\right)\rho,
\end{aligned}
\]

(4.17)

where \(\xi^- = (\frac{p^+}{\beta^+}, \frac{c^+}{\beta^+})\).

Note that (a), (c), (e), and (f) follow from (4.16), where \(2\rho\) in (4.16.6) can be adjusted to \(\rho\) as in (f). By the uniform continuity of \(\sigma\) on the set \(J = \{p' \in \mathbb{R}^n \mid |p'| \leq m_+(\delta)+1\}\), we can find a \(\rho' > 0\) such that \(|\sigma(p') - \sigma(p'')| < \frac{1}{3|Q \times I|}\) whenever \(p', p'' \in J\) and \(|p' - p''| < \rho', \text{ where } m_+(\delta) > 0\) is the number defined in Lemma 4.3. We then choose a \(\tau > 0\) so small that \(C \tau < \rho'\) and \(C|Q \times I|\tau < \frac{\delta}{4}\). Since \(p^+, p^+ \in J\) and \(|p^+ - p^+| \leq C \tau < \rho'\), it follows from (4.17) that (d) holds for any choice of \(\rho > 0\) with \((C + \rho + \frac{1}{2})\rho < \frac{1}{4|Q \times I|}\). Next, we choose a \(\rho_0 > 0\) such that \(\rho_0 < \frac{\delta}{2M}\). If \(0 < \rho < \rho_0\), then by (4.16.1) and (4.16.3), for all \(z \in Q \times I\) and \(|(p', \beta') - (p, \beta)| \leq \rho_0\),

\[
\xi(p', \beta') + \nabla \omega(z) \in S_\delta, \quad \text{dist}(\xi(p', \beta') + \nabla \omega(z), H_\tau) < d_\tau,
\]

and so \(\xi(p', \beta') + \nabla \omega(z) \in S_\delta\), that is, \((p' + D\varphi(z), \beta' + \psi_t(z)) \in S_\delta\). Thus (b) holds.

The proof is now complete.

5. Construction of admissible class \(U\). In this section, we construct a suitable admissible class \(U\) as defined in section 3.

Assume \(\Omega\) and \(u_0\) are as given in Theorem 1.1, with (1.4) fulfilled in addition. Let \(M = \|Du_0\|_{L^\infty(\Omega)} > 0\).

5.1. The modified uniformly parabolic problem. We first apply Lemma 2.2 to construct the function \(f \in C^3([0, \infty))\) with a suitable choice of \(\delta \in (0, 1/2)\) and \(1 < \Lambda < m_+(\delta)\) according to the value of \(M\) as follows (see Figure 1):

(i) If \(0 < M < 1\), we select \(0 < \delta < \frac{M}{1+M^2}\) and arbitrary \(1 < \Lambda < m_+(\delta)\).
(ii) If $M \geq 1$ and $\lambda > 0$, we select $\delta = \frac{M + \lambda}{M + \lambda + M}$ and arbitrary $\Lambda \in (M, M + \lambda)$; in this case, $m_+(\delta) = M + \lambda$.

Note that in both cases we have $M < \Lambda$ and that the choice of $\delta$ in the case (i) is different from the one used earlier in the proof of Theorem 1.2.

Once $f \in C^3([0, \infty))$ is constructed, we define

$$A(p) = f(|p|^2)p \quad (p \in \mathbb{R}^n).$$

By Lemma 2.2, equation $u_t = \text{div}(A(Du))$ is uniformly parabolic. We have the following result.

**Lemma 5.1.** With $\delta$ selected above and $\mathcal{S}_\delta$ defined by (4.12), one has

$$(p, A(p)) \in \mathcal{S}_\delta \quad \forall \, m_-(\delta) < |p| \leq M.$$

**Proof.** From the definition of set $\mathcal{S}_\delta$, it follows that $(p, A(p)) \in \mathcal{S}_\delta$ if and only if

$$\frac{\delta}{|p|} < f(|p|^2) < \frac{1}{1 + |p|^2}; \text{ namely, } \rho(m_-(\delta)) = \delta < \rho^*(|p|) < \rho(|p|).$$

By Lemma 2.2, this condition is satisfied if $m_-(\delta) < |p| \leq \Lambda$. □

**5.2. The suitable boundary function $\Phi$.** By Theorem 2.1, the initial-Neumann boundary value problem

$$\begin{cases}
\begin{align*}
u_t^* &= \text{div}(A(Du^*)) \quad &\text{in } \Omega_T, \\
\partial u^*/\partial n &= 0 \quad &\text{on } \partial \Omega \times [0, T], \\
u^*(x, 0) &= u_0(x), &x \in \Omega,
\end{align*}
\end{cases}$$

(5.1)

admits a unique classical solution $u^* \in C^{2+\alpha, \frac{2+\alpha}{n}}(\bar{\Omega}_T)$ satisfying

$$|D u^*(x, t)| \leq M, \quad (x, t) \in \Omega_T.$$  

From conditions (1.2) and (1.4), we can find a function $h \in C^{2+\alpha}(\bar{\Omega})$ satisfying

$$\Delta h = u_0 \quad \text{in } \Omega, \quad \partial h/\partial n = 0 \quad \text{on } \partial \Omega.$$

Now let $v_0 = Dh \in C^{1+\alpha}((\Omega; \mathbb{R}^n)$ and define, for $(x, t) \in \Omega_T$,

$$v^*(x, t) = v_0(x) + \int_0^t A(Du^*(x, s)) \, ds.$$  

(5.2)

Define $\Phi = (u^*, v^*) \in C^1(\Omega_T; \mathbb{R}^{1+n})$. Then it is easy to see that $\Phi$ satisfies condition (1.6) above; i.e.,

$$\begin{cases}
u^*(x, 0) &= u_0(x) \quad (x \in \Omega), \\
\text{div } v^* &= u^* \quad \text{in } \Omega_T, \\
v^*(\cdot, t) \cdot n|_{\partial \Omega} &= 0 \quad \forall \, t \in [0, T].
\end{cases}$$

**Lemma 5.2.** Let $\mathcal{K}_\delta = \{(p, \sigma(p)) \mid |p| \leq m_-(\delta)\}$. 


Then

\[(Du^*(x, t), v_t^*(x, t)) \in \mathcal{S}_\delta \cup \mathcal{K}_\delta \quad \forall (x, t) \in \Omega_T.\]

**Proof.** Given \((x, t) \in \Omega_T\), let \(p = Du^*(x, t)\); then \(|p| \leq M\). By (5.2), \(v_t^*(x, t) = A(p)\). If \(|p| \leq m_-(\delta)\), then \(A(p) = \sigma(p)\), and hence

\[ (Du^*(x, t), v_t^*(x, t)) = (p, A(p)) = (p, \sigma(p)) \in \mathcal{K}_\delta. \]

If \(m_-(\delta) < |p| \leq M\), then by Lemma 5.1,

\[ (Du^*(x, t), v_t^*(x, t)) = (p, A(p)) \in \mathcal{S}_\delta. \]

Hence \((Du^*, v_t^*) \in \mathcal{S}_\delta \cup \mathcal{K}_\delta\) in \(\Omega_T\).

5.3. The admissible class \(\mathcal{U}\). In what follows, we say that a function \(u\) is piecewise \(C^1\) in \(\Omega_T\) and write \(u \in C^1_{pc}(\Omega_T)\) provided that there exists a sequence of disjoint open sets \(\{G_j\}_{j=1}^\infty\) in \(\Omega_T\) with \(|\partial G_j| = 0\) such that

\[ u \in C^1(G_j) \quad \forall j \in \mathbb{N} = \{1, 2, \ldots\}, \quad |\Omega_T \setminus \bigcup_{j=1}^\infty G_j| = 0. \]

(For our purpose it is also acceptable to allow only finitely many pieces in this definition.)

**Definition 5.3.** Let \(\mu = \|u_t^*\|_{L^\infty(\Omega_T)} + 1\). We define the admissible class

\[ \mathcal{U} = \left\{ u \in C^1_{pc} \cap W^{1,\infty}(\Omega_T) \mid \|u_t\|_{L^\infty} < \mu; \exists v \in C^1_{pc} \cap W^{1,\infty}(\Omega_T; \mathbb{R}^n) \right\}, \]

such that \(\text{div} v = u\) and \((Du, v_t) \in \mathcal{S}_\delta \cup \mathcal{K}_\delta\) a.e. in \(\Omega_T\).

For each \(\epsilon > 0\), let \(\mathcal{U}_\epsilon\), be defined by

\[ \mathcal{U}_\epsilon = \left\{ u \in \mathcal{U} \mid \exists v \in C^1_{pc} \cap W^{1,\infty}(\Omega_T; \mathbb{R}^n) \text{ such that div} v = u \text{ and} \right\} \]

\[ (Du, v_t) \in \mathcal{S}_\delta \cup \mathcal{K}_\delta \text{ a.e. in } \Omega_T, \text{ and } \int_{\Omega_T} |v_t - \sigma(Du)| \, dx \, dt \leq \epsilon |\Omega_T|. \]

**Remark 5.1.** Clearly, \(u^* \in \mathcal{U}\); so \(\mathcal{U}\) is nonempty. Also \(\mathcal{U}\) is a bounded subset of \(W^{1,\infty}(\Omega_T)\) as \(\mathcal{S}_\delta \cup \mathcal{K}_\delta\) is bounded. Moreover, by Corollary 4.4, for each \(u \in \mathcal{U}\), its corresponding vector function \(v\) satisfies \(\|v_t\|_{L^\infty(\Omega_T)} \leq 1/2\). Finally, note that \(m_-(\delta) < |Du^*| \leq M\) on some nonempty open subset of \(\Omega_T\) and \(A(Du^*) \neq \sigma(Du^*)\) on this set; hence \(u^*\) itself is not a weak solution to (1.1).

6. Density property and Proof of Theorem 1.1. In this final section, we prove the density property of the sets \(\mathcal{U}\), and then complete the proof of Theorem 1.1.

6.1. The density property of \(\mathcal{U}\). Let \(\mathcal{U}\) and \(\mathcal{U}_\epsilon\) be as defined in section 5. We establish the density property (3.3).

**Theorem 6.1.** For each \(\epsilon > 0\), \(\mathcal{U}_\epsilon\) is dense in \(\mathcal{U}\) under the \(L^\infty\)-norm.

**Proof.** Let \(u \in \mathcal{U}\), \(\eta > 0\). The goal is to show that there exists a function \(\tilde{u} \in \mathcal{U}_\epsilon\) such that \(\|\tilde{u} - u\|_{L^\infty(\Omega_T)} < \eta\). For clarity, we divide the proof into several steps.

1. Note that \(\|u_t\|_{L^\infty(\Omega_T)} < \mu - \tau_0\) for some \(\tau_0 > 0\) and there exists a function \(v \in C^1_{pc} \cap W^{1,\infty}(\Omega_T; \mathbb{R}^n)\) such that \(\text{div} v = u\) and \((Du, v_t) \in \mathcal{S}_\delta \cup \mathcal{K}_\delta\) a.e. in \(\Omega_T\). Since both \(u\) and \(v\) are piecewise \(C^1\) in \(\Omega_T\), there exists a sequence of disjoint open sets \(\{G_j\}_{j=1}^\infty\) in \(\Omega_T\) with \(|\partial G_j| = 0\) for all \(j \geq 1\) such that

\[ u \in C^1(G_j), \quad v \in C^1(G_j; \mathbb{R}^n) \quad \forall j \geq 1, \quad |\Omega_T \setminus \bigcup_{j=1}^\infty G_j| = 0. \]
2. Let \( j \in \mathbb{N} \) be fixed. Note that \( (Du(z), v_t(z)) \in S_\delta \cup K_\delta \) for all \( z \in G_j \) and that \( H_j = \{ z \in G_j | (Du(z), v_t(z)) \in \partial \mathcal{S}_\delta \} \) is a (relatively) closed set in \( G_j \) with measure zero. So \( \tilde{G}_j = G_j \setminus H_j \) is an open subset of \( G_j \) with \( |\tilde{G}_j| = |G_j| \), and \( (Du(z), v_t(z)) \in S_\delta \cup K_\delta \) for all \( z \in \tilde{G}_j \).

3. For each \( \tau > 0 \), let \( \mathcal{G}_\tau = \{ (p, \beta) \in \mathcal{S}_\delta | |\beta - \sigma(p)| > \tau, \text{dist}((p, \beta), \partial \mathcal{S}_\delta) > \tau \} \); then \( \mathcal{G}_\tau \subset \subset \mathcal{S}_\delta \). We can find a \( \tau_j > 0 \) such that

\[
\int_{F_j} |v_t(z) - \sigma(Du(z))|\,dz < \frac{\epsilon}{3 \cdot 2^j} |\Omega_T|,
\]

where \( z = (x, t) \) and \( F_j = \{ z \in \tilde{G}_j | (Du(z), v_t(z)) \notin \mathcal{G}_{\tau_j} \} \). To check this, note that \( F_j = F^1_j \cup F^2_j \), where \( F^1_j = \{ z \in \tilde{G}_j | |v_t(z) - \sigma(Du(z))| \leq \tau \} \) and \( F^2_j = \{ z \in \tilde{G}_j | \text{dist}((Du(z), v_t(z)), \partial \mathcal{S}_\delta) \leq \tau \} \). Clearly, \( \int_{F^1_j} |v_t - \sigma(Du)|\,dz \leq \tau |\Omega_T| \to 0 \) as \( \tau \to 0^+ \). Since \( F^2_j \) is decreasing as \( \tau \to 0^+ \) and \( v_t = \sigma(Du) \) on \( \cap_{\tau > 0} F^2_j \), it follows that \( \int_{F^2_j} |v_t - \sigma(Du)|\,dz \to 0 \) as \( \tau \to 0^+ \).

4. Let \( O_j = \tilde{G}_j \setminus F_j \). Furthermore, we may require that the number \( \tau_j \) be chosen in such a way that \( O_j \) is empty or \( O_j \) is a nonempty open set with \( |\partial O_j| = 0 \) (see [32]). Let \( J \) be the set of all \( j \in \mathbb{N} \) with \( O_j \neq \emptyset \). Then for \( j \notin J \), \( F_j = \tilde{G}_j \).

5. We now fix a \( j \in J \). Note that \( O_j = \{ z \in \tilde{G}_j | (Du(z), v_t(z)) \in \mathcal{G}_{\tau_j} \} \) and that \( K_j := \tilde{G}_{\tau_j} \) is a compact subset of \( \mathcal{S}_\delta \). Let \( Q \subset \mathbb{R}^n \) be an open box with \( \Omega \subset Q \) and \( I = (0, T) \). Applying Theorem 4.6 to box \( \tilde{Q} \times I, K_j \), and \( \epsilon' = \frac{\epsilon |\Omega|}{12} \), we obtain a constant \( \rho_j > 0 \) that satisfies the conclusion of the theorem. By the uniform continuity of \( \sigma \) on compact subsets of \( \mathbb{R}^n \), we can find a \( \theta = \theta_{\delta, \rho} > 0 \) such that

\[
|\sigma(p) - \sigma(p')| < \frac{\epsilon}{12}
\]

if \( |p|, |p'| \leq m_+ (\delta) \) and \( |p - p'| \leq \theta \). Also by the uniform continuity of \( u, v \), and their gradients on \( \tilde{G}_j \), there exists a \( v_j > 0 \) such that

\[
|u(z) - u(z')| + |\nabla u(z) - \nabla u(z')| + |v(z) - v(z')| + |\nabla v(z) - \nabla v(z')| < \min\{\frac{\epsilon}{12}, \frac{1}{12}, \theta\}
\]

whenever \( z, z' \in \tilde{G}_j \) and \( |z - z'| \leq v_j \). We now cover \( O_j \) (up to measure zero) by a sequence of disjoint open cubes \( \{ Q^i_j \times I^i_j \}_{i=1}^{\infty} \) in \( \Omega_T \) whose sides are parallel to the axes with center \( z^*_i \) and diameter \( l^i_j < v_j \).

6. Fix \( i \in \mathbb{N} \), and write \( w = (u, v), \xi = \left( \frac{p}{\beta}, \epsilon \right) = (\nabla w(z^*_i)) = (\frac{Du(z)}{Dv(z)})(z^*_i) \). By the choice of \( \rho_j > 0 \) in step 5 via Theorem 4.6, since \( Q^i_j \times I^i_j \subset \tilde{Q} \times \tilde{I} \) and \( (p, \beta) \in K_j \), for all sufficiently small \( \rho > 0 \), there exists a function \( \omega^i_j = (\varphi^i_j, \psi^i_j) \in C^\infty_c(Q^i_j \times I^i_j; \mathbb{R}^{1+n}) \) satisfying the following properties:

(a) \( \text{div} \varphi^i_j = 0 \) in \( Q^i_j \times I^i_j \);
(b) \( (p^i_j + D\varphi^i_j)(z), \beta^i_j + (\psi^i_j)_t(z) \in S_\delta \) for all \( z \in Q^i_j \times I^i_j \), and all \( (p^i_j, \beta^i_j) - (p, \beta) \leq \rho_j \),
(c) \( \|\omega^i_j\|_{L^\infty(Q^i_j \times I^i_j)} < \rho \),
(d) \( \int_{Q^i_j \times I^i_j} |\beta + (\psi^i_j)_t(z) - \sigma(p + D\varphi^i_j(z))|\,dz < \epsilon' |Q^i_j \times I^i_j|/|\tilde{Q} \times \tilde{I}| \),
(e) \( \int_{Q^i_j} \varphi^i_j(x, t)\,dx = 0 \) for all \( t \in I^i_j \),
(f) \( \|\varphi^i_j\|_{L^\infty(Q^i_j \times I^i_j)} < \rho \).

Here, we let \( 0 < \rho \leq \min\{\tau_0, \frac{\epsilon}{27}, \frac{\epsilon}{127}, \eta\} \), where \( C_n \) is the constant in Theorem
2.3 and $C$ is the product of $C_n$ and the sum of the lengths of all sides of $Q$. By (e), we can apply Theorem 2.3 to $\varphi_j^i$ on $Q_j^i \times I_j^i$ to obtain a function $g_j^i = R\varphi_j^i \in C^1(Q_j^i \times I_j^i; \mathbb{R}^n) \cap W^{1,\infty}_0(Q_j^i \times I_j^i; \mathbb{R}^n)$ such that $\text{div } g_j^i = \varphi_j^i$ in $Q_j^i \times I_j^i$ and

$$\|g_j^i\|_{L^\infty(Q_j^i \times I_j^i)} \leq C\|\varphi_j^i\|_{L^\infty(Q_j^i \times I_j^i)} \leq \frac{\rho_j}{2}. \quad (6.4)$$

7. As $|v_t|, |\sigma(Du)| \leq 1/2$ a.e. in $\Omega_T$, we can select a finite index set $I \subset J \times N$ so that

$$\int_{\bigcup_{(j,i) \in (J \times N) \setminus I} Q_j^i \times I_j^i} |v_t(z) - \sigma(Du(z))|dz \leq \frac{\xi}{3} |\Omega_T|. \quad (6.5)$$

We finally define

$$(\tilde{u}, \tilde{v}) = (u, v) + \sum_{(j,i) \in I} \chi_{Q_j^i \times I_j^i}(\varphi_j^i, \psi_j^i + g_j^i) \quad \text{in } \Omega_T.$$ As a side remark, note here that only finitely many functions $(\varphi_j^i, \psi_j^i + g_j^i)$ are disjointly patched to the original $(u, v)$ to obtain a new function $(\tilde{u}, \tilde{v})$ towards the goal of the proof. The reason for using only finitely many pieces of gluing is due to the lack of control over the spatial gradients $D(\psi_j^i + g_j^i)$, and overcoming this difficulty is at the heart of this paper.

8. Let us now check that $\tilde{u}$ together with $\tilde{v}$ indeed gives the desired result. By construction, it is clear that $\tilde{u} \in C^1_{pc} \cap W^{1,\infty}_0(\Omega_T)$ and $\tilde{v} \in C^1_{pc} \cap W^{1,\infty}_0(\Omega_T; \mathbb{R}^n)$. By the choice of $\rho$ in (f) as $\rho \leq \tau_0$, we have $\|\tilde{u}\|_{L^\infty(\Omega_T)} < \mu$. Next, let $(j, i) \in I$, and observe that for $z \in Q_j^i \times I_j^i$, with $(p, \beta) = (Du(z_j^i), v_t(z_j^i)) \in \mathcal{G}_{r_j}$, since $|z - z_j^i| < I_j^i < \nu_j$, it follows from (6.3) and (6.4) that

$$|(Du(z), v_t(z) + (g_j^i)_t(z)) - (p, \beta)| \leq \rho_j,$$

and so $(D\tilde{u}(z), \tilde{v}_t(z)) \in \mathcal{S}_\delta$, by (b) above. From (a) and $\text{div } g_j^i = \varphi_j^i$, for $z \in Q_j^i \times I_j^i$,

$$\text{div } \tilde{v}(z) = \text{div } (v + \psi_j^i + g_j^i)(z) = u(z) + 0 + \varphi_j^i(z) = \tilde{u}(z).$$

Therefore, $\tilde{u} \in \mathcal{U}$. Next, observe that

$$\int_{\Omega_T} |\tilde{v}_t - \sigma(D\tilde{u})|dz = \int_{\bigcup_{j \in N} F_j} |v_t - \sigma(Du)|dz$$

$$+ \int_{\bigcup_{(j,i) \in J \times N \setminus I} Q_j^i \times I_j^i} |v_t - \sigma(Du)|dz + \int_{\bigcup_{(j,i) \in I} Q_j^i \times I_j^i} |\tilde{v}_t - \sigma(D\tilde{u})|dz$$

$$= I_1 + I_2 + I_3.$$ From (6.1) and (6.5), we have $I_1 + I_2 \leq \frac{2\xi}{3} |\Omega_T|$. Note also that for $(j, i) \in I$ and $z \in Q_j^i \times I_j^i$, from (6.3), (6.4), and (f),

$$|\tilde{v}_t(z) - \sigma(D\tilde{u}(z))| = |v_t(z) + (\psi_j^i)_t(z) + (g_j^i)_t(z) - \sigma(Du(z) + D\varphi_j^i(z))|$$

$$\leq |v_t(z) - v_t(z_j^i)| + |v_t(z_j^i) + (\psi_j^i)_t(z) - \sigma(Du(z_j^i) + D\varphi_j^i(z))|$$
Here, as (\(\tilde{D}(z), \tilde{v}(z)\)) \(\in \mathcal{S}_\delta\), we have |\(D(u(z)) + D\varphi_j^i(z)\)| = |\(D\tilde{u}(z)\)| \(\leq m_+(\delta)\), and by (6.3), |\(D(u(z)) - D\tilde{u}(z)\)| < \(\theta\). From (6.2) we now have

\[
|\sigma(D(u(z)) + D\varphi_j^i(z)) - \sigma(D(u) + D\varphi_j^i(z))| < \frac{\epsilon}{12}.
\]

Integrating the above inequality over \(Q_j^i \times I_j^i\), we thus obtain from (d) that

\[
\int_{Q_j^i \times I_j^i} |\tilde{v}(z) - \sigma(D\tilde{u}(z))|dz \leq \frac{\epsilon}{4}|Q_j^i \times I_j^i| + \frac{\epsilon|\Omega_T| |Q_j^i \times I_j^i|}{12 |Q \times I|}
\]

\[
\leq \frac{\epsilon}{3}|Q_j^i \times I_j^i|,
\]

which yields that \(I_3 \leq \frac{\epsilon}{3}|\Omega_T|\). Hence \(I_1 + I_2 + I_3 \leq \epsilon|\Omega_T|\), and so \(\tilde{u} \in \mathcal{U}\). Finally, from (c) with \(\rho \leq \eta\) and the definition of \(\tilde{u}\), we have

\[
\|\tilde{u} - u\|_{L^\infty(\Omega_T)} < \eta.
\]

This completes the proof. \(\square\)

### 6.2. Completion of the Proof of Theorem 1.1.

The existence of infinitely many weak solutions to problem (1.1) readily follows from a combination of Remark 5.1, Theorem 6.1, and Theorem 3.3.

To prove the last statement of Theorem 1.1, assume \(M = \|Du_0\|_{L^\infty(\Omega)} \geq 1\) and \(\lambda > 0\). Recall that in this case \(\delta = \frac{M + \lambda}{1 + (M + \lambda)^2}\). By the definition of \(\mathcal{U}\) and Corollary 4.4, we have |\(Du\)| \(\leq m_+(\delta) = M + \lambda\) a.e. in \(\Omega_T\) for all \(u \in \mathcal{U}\). On the other hand, following step 3 in the proof of Theorem 3.3, every weak solution \(u \in \mathcal{G}\) is the \(L^\infty\)-limit of some sequence \(u_j \in \mathcal{U}\) such that \(Du_j \rightarrow Du\) a.e. in \(\Omega_T\). So these weak solutions \(u \in \mathcal{G}\) must satisfy \(\|Du\|_{L^\infty(\Omega_T)} \leq M + \lambda\).

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