MTH 299 In Class and Recitation Problems

SUMMER 2016

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1 Week 1

1.1 In Class Problems

Example 1.1. Example 1.7 (from textbook)
Note: Spend some time on (iv) – perfect practice for explaining problems related to HW1 Q 7 and HW2 Q 6

Example 1.2. Example 1.9 (from textbook)

Example 1.3. Compute the cardinality of the set, $E$, where $E$ is defined as

$$E = \{ x \in \mathbb{R} : \sin(x) = 1/2 \text{ and } |x| \leq 5 \}$$

Example 1.4. Example 1.17 (from textbook)

Example 1.5. Define the sets

$$A = \{ n \in \mathbb{Z} : n = 2k, \text{ for some } k \in \mathbb{Z} \}$$

and

$$B = \{ m \in \mathbb{Z} : m = 2^i, \text{ for some } i \in \mathbb{N} \}.$$  

Give a careful explanation of why each of the following two statements are true: (i) $B \subseteq A$ and (ii) $A \not\subseteq B$. (Note that this means that $B$ is a proper subset of $A$, see p. 7.)

Example 1.6. Example 1.19 (from textbook)

Example 1.7. Example 1.22 (from textbook)

Example 1.8. Example 1.26 (from textbook)

Example 1.9. For $A = \{ x \in \mathbb{R} : |x - 1| \leq 2 \}$ and $B = \{ y \in \mathbb{R} : |y - 4| \leq 2 \}$, give a geometric description of the points in the $xy$-plane belonging to $A \times B$.

Example 1.10. Example 1.31 (from textbook)

Example 1.11. Example 1.32, parts (i) and (ii) (from textbook)

Example 1.12. Find the largest possible domain, $X \subseteq \mathbb{R}$, so that the formula,

$$f(x) = \frac{3x + 1}{5x - 2}$$

defines a function $f : X \rightarrow \mathbb{R}$. Justify your answer.

Example 1.13. Does the formula,

$$f(x) = \frac{3x + 1}{5x - 2}$$

define a function $f : \mathbb{Z} \rightarrow \mathbb{Q}$? Justify your answer.
Example 1.14. Write each of the following sets by listing its elements within braces. Find their cardinalities.

1. \( A = \{ n \in \mathbb{Z} : -4 < n \leq 4 \} \)
2. \( B = \{ n \in \mathbb{Z} : n^2 < 5 \} \)
3. \( C = \{ n \in \mathbb{N} : n^3 < 100 \} \)
4. \( D = \{ x \in \mathbb{R} : x^2 - x = 0 \} \)
5. \( E = \{ x \in \mathbb{R} : x^2 + 1 = 0 \} \)

Example 1.15. Let \( S = \{1, \{2, 3\}, 4\} \). Indicate whether each statement is true or false.

1. \( |S| = 4 \)
2. \( \{1\} \in S \)
3. \( \{2, 3\} \in S \)
4. \( \{1, 4\} \subseteq S \)
5. \( S = \{1, 4, \{2, 3\}\} \)
6. \( \emptyset \subseteq S \)

Example 1.16. Let \( U = \{1, 3, \ldots, 15\} \) be the universal set, \( A = \{1, 5, 9, 13\} \) and \( B = \{3, 9, 13\} \). Determine the following:

1. \( A \cup B \)
2. \( A \cap B \)
3. \( A \setminus B \)
4. \( B \setminus A \)
5. \( A^c \)
6. \( A \cap B^c \).

Example 1.17. Let \( U = \{1, 2, 3\} \) be the universal set and let \( A = \{1, 2\}, B = \{2, 3\} \) and \( C = \{1, 3\} \). Determine the following:

1. \( (A \cup B) \setminus (B \cap C) \)
2. \( A^c \)
3. \( (B \cup C)^c \)
4. \( A \times B \).
Example 1.18. For \( A = \{-1, 0, 1\} \) and \( B = \{x, y\} \), determine \( A \times B \).

Example 1.19. Let \( z = (x, y) \) be a point on the unit circle in the plane. Define the assignment \( f(z) \) is the point in the plane which results from rotating \( z \) by \( \pi/4 \) radians anti-clockwise. Does this assignment, \( f \), properly define a function? Give an appropriate domain and co-domain so that \( f \) does define a function. As a bonus, try to represent this situation using an angle, \( \theta \), as well as \( \sin(\theta) \) and \( \cos(\theta) \).

Example 1.20. Attempt to define an assignment for a real number, \( x \in \mathbb{R} \), as \( f(x) = \begin{cases} x^2 & \text{if } x \text{ is even} \\ \frac{x+1}{2} & \text{if } x \text{ is odd}. \end{cases} \) Does this assignment make sense for \( x \in \mathbb{R} \)? Justify your answer. Does this assignment define a function \( f : \mathbb{R} \to \mathbb{R} \)? Justify your answer.

Example 1.21. Let \( f : \mathbb{R} \to \mathbb{Z} \) be defined via the assignment
\[
f(x) = |x| := \max\{z \in \mathbb{Z} : z \leq x\}.
\]
(i) does this assignment define a function? Justify your answer.
(ii) evaluate \( f(0.1), f(4.7), f(\pi), f(-e) \).

Example 1.22. Suppose \( A = \{a, b, c, d, e, f, g\}, B = \{1, 2, 3, 4, 5, 6\} \) and \( f \) defined as
\[
f(a) = 2, \ f(b) = 3, \ f(c) = 4, \ f(d) = 5.
\]
State the domain and range of \( f \). Find \( f(b) \) and \( f(d) \).

Example 1.23. For each of the following, determine the largest set \( A \subseteq \mathbb{R} \), such that \( f : A \to \mathbb{R} \) defines a function. Next, determine the range, \( f(A) := \{y \in \mathbb{R} : f(x) = y, \text{ for some } x \in A\} \).

(i) \( f(x) = 1 + x^2 \),
(ii) \( f(x) = 1 - \frac{1}{x} \),
(iii) \( f(x) = \sqrt{3x - 1} \),
(iv) \( f(x) = x^3 - 8 \),
(v) \( f(x) = \frac{x}{x-3} \).

Example 1.24. Let the function \( f : [-1, 1] \to [0, 1] \) be defined by \( f(x) = |x| \). Is this function injective? How about if its domain is \([0, 1], \) instead of \([-1, 1]\)?

Example 1.25. Let \( A = \{1, 2, 3, 4\} \) and \( B = \{a, b, c\} \). Give an example of a function \( f : A \to B \) that is neither injective nor surjective.

Example 1.26. A function \( f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \) is defined as \( f((m, n)) = 2n - 4m \). Verify whether this function is injective and whether it is surjective.
Example 1.27. There are four different functions $f : \{a, b\} \to \{0, 1\}$. List them all.

Example 1.28. Suppose $A = \{-101, -10, -5, 0, 1, 2, 3, 4, 5, 6, 10\}, B = \{1, 2, 3, 4, 5, 6\}$ and $f$ is defined by the table:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-10$</td>
<td>3</td>
</tr>
<tr>
<td>$-5$</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
</tr>
</tbody>
</table>

State the domain and range of $f$. Find $f(2)$ and $f(-5)$.

Example 1.29. Let $f : \mathbb{Z} \to \mathbb{Z}$ be the assignment defined by the relationship

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ \frac{n+1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

(i) Does this description of $f$ define a function? Justify your answer.

(ii) Is $f$ injective? Justify your answer.

(iii) Is $f$ surjective? Justify your answer.

Example 1.30. Let $A = \{x|x \in \mathbb{R} \text{ and } x \neq 2\}$ and the function $f : A \to \mathbb{R}$ be defined by $f(x) = \frac{4x}{x-2}$. Prove that the function $f$ is injective.

Example 1.31. A function $f : \mathbb{Z} \to \mathbb{Z}$ is defined as $f(n) = 2n + 1$. Verify whether this function is injective and whether it is surjective.

Example 1.32. Prove that the function $f : \mathbb{R} \setminus \{2\} \to \mathbb{R} \setminus \{5\}$ defined by $f(x) = \frac{5x+1}{x-2}$ is bijective.

Example 1.33. Assume that you know for any real numbers, $a, b, c \in \mathbb{R}$ that if $a \geq 0$ and $b \leq c$, then it holds that $ab \leq ac$. Carefully justify the statement: for any real number, $x$, it holds that $x^2 \geq 0$.

Example 1.34. Use the limit rules from MTH 132 to carefully justify that if $f(x) = x^2$, then $f'(x) = 2x$.

Example 1.35. Assume that you know that $x < y$. Carefully justify the statement that

$$x < \frac{x+y}{2} < y.$$ 

Example 1.36. Carefully justify both of these two claims under the assumption that $x$ and $y$ are real numbers.

(i) $\max\{x, -x\} = |x|$.

(ii) $\max\{x, y\} = \frac{1}{2}(|x - y| + x + y)$

Note, it may be useful to recall that

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$
Example 1.37. Decide whether or not the following are statements. In the case of a statement, say if it is true or false.

(i) Every real number is an even integer.
(ii) If $x$ and $y$ are real numbers and $5x = 5y$, then $x = y$.
(iii) The integer $x$ is a multiple of 7.
(iv) Either $x$ is a multiple of 7, or it is not.
(v) Sets $\mathbb{Z}$ and $\mathbb{N}$ are infinite.
(vi) The derivative of any polynomial of degree 5 is a polynomial of degree 6.

Example 1.38. Consider the sets $A, B, C$ and $D$ below. Which of the following statements are true? Give an explanation for each false statement.

\[ A = \{1, 4, 7, 10, 13, 16, \ldots \} \quad C = \{x \in \mathbb{Z} : x \text{ is prime and } x \neq 2\} \]
\[ B = \{x \in \mathbb{Z} : x \text{ is odd}\} \quad D = \{1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots \} \]

(a) $25 \in A$  (b) $33 \in D$  (c) $22 \notin A \cup D$  (d) $C \subseteq B$  (e) $\emptyset \in B \cap D$  (f) $53 \notin C$.

Example 1.39. Consider the conditional statement $P(x) : x(x - 1) = 6$ over the domain $\mathbb{R}$.

(a) For what values of $x$ is $P(x)$ a true statement?
(b) For what values of $x$ is $P(x)$ a false statement?

Example 1.40. Consider the conditional statement $P(x) : 3x - 2 > 4$ over the domain $\mathbb{Z}$.

(a) For what values of $x$ is $P(x)$ a true statement?
(b) For what values of $x$ is $P(x)$ a false statement?

Example 1.41. State the negation of each of the following statements.

(a) $\sqrt{2}$ is a rational number.
(b) 0 is not a negative number.
(c) 111 is a prime number.
Example 1.42. State the negation of each of the following statements.

(a) The real number \( r \) is at most \( \sqrt{2} \).
(b) The absolute value of the real number \( a \) is less than 3.
(c) Two angles of the triangle are \( 45^\circ \).
(d) The area of the circle is at least \( 9\pi \).
(e) Two sides of the triangle have the same length.
(f) The point \( P \) in the plane lies outside of the circle \( C \).

Example 1.43. If \( n \in \mathbb{Z} \) is even, and \( m \in \mathbb{Z} \) is odd, prove that \( n + m \) is odd.

Example 1.44. Complete the following truth table

<table>
<thead>
<tr>
<th>( P )</th>
<th>( Q )</th>
<th>( \neg P )</th>
<th>( \neg Q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td></td>
<td></td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td></td>
<td></td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td></td>
<td></td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Example 1.45. Complete the following truth table

<table>
<thead>
<tr>
<th>( P )</th>
<th>( Q )</th>
<th>( \neg Q )</th>
<th>( P \land (\neg Q) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td></td>
<td></td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td></td>
<td></td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td></td>
<td></td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Example 1.46. Decide whether the following pairs of statements are logically equivalent.

(i) \( P \land Q \) and \( \neg(\neg P \lor \neg Q) \)
(ii) \( P \lor (Q \land R) \) and \( (P \lor Q) \land R \)

Example 1.47. For the sets \( A = \{1, 2, \ldots, 10\} \) and \( B = \{2, 4, 6, 9, 12, 25\} \), consider the statements

\[ P : A \subseteq B, \quad Q : |A \setminus B| = 6. \]

Determine which of the following statements are true.

(a) \( P \lor Q \)  (b) \( P \lor (\neg Q) \)  (c) \( P \land Q \)  (d) \( (\neg P) \land Q \)  (e) \( (\neg P) \lor (Q) \).

Example 1.48. Let \( P : 15 \) is odd. and \( Q : 21 \) is prime. State each of the following in words, and determine whether they are true or false.

(a) \( P \lor Q \)  (b) \( P \land Q \)  (c) \( (\neg P) \lor Q \)  (d) \( P \land (\neg Q) \)
Example 1.49. Prove or disprove that the function \( f \) is surjective

(i) \( f : \mathbb{R} \rightarrow \mathbb{R} \) defined by
\[
f(x) = \begin{cases} 
x^2 & \text{if } x > 0 \\
x & \text{if } x \leq 0,
\end{cases}
\]

(ii) \( f : \mathbb{Z} \rightarrow \mathbb{N} \cup \{0\} \) defined by \( f(x) = |x| \).

(iii) \( f : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) defined by \( f(x, y) = (x + y, x - y) \).

(iv) Let \( A = \{x \mid x \in \mathbb{R} \text{ and } 0 \leq x \leq 5\} \) and \( B = \{x \mid x \in \mathbb{R} \text{ and } 2 \leq x \leq 8\} \). Let \( f : A \rightarrow B \) be defined by \( f(x) = x + 2 \).
1.2 Recitation Problems

Example 1.50. Find the intersection of \( \{2k \mid k \in \mathbb{N}\} \) and \( \{3k \mid k \in \mathbb{N}\} \).

Example 1.51. Let \( A = \{(x, y) \mid x^2 + y^2 \leq 1\} \) and \( B = \{(x, y) \mid (x - 1)^2 + y^2 \leq 1\} \). Let the universal set be the Euclidean plane \( \mathbb{R}^2 = \{(x, y) \mid x, y \in \mathbb{R}\} \).

1. Sketch the regions \( A \) and \( B \). (Both are closed discs: circles together with the points they enclose.)
2. Sketch the region \( (A \cup B)^c \)
3. Sketch the region \( A^c \cap B^c \)
4. Formulate a conjecture which explains your observations in the previous two steps.

Example 1.52. For \( A = \{a \in \mathbb{R} : |a| \leq 1\} \) and \( B = \{b \in \mathbb{R} : |b| = 1\} \), give a geometric description of the points in the \( xy\)-plane belonging to \( (A \times B) \cup (B \times A) \).

Example 1.53. Find the cardinalities of the following sets.

1. \( \emptyset, \{\emptyset\}, \{\{\emptyset\}\} \)
2. \( \{n \in \mathbb{Z} \mid -2014 \leq n \leq 2014\} \)
3. \( \{n \in \mathbb{N} \mid n \text{ has exactly 3 digits}\} \)
   For example, 7 has one digit and 107 has three digits.

Example 1.54. Find the union of of \( \{2k \mid k \in \mathbb{N}\} \) and \( \{2k + 1 \mid k \in \mathbb{N}\} \).

Example 1.55 (extra). Describe each set below in the form

\[ \{x \in X : x \text{ satisfies property } P\} \]

1. \( D = \{3, 5, 7, 9, \ldots\} \)
2. \( E = \left\{ \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots \right\} \)
3. \( F = \{2, -4, 8, -16, \ldots \} \)

Example 1.56 (extra). List the elements of the set \( S = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} : |x| + |y| = 3\} \). Plot the corresponding points in the Euclidean \( xy\)-plane.
Example 1.57. List the elements of each set below and then sketch a Venn diagram to illustrate how they are related. For this problem, \( \mathcal{U} = \{1, 2, 3, \ldots, 10\} \) is the universal set.

\[
A = \{n \mid n \text{ is even}\}, \quad B = \{n \mid n \text{ is the square of an integer}\}, \quad C = \{n \mid n^2 - 3n + 2 = 0\}
\]

Shade each of the following sets:

(i) \( (A \cup B)^c \)

(ii) \( A^c \cap B^c \)

(iii) \( (A \cap B)^c \)

(iv) \( A^c \cup B^c \).

Example 1.58. Show that the function \( f : \mathbb{R} \to \mathbb{R} \) given by \( f(x) = x^3 + 1 \) is injective.

Example 1.59. The function \( f : \mathbb{R} \to \mathbb{R} \) defined as \( f(x) = \pi x - e \) is bijective. Find its inverse.

Example 1.60. Prove or disprove that the function \( f : \mathbb{R} \to \mathbb{R} \) defined by \( f(x) = x^3 - x \) is injective.

Example 1.61. Consider the cosine function \( \cos : \mathbb{R} \to \mathbb{R} \). Decide whether this function is injective and whether it is surjective. What if it had been defined as \( \cos : \mathbb{R} \to [-1, 1] \)?

Example 1.62. Check that the function \( f : \mathbb{Z} \to \mathbb{Z} \) defined by \( f(n) = 6 - n \) is bijective. Then compute \( f^{-1} \).

Example 1.63. Let \( A = \mathbb{R} \setminus \{1\} \) and define \( f : A \to A \) by \( f(x) = \frac{x}{x-1} \) for all \( x \in A \).

(i) Prove that \( f \) is bijective.

(ii) Determine \( f^{-1} \).

Example 1.64. Define the operation \( f(p) := \frac{d}{dx} p \).

Does \( f \) define a function from \( \mathbb{P}_4 \) to \( \mathbb{P}_4 \)? Justify your answer. Is \( f \) an injective function from \( \mathbb{P}_4 \) to \( \mathbb{P}_4 \)? Justify your answer. Is \( f \) a surjective function from \( \mathbb{P}_4 \) to \( \mathbb{P}_4 \)? Justify your answer. (\( \mathbb{P}_n \) is the collection of polynomials or order at most \( n \), see the supplementary document.)

Example 1.65. The function \( f : \mathbb{R} \to \mathbb{R} \) is defined by

\[
f(x) = \begin{cases} 
\frac{1}{1-x} & \text{if } x < 1 \\
\frac{1}{\sqrt{x-1}} & \text{if } x \geq 1.
\end{cases}
\]

(i) Show that \( f \) is a bijection.

(ii) Determine the inverse \( f^{-1} \) of \( f \).
Example 1.66. Which of the following sentences are statements? For those that are, indicate the truth value.

(i) The integer 123 is prime.
(ii) The integer 0 is even.
(iii) Is $5 \times 2 = 10$?
(iv) $x^2 - 4 = 0$.
(v) Multiply $5x + 2$ by 3.
(vi) $5x + 3$ is an odd integer.
(vii) What an impossible question!

Example 1.67. Which of the following statements are true? Give an explanation for each false statement.
(a) $\emptyset \in \emptyset$
(b) $\emptyset \in \{\emptyset\}$
(c) $\{1, 3\} = \{3, 1\}$
(d) $\emptyset = \{\emptyset\}$
(e) $\emptyset \subset \{\emptyset\}$
(f) $1 \subseteq \{1\}$.

Example 1.68. Find a conditional statement $P(n)$ over the domain $S = \{3, 5, 7, 9\}$ such that $P(n)$ is true for half of the integers in $S$ and false for the other half.

Example 1.69. For a conditional statement $P(A) : A \subseteq \{1, 2, 3\}$ over the domain $S = \mathcal{P}\{\{1, 2, 4\}\}$ determine:

(a) all $A \in S$ for which $P(A)$ is true.
(b) all $A \in S$ for which $P(A)$ is false.
(c) all $A \in S$ for which $A \cap \{1, 2, 3\} = \emptyset$. 
2 Week 2

2.1 In Class Problems

Example 2.1. Consider the statements \( P : 17 \) is even. and \( Q : 19 \) is prime. Write each of the following statements in words and indicate whether it is true or false.

(a) \( \neg P \)  
(b) \( P \lor Q \)  
(c) \( P \land Q \)  
(d) \( P \Rightarrow Q \)

Example 2.2. For statements \( P \) and \( Q \), construct a truth table for \((P \Rightarrow Q) \Rightarrow (\neg P)\).

Example 2.3. Consider the statements \( P : \sqrt{2} \) is rational. and \( Q : \frac{22}{7} \) is rational. Write each of the following statements in words and indicate whether it is true or false.

(a) \( P \Rightarrow Q \)  
(b) \( Q \Rightarrow P \)  
(c) \((\neg P) \Rightarrow (\neg Q)\)  
(d) \((\neg Q) \Rightarrow (\neg P)\)

Example 2.4. Decide whether the following pairs of statements are logically equivalent.

(i) \( \neg(P \Rightarrow Q) \) and \( P \land \neg Q \)

(ii) \((P \Rightarrow Q) \lor R \) and \( \neg((P \land \neg Q) \land \neg R) \)

(iii) \((\neg P) \land (P \Rightarrow Q) \) and \( \neg(Q \Rightarrow P) \)

(iv) \( P \land (Q \lor \neg Q) \) and \( (\neg P) \Rightarrow (Q \land \neg Q) \)

Example 2.5. Use truth tables to show that the following statements are logically equivalent.

(i) \( P \Rightarrow Q \equiv (\neg P) \lor Q \)

(ii) \( P \Rightarrow Q \equiv (P \land \neg Q) \Rightarrow (Q \land \neg Q) \)

Example 2.6. Write a truth table for the logical statements:

(i) \( P \lor (Q \Rightarrow R) \)

(ii) \( (P \land \neg P) \Rightarrow Q \)

(iii) \( \neg(P \Rightarrow Q) \)

Example 2.7. In each of the following, two conditional statements \( P(x) \) and \( Q(x) \) over a domain \( S \) are given. Determine the truth values of \( P(x) \Rightarrow Q(x) \) for each \( x \in S \).

(a) \( P(x) : |x| = 4 \); \( Q(x) : x = 4 \); \( S = \{-4, -3, 1, 4, 5\} \).

(b) \( P(x) : x^2 = 16 \); \( Q(x) : |x| = 4 \); \( S = \{-6, -4, 0, 3, 4, 8\} \).

(c) \( P(x) : x > 3 \); \( Q(x) : 4x - 1 > 12 \); \( S = \{0, 2, 3, 4, 6\} \).

Example 2.8. Assume you know the following facts for \( f(x) = x \) and \( g(x) = x^2 \); \( f'(x) = 1 \) and \( g'(x) = 2x \). Use the product rule for derivatives to carefully justify the statement that if \( h(x) = x^3 \) and \( j(x) = x^4 \), then \( h'(x) = 3x^2 \) and \( j'(x) = 4x^3 \).

Example 2.9. Use truth tables to decide whether the following are logically equivalent:
1. \( p \Rightarrow (q \Rightarrow r) \) and \( (p \land q) \Rightarrow r \),

2. \( f \Rightarrow (r \land h) \) and \( (f \Rightarrow r) \land h \).

**Example 2.10.** For statements \( P \) and \( Q \), show that \( (P \land (P \Rightarrow Q)) \Rightarrow Q \) is a tautology. Then state this compound statement in words. (This is an important logical argument form, called **modus ponens**.)

**Example 2.11.** For statements \( P \) and \( Q \), show that \( [\neg Q \land (P \Rightarrow Q)] \Rightarrow \neg P \) is a tautology. (This is another important logical argument form, called **modus tollens**.)

**Example 2.12.** Show that \( (P \Rightarrow Q) \lor (Q \Rightarrow P) \) is a tautology

**Example 2.13.** Prove that \( (P \land (P \Rightarrow Q)) \land (\neg Q) \) is a contradiction.

**Example 2.14.** Let \( P \), \( Q \), and \( R \) be statements. Prove that \( (P \Rightarrow Q) \Rightarrow R \) is equivalent to \((P \land (\neg Q)) \lor R \).

**Example 2.15.** Does the implication operator satisfy the associative property, i.e. is

\[
((P \Rightarrow Q) \Rightarrow R) \equiv (P \Rightarrow (Q \Rightarrow R))?
\]

### 2.2 Recitation Problems

**Example 2.16.** State the negation of each of the following statements.

(a) At least two of my library books are overdue.

(b) One of my two friends misplaced his homework assignment.

(c) No one expected that to happen.

(d) It’s not often that my instructor teaches that course.

(e) It’s surprising that two students received the same exam score.

**Example 2.17.** Express each statement as one of the forms \( P \land Q \), \( P \lor Q \), or \( \neg P \). Be sure to also state exactly what statements \( P \) and \( Q \) stand for.

(i) The number 8 is both even and a power of 2.

(ii) The number \( x \) equals zero, but the number \( y \) does not.

(iii) \( x \in A \setminus B \).

(iv) \( x \neq y \).

(v) \( y \geq x \).

(vi) \( A \in \{X \in \mathcal{P}(\mathbb{N}) : |X^c| < \infty\} \).

**Example 2.18.** Write a truth table for the following:
Example 2.19. Use truth tables to show that the following statements are logically equivalent.

(i) \( P \land (Q \lor R) \equiv (P \lor Q) \land (P \lor R) \)

(ii) \( \neg (P \lor Q \lor R) \equiv \neg P \land \neg Q \land \neg R \)

(iii) \( \neg (P \land Q \land R) \equiv \neg P \lor \neg Q \lor \neg R \)

(iv) \( \neg (P \lor Q) \equiv \neg P \land \neg Q \)
Example 2.20. Let $S = \{1, 2, \ldots, 6\}$ and let
\[ P(A) : A \cap \{2, 4, 6\} = \emptyset \quad \text{and} \quad Q(A) : A \neq \emptyset. \]
be conditional statements over the domain $\mathcal{P}(S)$.

(a) Determine all $A \in \mathcal{P}(S)$ for which $P(A) \land Q(A)$ is true.

(b) Determine all $A \in \mathcal{P}(S)$ for which $P(A) \lor (\neg Q(A))$ is true.

(c) Determine all $A \in \mathcal{P}(S)$ for which $(\neg P(A)) \land (\neg Q(A))$ is true.

Example 2.21. Use truth tables to determine the following questions.

(i) Is the statement $(\neg P) \land (P \Rightarrow Q)$ logically equivalent to the statement $\neg (Q \Rightarrow P)$?

(ii) Is the statement $P \Rightarrow Q$ logically equivalent to the statement $(\neg P) \lor Q$?

Example 2.22. Suppose $A = \{\pi, e, 0\}$ and $B = \{0, 1\}$. Write out the indicated sets by listing their elements.

(a) $A \times B$  (b) $B \times A$  (c) $A \times A$  (d) $B \times B$  (e) $A \times \emptyset$  (f) $A \times B \times B$

Example 2.23. Decide if the following statements are true or false. Explain.

1. $\mathbb{R}^2 \subseteq \mathbb{R}^3$

2. $\{(x, y) : x - 1 = 0\} \subseteq \{(x, y) : x^2 - x = 0\}$

Example 2.24. Prove that $\{x \in \mathbb{Z} : x \text{ is divisible by } 18\} \subset \{x \in \mathbb{Z} : x \text{ is divisible by } 6\}$.

Example 2.25. Suppose $A = \{0, 2, 4, 6, 8\}$, $B = \{1, 3, 5, 7\}$ and $C = \{2, 8, 4\}$. Find:

(a) $A \cup B$  (b) $A \setminus C$  (c) $B \setminus A$  (d) $B \cap C$  (e) $C \setminus B$

Example 2.26. Sketch the set $X = \{(x, y) \in \mathbb{R}^2 : 1 \leq x^2 + y^2 \leq 4\}$ on the plane $\mathbb{R}^2$. On a separate drawing, shade in the set $X^c$.

Example 2.27. Consider the function $f : \mathbb{R} \times \mathbb{N} \to \mathbb{N} \times \mathbb{R}$ defined as $f(x, y) = (y, 3xy)$. Check that this is bijective; find its inverse. Carefully justify that your answer does indeed yield the inverse function.

Example 2.28. Define the function
\[ f : A \to \mathcal{P}(A), \quad f(x) := \{x\}. \]

Example 2.29. Use the limit rules from MTH 132 to carefully justify that if $f(x) = x^2$, then $f'(x) = 2x$.

Example 2.30. If $n$ is an even integer, show that $n^2$ is also an even integer.

Example 2.31. Make a truth table for $\neg[(\neg p \land p) \land (\neg q \lor p)]$. Is this statement a tautology?
3 Week 3

3.1 In Class Problems

Example 3.1. Without changing their meanings, convert each of the following sentences into a sentence having the form “If P, then Q.”

1. The quadratic equation $ax^2 + bx + c = 0$ has real roots provided that $b^2 - 4ac \geq 0$.
2. A function is rational if it is a polynomial.
3. An integer is divisible by 8 only if it is divisible by 4.
4. Whenever a circle, $C$, has a circumference of $4\pi$, it has an area of $4\pi$.
5. The integer $n^3$ is even only if $n$ is even.

Example 3.2. Write the inverse of each of the following implications:

1. If $n$ is an integer, then $2n$ is an even integer.
2. You can work here only if you have a college degree.
3. The car will not run whenever you are out of gas.
4. Continuity is a necessary condition for differentiability.

Example 3.3. Write down the contrapositive of the implication “If $7m$ is an odd number, then $m$ is an odd number”.

Example 3.4. Write down the contrapositive of the following:

1. If $n$ is an integer, then $2n$ is an even integer.
2. You can work here only if you have a college degree.
3. The car will not run whenever you are out of gas.
4. Continuity is a necessary condition for differentiability.
Example 3.5. Here are four conditional statements $P(x)$, $Q(x)$, $A(x)$, and $B(x)$ over the domain $S = \{-4, -3, 1, 3, 4, 5\}$.

- $P(x) : |x| = 5$;
- $Q(x) : x = 5$;
- $A(x) : x^2 = 9$;
- $B(x) : x = 3$.

Determine the truth value of both $P(x) \implies Q(x)$ as well as $A(x) \implies B(x)$ for each $x \in S$.

Example 3.6. Prove that if $x, y \in \mathbb{R}$, then $\frac{3}{4}x^2 + \frac{1}{3}y^2 \geq xy$.

Example 3.7. Suppose the statement $((P$ and $Q) \text{ or } R) \implies (R$ or $S)$ is false. Find the truth values of $P, Q, R$ and $S$. (This can be done without a truth table.)

Example 3.8. Consider the implication “If $m^2 > 0$, then $m > 0$”. Is it true or false? Write down its converse and determine its truth value.

Example 3.9. Write the converse of each of the following implications:

1. If $n$ is an integer, then $2n$ is an even integer.
2. You can work here only if you have a college degree.
3. The car will not run whenever you are out of gas.
4. Continuity is a necessary condition for differentiability.

Example 3.10. Use a truth table to show that the inverse and converse of $P \implies Q$ are logically equivalent.

Example 3.11. Without changing their meanings, convert each of the following sentences into a sentence having the form “$P$ if and only if $Q$.”

1. If $xy = 0$ then $x = 0$ or $y = 0$, and conversely.
2. If $a \in \mathbb{Q}$ then $5a \in \mathbb{Q}$, and if $5a \in \mathbb{Q}$ then $a \in \mathbb{Q}$.
3. For an occurrence to become an adventure, it is necessary and sufficient for one to recount it. (Jean-Paul Sartre)

Example 3.12. Let $P : 18$ is odd, and $Q : 25$ is even. State $P \iff Q$ in words. Is $P \iff Q$ true or false?

Example 3.13. Suppose $P$ is false and that the statement $(R \implies S) \iff (P$ and $Q)$ is true. Find the truth values of $R$ and $S$. (This can be done without a truth table)
Example 3.14. Let $S = \{1, 2, 3, 4\}$. Consider the following conditional statements over the domain $S$:

$P(n) : \frac{n(n-1)}{2}$ is even.

$Q(n) : 2^{n-2} - (-2)^{n-2}$ is even.

$R(n) : 5^{n-1} + 2^n$ is prime.

Determine the distinct elements $a, b, c, d$ in $S$ such that

(i) $P(a) \implies Q(a)$ is false;  \hspace{1cm} (ii) $Q(b) \implies P(b)$ is true;

(iii) $P(c) \iff R(c)$ is true;  \hspace{1cm} (iv) $Q(d) \iff R(d)$ is false;

Example 3.15. If $n$ is an integer, show that $n^2 + n^3$ is an even number.

Example 3.16. Write the following as English sentences. Say whether they are true or false.

1. $\forall x \in \mathbb{R}, x^2 > 0$
2. $\forall x \in \mathbb{R}, \exists n \in \mathbb{N}, x^n \geq 0$
3. $\exists a \in \mathbb{R}, \forall x \in \mathbb{R}, \alpha x = x$

Example 3.17. Rewrite the following using logical connectives and quantifiers

1. If $f$ is a polynomial and its degree is greater than 2, then $f'$ is not constant.
2. The number $x$ is positive but the number $y$ is not positive.
3. If $x$ is prime then $\sqrt{x}$ is not a rational number.

Example 3.18. Rewrite the following using logical connectives and quantifiers

1. For every prime number $p$ there is another prime number $q$ with $q > p$.
2. For every positive number $\epsilon$, there is a positive number $\delta$ for which $|x-a| < \delta$ implies $|f(x) - f(a)| < \epsilon$.
3. For every positive number $\epsilon$ there is a positive number $M$ for which $|f(x) - b| < \epsilon$, whenever $x > M$.

Example 3.19. Determine the truth value of each of the following statements.

1. $\exists x \in \mathbb{R}, x^2 - x = 0$.
2. $\forall x \in \mathbb{R}, \sqrt{x^2} = x$. 
Example 3.20. Let the sets $A$ and $B$ be given as $A = \{3, 5, 8\}$ and $B = \{3, 6, 10\}$. Show that the following quantified statement is indeed true:
\[ \exists b \in B, \forall a \in A, a - b < 0. \]

Example 3.21. Define the function, $f : \mathbb{P}_3 \to \mathbb{R}$ via the operation
\[ f(p) := \int_0^1 p(x)dx. \]
Is $f$ injective and or surjective from $\mathbb{P}_3$ to $\mathbb{R}$? Justify your answer.

Example 3.22. In each of the following cases explain what is meant by the statement and decide whether it is true or false.
1. $\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, x + y = 1.$
2. $\exists y \in \mathbb{Z}, \forall x \in \mathbb{Z}, x + y = 1.$
3. $\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, xy = x.$
4. $\exists y \in \mathbb{Z}, \forall x \in \mathbb{Z}, xy = x.$

Example 3.23. Which of the following best identifies $f$ as a constant function, where $x$ and $y$ are real numbers.
(a) $\exists x, \forall y, f(x) = y.$
(b) $\forall x, \exists y, f(x) = y.$
(c) $\exists y, \forall x, f(x) = y.$
(d) $\forall y, \exists x, f(x) = y.$

Example 3.24. State the negations of the following quantified statements:
1. For every rational number $r$, the number $1/r$ is rational.
2. There exists a rational number $r$ such that $r^2 = 2$.

Example 3.25. Define the sets
\[ E := \{-4, -3, -2, -1, \mathbb{N}\} \]
and
\[ F := \mathbb{N} \bigcup \{0\} \bigcup \{\mathbb{Z}\}. \]
(i) Solve the equation $2x^3 + 4x^2 = 0$ for $x \in E$.
(ii) Solve the equation $2x^3 + 4x^2 = 0$ for $x \in F$.
(iii) Now, go express $F$ in a more elementary form, such as (just for example!)
\[ \{A, \{0, 1\}, a, b, c, \ldots\}, \]
and then go back and do (ii) over again.
Example 3.26. Negate the following statements:

1. $\exists y \in \mathbb{Z}, \forall x \in \mathbb{Z}, x + y = 1$.
2. $\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, xy = x$.

Example 3.27. Negate the following statements:

1. $\forall \varepsilon > 0, \exists \delta > 0, 0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon$.
   (This is a formal definition of continuity of a function $f$ at a point $x = c$. We say that $\lim_{x \to c} f(x) = L$. Refer back to your calculus text for more details.)
2. $\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq N \Rightarrow |a_n - L| < \varepsilon$.
   (This is a definition of the limit of a sequence. We say that $\lim_{n \to \infty} a_n = L$. We will discuss this later in the semester. Also see Supplementary Material for a more detailed discussion.)
3. $\forall y \in B \exists x \in A, f(x) = y$.
   Recall that this is the formal definition of a surjective function $f : A \to B$.

Example 3.28. Prove the following:

$$\neg(\exists x \ P(x)) \iff \forall x (\neg P(x)).$$

Example 3.29. The following statements give certain properties of functions. You are to do two things:

(a) rewrite the defining conditions using quantifiers and logical connectives, as appropriate

(b) write the negation of part (a) using the same symbolism.

Example: A function $f$ is odd if for every $x$, $f(-x) = -f(x)$.

(a) defining condition: $\forall x, f(-x) = -f(x)$.

(b) negation: $\exists x$ such that $f(-x) \neq -f(x)$.

It is not necessary that you understand precisely what each term means.

1. A function $f$ is even if for every $x$, $f(-x) = f(x)$.
2. A function $f$ is periodic if there exists a $k > 0$ such that for every $x$, $f(x + k) = f(x)$.
3. A function $f$ is increasing if for every $x$ and $y$, if $x \leq y$, then $f(x) \leq f(y)$.

Example 3.30. Show that if $n \in \mathbb{N}$, then $1 + (-1)^n(2n - 1)$ is a multiple of 4. Next, show the converse, i.e. that if $k$ is a multiple of 4, then $k = 1 + (-1)^m(2m - 1)$ for some $m \in \mathbb{N}$. 

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Example 3.31. Let

\[ p : \text{John is a student of the Mathematics Department at MSU}. \]
\[ q : \text{John takes course MTH299}. \]

(a) Write the converse, inverse, and contrapositive forms for the statement \( p \implies q \) in words.

(b) Write the statements \( \neg p \lor q \) and \( \neg q \lor p \) in words.

Example 3.32. Suppose \( x \in \mathbb{Z} \). Assume that \( x^2 - 6x + 5 \) is even, show that this implies \( x \) is odd.
(Hint: Use the fact that an implication and its contrapositive are logically equivalent. You may want to write the desired implication as \( A \implies B \) so that you do the contrapositive correctly.)

Example 3.33. Define the set \( U_+ = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 \geq 0\} \subset \mathbb{R}^2 \). Define the function \( h : \mathbb{R}^2 \to U_+ \) via the assignment

\[ h(x_1, x_2) = (x_1, x_1^2 + x_2^2). \]

Prove that \( h \) is not an injective function. Prove that \( h \) is not a surjective function. Is \( h \) a bijective function? Does an inverse function exist for \( h \)?

Example 3.34. Let \( g : [-\pi/2, 0] \to [-1, 1] \) be the function defined via the assignment \( g(x) = \sin(x) \). Prove that \( g \) is not a surjective function. Does it make sense to talk about an inverse function for say \( g \)?

Example 3.35. Let \( G : [-\pi, 0] \to [-1, 0] \) be the function defined via the assignment \( G(x) = \sin(x) \). Prove that \( G \) is not an injective function. Prove that \( G \) is a surjective function. Does it make sense to talk about an inverse function for say \( G \)? (You had best consult p. 218-223 of the text before answering this last question.)
3.2 Recitation Problems

Example 3.36. Consider the open sentences,

\[ P(x, y) : x^2 + y^2 = 4 \quad \text{and} \quad Q(x, y) : \frac{y}{x} \in \mathbb{Z} \]

on the domain \( A \times B \), where \( A = \{1, 2\} \) and \( B = \{0, \sqrt{3}\} \). Determine for what elements in the domain the statement \( P(x, y) \Rightarrow Q(x, y) \) is true.

Example 3.37. Find the contrapositive of the following statements.

1. If Jane has grandchildren, then she has children.
2. If \( x = 1 \), then \( x \) is a solution to \( x^2 - 3x + 2 = 0 \).
3. If \( x \) is a solution to \( x^2 - 3x + 2 = 0 \), then \( x = 1 \) or \( x = 2 \).

Example 3.38. Fill in the blank with necessary, sufficient or necessary and sufficient.

1. The number \( x > 1 \) is ________________ for \( x^2 > 1 \).
2. The number \( x \in \mathbb{N} \) is ________________ for \( x \geq 0 \).
3. The number \( |x| > 1 \) is ________________ for \( x^2 > 1 \).

Example 3.39. Fill in the blank with necessary, sufficient or necessary and sufficient.

1. “Mary earned an A in MTH299.” is ________________ for “Mary passed MTH299.”
2. “Mary passed MTH299” is ________________ for “Mary earned an A in MTH299.”
3. “The function \( f \) is continuous at \( x = c \)” is ________________ for “the function \( f \) has a derivative at \( x = c \).”

Example 3.40. Consider the conditional statements

\[ P(x) : x \text{ is prime} \]
\[ Q(x) : x \text{ is odd} \]

over the domain \( \mathbb{Z}_{>0} = \{x \in \mathbb{Z} : x > 0\} \)

(a) Determine the set \( S \) whose elements make the statement \( P \Rightarrow Q \) false. Explain your answer.

(b) Write down in words the contrapositive of \( P \Rightarrow Q \) and determine the set \( T \) for which the contrapositive is a true statement.

(c) Is there any relation between the sets \( S \) and \( T \)? Explain your answer.

Example 3.41. Let \( x \in \mathbb{R} \) and consider the statement whose converse is

\[ \text{If } x = 2 \text{ then } x^2 = 4. \]
(a) Write in words the original statement.

(b) Determine if your statement in part [(a)] is true or false. Explain your answer.

(c) Write down the contrapositive of the statement you provided in part [(a)] and determine if it is true or false.

Example 3.42. Let $m, n \in \mathbb{Z}$. Consider

\[
S_1 : 3|m \text{ and } 3|n \\
S_2 : \gcd(m, n) = 3
\]

is true for all $m, n \in \mathbb{Z}$

(a) Determine whether the condition $S_1$ is necessary or sufficient for $S_2$.

(b) Using (a), write the implication obtained from $S_1, S_2$.

(c) Write the converse of the implication you obtained in (b).

(d) Determine whether the converse you obtained in (c) is true or false.

Example 3.43. Consider the statements

\[
P : \sqrt{2} \text{ is rational} \\
Q : \frac{2}{3} \text{ is rational} \\
R : \sqrt{3} \text{ is rational}
\]

(a) Write in words the statement $(P \lor Q) \Rightarrow \neg R$ and determine whether it is true or false.

(b) Write in words the contrapositive of the statement given in (a) and determine whether it is true or false by means of logical equivalences.

(c) Write in words the converse of $(P \lor Q) \Rightarrow \neg R$ and conclude whether it is true or false.

Example 3.44. Define the assignment, for $p \in \mathbb{P}_2$,

\[f(p) := (x_1, x_2), \text{ where } x_1 \leq x_2, \text{ and } x_1 \text{ and } x_2 \text{ are the real roots of } p.\]

(i) Does this assignment define a function from $\mathbb{P}_2$ to $\mathbb{R}^2$? Justify your answer.

(ii) Define $U = \{p \in \mathbb{P}_2 : p(x) = (x - a)^2 - b, \text{ for } a, b \in \mathbb{R} \text{ and } b \geq 0\}$. Explain why this assignment, $f$, does indeed define a function, $f : U \to \mathbb{R}^2$.

(iii) Define the “half-space”, $H = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \leq x_2\} \subseteq \mathbb{R}^2$. Is this function, $f : U \to H$, surjective? Give a justification of your answer. As a hint, you should draw a picture of what the graphs of the elements of $U$ look like (note, $p \in U$ means that $p$ is a quadratic function, and you know how to graph $p$. This should give you a hint of how to proceed. If I tell you to make up a quadratic function that crosses the x-axis at two specified points on the x-axis, can you do that?)
Example 3.45. Rewrite each statement using ∀, ∃, as appropriate.

1. There exists a positive number \( x \) such that \( x^2 = 5 \).

2. For every positive number \( M \), there is a positive number \( N \) such that \( N < 1/M \).

3. If \( n \geq N \), then \( |f_n(x) - f(x)| \leq 3 \) for all \( x \in A \).

4. No positive number \( x \) satisfies the equation \( f(x) = 5 \).

Example 3.46. Let \( A = \{1, 2, 3, 4, 5\} \) and \( B = \{1, 2, 3, 4\} \).

(a) Give an example of a function \( f : A \rightarrow B \) such that \( \forall y \in B \ \exists x \in A, f(x) = y \).

(b) Give an example of a function \( g : A \rightarrow B \) such that \( \exists y \in B \ \forall x \in A, g(x) = y \).

(c) Give an example of a function \( h : B \rightarrow A \) such that \( \forall x, y \in B, h(x) = h(y) \implies x = y \).

Example 3.47. In each of the following cases explain what is meant by the statement and decide whether it is true or false.

1. \( \lim_{x \to c} f(x) = L \) if \( \forall \varepsilon > 0 \ \exists \delta > 0 \) such that \( 0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon \).

2. \( \lim_{x \to c} f(x) = L \) if \( \exists \delta > 0 \ \forall \varepsilon > 0 \) such that \( 0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon \).

3. \( f : A \rightarrow B \) is surjective provided \( \forall y \in B, \exists x \in A \) such that \( f(x) = y \).

Example 3.48. State the negations of the following quantified statements, where all sets are subsets of some universal set \( U \):

1. For every set \( A \), \( A \cap A^c = \emptyset \).

2. There exists a set \( A \) such that \( A^c \subseteq A \).

3. For every \( x \) in \( A \), \( f(x) > 5 \).

4. There exists a positive number \( y \) such that \( 0 < g(y) \leq 1 \).

5. For all \( x \) and \( y \) in \( A \), there exists \( z \) in \( B \) such that \( x = y = z \).

6. \( \forall \varepsilon > 0, \exists N \) such that \( \forall n, \text{if } n \geq N, \text{ then } \forall x \text{ in } S, |f_n(x) - f(x)| < \varepsilon \).

Example 3.49. Negate each statement in Question 3.46 and give an example of a function \( f, g, h \) satisfying the corresponding negation, if possible. If no example can be provided explain why.
Example 3.50. The following statements give certain properties of functions. You are to do two things:

(a) rewrite the defining conditions using quantifiers and logical connectives, as appropriate

(b) write the negation of part (a) using the same symbolism.

Example: A function \( f \) is odd if for every \( x \), \( f(-x) = -f(x) \).

(a) defining condition: \( \forall x, f(-x) = -f(x) \).

(b) negation: \( \exists x \) such that \( f(-x) \neq -f(x) \).

It is not necessary that you understand precisely what each term means.

1. A function \( f \) is strictly decreasing if for every \( x \) and \( y \), if \( x < y \), then \( f(x) > f(y) \).

2. A function \( f : A \to B \) is injective if for every \( x \) and \( y \) in \( A \), if \( f(x) = f(y) \), then \( x = y \).

3. A function \( f \) is surjective if for every \( y \) in \( B \) there exists an \( x \) in \( A \) such that \( f(x) = y \).

Example 3.51. Prove or disprove: There is a real number solution of the equation

\[ x^6 + 2x^2 + 1 = 0. \]

Example 3.52. Suppose \( a, b, c \in \mathbb{Z} \). If \( a \mid b \), and \( a \mid c \), prove that \( a \mid (b + c) \).

Example 3.53. Show that there exists a positive even integer \( m \) such that for every positive integer \( n \),

\[ \left| \frac{1}{m} - \frac{1}{n} \right| \leq \frac{1}{2}. \]
4 Week 4

4.1 In Class Problems

Example 4.1. Let $S = \{1, \{2, 3\}, 4\}$. Indicate whether each statement is true or false.

1. $|S| = 4$
2. $\{1\} \in S$
3. $\{2, 3\} \in S$
4. $\{1, 4\} \subset S$
5. $S = \{1, 4, \{2, 3\}\}$
6. $\emptyset \subset S$

Example 4.2. Let $U = \{1, 2, 3\}$ be the universal set and let $A = \{1, 2\}$, $B = \{2, 3\}$ and $C = \{1, 3\}$. Determine the following:

1. $(A \cup B) \setminus (B \cap C)$
2. $A^c$
3. $(B \cup C)^c$
4. $A \times B$

Example 4.3. Compute the cardinality of the set, $E$, where $E$ is defined as

$$E = \{x \in \mathbb{R} : \sin(x) = 1/2 \text{ and } |x| \leq 5\}$$

Example 4.4. Let us define the following sets:

$$A_0 = \{z \in \mathbb{Z} : z = 3k, \text{ for some } k \in \mathbb{Z}\},$$
$$A_1 = \{z \in \mathbb{Z} : z = 3k + 1, \text{ for some } k \in \mathbb{Z}\},$$
$$A_2 = \{z \in \mathbb{Z} : z = 3k + 2, \text{ for some } k \in \mathbb{Z}\}.$$

Does $\mathbb{Z} = A_0 \cup A_1 \cup A_2$. Carefully justify your answer.

Example 4.5. Does the formula,

$$f(x) = \frac{3x + 1}{5x - 2}$$

define a function $f : \mathbb{Z} \to \mathbb{Q}$? Justify your answer.

Example 4.6. Consider the cosine function $\cos : \mathbb{R} \to \mathbb{R}$. Decide whether this function is injective and whether it is surjective. What if it had been defined as $\cos : \mathbb{R} \to [-1, 1]$?

Example 4.7. A function $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ is defined as $f((m, n)) = 2n - 4m$. Verify whether this function is injective and whether it is surjective.
Example 4.8. The function $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} \frac{1}{1-x} & \text{if } x < 1 \\ \sqrt{x-1} & \text{if } x \geq 1. \end{cases}$$

(i) Show that $f$ is a bijection.

(ii) Determine the inverse $f^{-1}$ of $f$. Carefully justify that your answer does indeed yield the inverse function.

Example 4.9. Define the operation

$$f(p) := \frac{d}{dx}p.$$ 

Does $f$ define a function from $\mathbb{P}_4$ to $\mathbb{P}_4$? Justify your answer. Is $f$ an injective function from $\mathbb{P}_4$ to $\mathbb{P}_4$? Justify your answer. Is $f$ a surjective function from $\mathbb{P}_4$ to $\mathbb{P}_4$? Justify your answer. ($\mathbb{P}_n$ is the collection of polynomials or order at most $n$, see the supplementary document.)

Example 4.10. Consider the conditional statement $P(x) : 3x - 2 > 4$ over the domain $\mathbb{Z}$.

(a) For what values of $x$ is $P(x)$ a true statement?

(b) For what values of $x$ is $P(x)$ a false statement?

Example 4.11. Let $P : 15$ is odd. and $Q : 21$ is prime. State each of the following in words, and determine whether they are true or false.

(a) $P \lor Q$  
(b) $P \land Q$  
(c) $(\neg P) \lor Q$  
(d) $P \land (\neg Q)$

Example 4.12. Use truth tables to decide whether the following are logically equivalent:

1. $p \Rightarrow (q \Rightarrow r)$ and $(p \land q) \Rightarrow r$,

2. $f \Rightarrow (r \land h)$ and $(f \Rightarrow r) \land h$.

Example 4.13. Write the inverse, converse and contrapositive of each of the following implications:

1. If $n$ is an integer, then $2n$ is an even integer.

2. You can work here only if you have a college degree.

3. The car will not run whenever you are out of gas.

4. Continuity is a necessary condition for differentiability.

Example 4.14. Suppose the statement $((P \land Q) \lor R) \Rightarrow (R \lor S)$ is false. Find the truth values of $P, Q, R$ and $S$. (This can be done without a truth table.)
Example 4.15. The following statements give certain properties of functions. You are to do two things:

(a) rewrite the defining conditions using quantifiers and logical connectives, as appropriate
(b) write the negation of part (a) using the same symbolism.

Example: A function \( f \) is \textit{odd} if for every \( x \), \( f(-x) = -f(x) \).

(a) defining condition: \( \forall x, f(-x) = -f(x) \).
(b) negation: \( \exists x \) such that \( f(-x) \neq -f(x) \).

It is not necessary that you understand precisely what each term means.

1. A function \( f \) is \textit{even} if for every \( x \), \( f(-x) = f(x) \).

2. A function \( f \) is \textit{periodic} if there exists a \( k > 0 \) such that for every \( x \), \( f(x+k) = f(x) \).

3. A function \( f \) is \textit{increasing} if for every \( x \) and \( y \), if \( x \leq y \), then \( f(x) \leq f(y) \).

Example 4.16. If \( n \in \mathbb{Z} \) is even, and \( m \in \mathbb{Z} \) is odd, prove that \( n + m \) is odd.

4.1.1 Proofs Involving Sets

Example 4.17. Let \( B_1 = \{1,2\}, B_2 = \{2,3\}, \ldots, B_{10} = \{10,11\} \); that is, \( B_i = \{i,i+1\} \) for some \( i = 1,2,\ldots,10 \). Determine the following:

\[
(i) \bigcup_{i=1}^{5} B_i \quad (ii) \bigcup_{i=1}^{10} B_i \quad (iii) \bigcup_{i=3}^{7} B_i \quad (iv) \bigcup_{i=j}^{k} B_i, \text{ where } 1 \leq j \leq k \leq 10.
\]

\[
(v) \bigcap_{i=1}^{10} B_i \quad (vi) B_i \cap B_{i+1} \quad (vii) \bigcap_{i=j}^{j+1} B_i, \text{ where } 1 \leq j < 10,
\]

\[
(viii) \bigcap_{i=j}^{k} B_i, \text{ where } 1 \leq j \leq k \leq 10.
\]

(Note, see section 1 of the supplementary material for a definition of this strange notation you are most likely encountering for the first time here.)

Example 4.18. Prove that \( \{12n : n \in \mathbb{Z}\} \subseteq \{2n : n \in \mathbb{Z}\} \cap \{3n : n \in \mathbb{Z}\} \).

Example 4.19. Prove that \( \{6n : n \in \mathbb{Z}\} = \{2n : n \in \mathbb{Z}\} \cap \{3n : n \in \mathbb{Z}\} \).

Example 4.20. Prove that \( \{12a + 4b : a,b \in \mathbb{Z}\} = \{4c : c \in \mathbb{Z}\} \).
4.1.2 Bounded Sets

**Example 4.21.** Discuss whether the following sets are bounded or not bounded.

1. \( A = \{-2, -1, 1/2\} \).
2. \( B = (-\infty, \sqrt{2}). \)
3. \( C = \{1/2, 3/2, 5/2, 7/2, 9/2, \ldots\} \). (Note, the Archimedean property of \( \mathbb{R} \) could be useful.)
4. \( D = \left\{ \frac{(-1)^n}{n} : n \in \mathbb{N} \right\} \) (Note, the Archimedean property of \( \mathbb{R} \) could be useful.)
5. \( E = \left\{ \frac{(-1)^n}{n} : n \in \mathbb{Q} \setminus \{0\} \right\} \) (Note, the Archimedean property of \( \mathbb{R} \) could be useful.)

4.1.3 Open Sets

**Example 4.22.** Discuss which of the following sets are open.

(i) \((2, 3)\).
(ii) \((-4, 8)\).
(iii) \([1, 3)\).
(iv) \((-\infty, \infty)\).
(v) \([1, 5] \cap [2, 3]\).

**Example 4.23.** Prove that \( \bigcap_{x \in \mathbb{N}} [3 - (1/x)^2, 5 + (1/x)^2] = [3, 5] \). See section 1 of the supplementary material for this notation with intersection of sets. (Note, the Archimedean property of \( \mathbb{R} \) will be helpful.)

4.1.4 Sequences

**Example 4.24.** Write out the first five terms of the following sequences:

i) \( \{2n - 5\}_{n \in \mathbb{N}} \).
ii) \( \left\{ \frac{1}{2n+5} \right\}_{n \in \mathbb{N}} \).

You may assume that each sequence is a function from \( \mathbb{N} \) to \( \mathbb{R} \).
Example 4.25. For each of the following, determine whether or not they converge. If they converge, what is their limit? No proofs are necessary, but provide some algebraic justification.

i) \( \left\{ \frac{3n+1}{7n-4} \right\}_{n \in \mathbb{N}} \)

ii) \( \left\{ \sin \left( \frac{2\pi}{n} \right) \right\}_{n \in \mathbb{N}} \)

iii) \( \left\{ (1 + 1/n)^2 \right\}_{n \in \mathbb{N}} \)

iv) \( \left\{ (-1)^n n \right\}_{n \in \mathbb{N}} \)

v) \( \left\{ \sqrt{n^2 + 1} - n \right\}_{n \in \mathbb{N}} \)

Example 4.26. Using the definition of convergence, that is, an \( \epsilon - N \) argument, prove that the following sequences converge to the indicated number:

i) \( \lim_{n \to \infty} \frac{1}{n} = 0. \)

ii) \( \lim_{n \to \infty} \frac{1}{\sqrt{n}} = 0. \)

iii) \( \lim_{n \to \infty} \frac{n}{2n+1} = \frac{1}{2}. \)
4.2 Recitation Problems

Example 4.27. Prove that \( \{9^n : n \in \mathbb{Z}\} \subseteq \{3^n : n \in \mathbb{Z}\} \), but \( \{9^n : n \in \mathbb{Z}\} \neq \{3^n : n \in \mathbb{Z}\} \).

Example 4.28. Prove that \( \{9^n : n \in \mathbb{Q}\} = \{3^n : n \in \mathbb{Q}\} \).

Example 4.29. For each \( n \in \mathbb{N} \), define \( A_n \) to be the closed interval \([-1/n, 1/n]\) of real numbers; that is,
\[
A_n = \left\{ x \in \mathbb{R} : -\frac{1}{n} \leq x \leq \frac{1}{n} \right\}.
\]
Determine:
\[
(i) \bigcup_{n \in \mathbb{N}} A_n \quad \text{and} \quad (ii) \bigcap_{n \in \mathbb{N}} A_n.
\]
See section 1 of the supplementary material for this notation about union and intersection.

Example 4.30. Discuss whether the following sets are bounded or not bounded:

(i) \( A = \{n \in \mathbb{N} : n^2 < 10\} \)

(ii) \( B = \{2, 4, 6, 8, \ldots\} \)

(iii) \( C = \{n/(n + m) : m, n \in \mathbb{N}\} \)

(iv) \( D = \{1 + (-1)^n(2n - 1) : n \in \mathbb{N}\} \)

Example 4.31. Which of the following sets are open?

\[
(i)(-3, 3) \quad (ii)(-4, 5) \quad (iii)(0, \infty) \quad (iv)\{(x, y) \in \mathbb{R}^2 : y > 0\}
\]

\[
(iv) \bigcup_{n=1}^{5} \left(-1 + \frac{1}{n}, 1 - \frac{1}{n}\right), \text{ where } n \in \mathbb{N} \quad (v)\{x \in \mathbb{R} : |x - 1| < 2\}.
\]


Example 4.33. Write out the first five terms of the following sequences:

i) \( \left\{ \frac{2^{n-1}}{2^n} \right\}_{n \in \mathbb{N}} \)

ii) \( \{a_n : a_1 = a_2 = 1, a_n = a_{n-1} + a_{n-2}, n \geq 3\}_{n \in \mathbb{N}} \)

iii) \( \left\{1 + \left(-\frac{1}{2}\right)^n \right\}_{n \in \mathbb{N}} \)

You may assume that each sequence is a function from \( \mathbb{N} \) to \( \mathbb{R} \).
**Example 4.34.** For each of the following, determine whether or not they converge. If they converge, what is their limit? No proofs are necessary, but provide some algebraic justification.

i) \( \left\{ 3 + \frac{(-1)^n 2}{n} \right\}_{n \in \mathbb{N}} \)

ii) \( \left\{ \frac{n^2 - 2n + 1}{n-1} \right\}_{n \in \mathbb{N}} \)

iii) \( \left\{ \frac{n}{n+1} \right\}_{n \in \mathbb{N}} \)

**Example 4.35.** Using the definition of convergence, that is, an \( \epsilon - N \) argument, prove that the following sequences converge to the indicated number:

i) \( \lim_{n \to \infty} \left( 3 + \frac{2}{n^2} \right) = 3 \)

ii) \( \lim_{n \to \infty} \frac{\sin(n)}{2n + 1} = 0 \)
5 Week 5

5.1 In Class Problems

Example 5.1. Which of the following are partitions of \( A = \{1, 2, 3, 4, 5\} \)? For each collection of subsets that is not a partition of \( A \), explain your answer.

(a) \( S_1 = \{\{1, 3\}, \{2, 5\}\} \).
(b) \( S_2 = \{\{1, 2\}, \{3, 4, 5\}\} \).
(c) \( S_3 = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}\} \).
(d) \( S_4 = A \).

Example 5.2. Give an example of a partition of \( \mathbb{Q} \) into three subsets.

Example 5.3. Decide if the following is a partition of \( \mathbb{R} \):

\[
\{[m, m+1) \}_{m \in \mathbb{Z}}.
\]

Justify your answer.

Example 5.4. Let \( A_1 \) and \( A_2 \) both be subsets of \( X \). Prove the following two statements.

(a) \( \left( \bigcup_{i \in \{1, 2\}} A_i \right)^\complement = \bigcap_{i \in \{1, 2\}} A_i^\complement \)
(b) \( \left( \bigcap_{i \in \{1, 2\}} A_i \right)^\complement = \bigcup_{i \in \{1, 2\}} A_i^\complement \)

Example 5.5. Prove the following:

(a) The real number \( \sqrt{2} \) is irrational.
(b) There are infinitely many prime numbers.
(c) There are no rational number solutions to the equation \( x^3 + x + 1 = 0 \).

(Hint: Develop a proof by contradiction)

Example 5.6. Which of the following sets are closed? Justify your answer.

\( (i) \ A = [2, 5] \) \quad \( (ii) \ B = (-1, 0) \cup (0, 1) \) \quad \( (iii) \ C = \{ x \in \mathbb{R} : |x - 1| < 2 \} \)

\( (iv) \ D = \{-2, -1, 0, 1, 2 \} \) \quad \( (v) \ Z \)

Example 5.7. Prove that the sequence \( \{(-1)^n\}_{n \in \mathbb{N}} \) does not converge.

Example 5.8. Use the formal definition of the limit of a sequence to prove that

\[
\lim_{n \to \infty} \frac{2n - 1}{3n + 2} = \frac{2}{3}.
\]
Example 5.9. Find an example of a convergent sequence \( \{ s_n \}_{n \in \mathbb{N}} \) of irrational numbers that has a rational number as a limit.

Example 5.10. Prove that the sequence \( \{ n - 4014 \}_{n \in \mathbb{N}} \) is not convergent.

Example 5.11. Prove that \( \{ \frac{3n^3 + 2}{n + 1} \}_{n \in \mathbb{N}} \) is not convergent.

Example 5.12. Use the formal definition of a limit to prove that \( \{ 2 + \frac{(-1)^n}{n} \}_{n \in \mathbb{N}} \) is a convergent sequence.

Example 5.13. Use the formal definition of a limit to prove that \( \lim_{n \to \infty} \frac{1}{n^{1/3}} = 0. \)

Example 5.14. Find the zero element for each of the following vector spaces defined over the scalar field \( \mathbb{R} \):

(a) The space of polynomials of degree three or less, \( \mathbb{P}_3 \).

(b) The space of \( 2 \times 2 \) matrices.

(c) The space \( \{ f : [0, 1] \to \mathbb{R} \mid f \text{ is continuous} \} \).

Assume that in each case, the natural definitions of addition and scalar multiplication hold. For example, in (a), if \( p_0(x) = a_0 + b_0x + c_0x^2 + d_0x^3 \) and \( p_1(x) = a_1 + b_1x + c_1x^2 + d_1x^3 \) are any two elements of the vector space, then \( (p_0 + p_1)(x) = (a_0 + a_1) + (b_0 + b_1)x + (c_0 + c_1)x^2 + (d_0 + d_1)x^3 \), and \( \lambda p_0(x) = \lambda a_0 + \lambda b_0x + \lambda c_0x^2 + \lambda d_0x^3 \), where \( a_i, b_i, x, \lambda \in \mathbb{R} \) and \( i = \{ 0, 1, 2, 3 \} \).

Example 5.15. Is the set of rational numbers a vector space over the scalar field \( \mathbb{R} \) under the usual addition and scalar multiplication operations? Justify your answer.

Example 5.16. Show that the set \( \mathbb{R}^2 \) over the scalar field \( \mathbb{R} \) is not a vector space under the following definitions for vector addition and scalar multiplication:

\[
\begin{align*}
    x + y &:= (x_1 - y_1, x_2 - y_2) \\
    \lambda x &:= (\lambda x_1, \lambda x_2),
\end{align*}
\]

where \( x = (x_1, x_2) \in \mathbb{R}^2, y = (y_1, y_2) \in \mathbb{R}^2, \) and \( \lambda \in \mathbb{R} \).

Example 5.17. Under the usual matrix operations, is the set

\[
\left\{ \begin{pmatrix} a & 1 \\ b & c \end{pmatrix} \mid a, b, c \in \mathbb{R} \right\}
\]

a vector space over the scalar field \( \mathbb{R} \)? Justify your answer.

(Use proposition 5.8 combined with definition 5.4 and proposition 5.5 to utilize the fact that you already know that the space of ALL 2x2 matrices is a vector space. This means that you only have to check that 2 properties are true instead of all 8 properties in the original definition of vector space. It should save you some time.)
Example 5.18. Define the function $A : \mathbb{P}^3 \to \mathbb{P}^5$ by

$$A(p)(x) = x^2 p(x) \quad \text{for } x \in \mathbb{R}.$$  

Is $A$ a linear function? Justify your answer.

Example 5.19. Suppose $b, c \in \mathbb{R}$. Define $T : \mathbb{R}^3 \to \mathbb{R}^2$ by

$$T(x, y, z) = (2x - 4y + 3z + b, 6x + cxyz).$$  

Show that $T$ is linear if and only if $b = c = 0$.  
(See remark 5.3 of the supplementary material for the notion of $\mathbb{R}^3$ as a vector space.)

Example 5.20. Use a proof by contradiction to prove that in a given vector space, the zero element is unique.

Example 5.21. Prove that $\sqrt{6}$ is an irrational number. Further, show that there are infinitely many positive integers $n$ such that $\sqrt{n}$ is irrational.

Example 5.22. Prove the following: If $n \in \mathbb{N}$, then $1 + 3 + 5 + 7 + \cdots + (2n - 1) = n^2$.

Example 5.23. Prove the following using mathematical induction:

For every integer $n \in \mathbb{N}$, it follows that $1^3 + 2^3 + 3^3 + 4^3 + \cdots + n^3 = \frac{n^2(n + 1)^2}{4}$.

Example 5.24. Assume that $f : \mathbb{R} \to \mathbb{R}$ is defined as $f(x) = x^n$, for $n \in \mathbb{N}$. Prove carefully that $f'(x) = nx^{n-1}$.

Example 5.25. If $n$ is a non-negative integer, use mathematical induction to show that $5 \mid (n^5 - n)$.

Example 5.26. Prove the following statement using (i) a proof by contrapositive, as well as (ii) a proof by contradiction.

Let $m \in \mathbb{Z}$. If $3 \nmid (m^2 - 1)$, then $3 \mid m$.

Example 5.27. Consider the set of polynomials of degree greater than or equal to two and smaller than or equal to 4, along with the zero polynomial. That is to say,

$$V = \{ p \in \mathbb{P}_4 : p(x) = a_2 x^2 + a_3 x^3 + a_4 x^4 \text{ for } a_2, a_3, a_4 \in \mathbb{R} \}.$$  

Is this a vector space over $\mathbb{R}$ under the usual polynomial addition and scalar multiplication operations? Justify your answer.  
(Please invoke HW10-Q0.2 which states that you have shown that $\mathbb{P}_4$ is a vector space. Use the notion of subspace, definition 5.4 and proposition 5.5 of the supplementary material.)
Example 5.28. Is the set $A = \{(x, y) \mid x, y \in \mathbb{R}\}$ a vector space over the scalar field $\mathbb{R}$ under the following operations?

1. $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$ and $r \cdot (x_1, y_1) = (rx_1, y_1)$.
2. $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$ and $r \cdot (x_1, y_1) = (rx_1, 0)$.

Here $(x_1, x_2), (y_1, y_2) \in A$ and $r \in \mathbb{R}$.

Example 5.29. Define the set $V = \{x \in \mathbb{R}^2 : x_2 \geq 2x_1 + 1\}$. Sketch a picture of the set $V$ inside of the plane, $\mathbb{R}^2$. Is $V$ a vector space over the scalar field $\mathbb{R}$? Justify your answer!

Example 5.30. Define $V = \{p \in P_2 : \forall x \in \mathbb{R}, p'(1) = 0\}$. Is $V$ a vector space over $\mathbb{R}$?

Example 5.31. For each of the following $L$, answer “yes” or “no”, and briefly justify your answer:

(i) Is $L : \mathbb{R} \to \mathbb{R}$, with $L(x) = \sin(x)$, a linear function?

(ii) Is $L : \mathbb{R} \to \mathbb{R}$, with $L(x) = x^{1/2}$, a linear function?

(iii) Is $L : \mathbb{R} \to \mathbb{R}$, with $L(x) = 51.5x$, a linear function?

Example 5.32. We know already that $\mathbb{R}^2$ and $P_4$ are both vector spaces over $\mathbb{R}$. Of the following two maps, decide which is linear and which is not. Justify your answer for both of them.

(i) $L : \mathbb{R}^2 \to P_4$, with the assignment for $x \in \mathbb{R}^2$, $L(x)(s) = x_1s^4 + x_2s^3 + x_2s^2 + 5$

(ii) $L : \mathbb{R}^2 \to P_4$, with the assignment for $x \in \mathbb{R}^2$, $L(x)(s) = x_1s^4 + x_2s^3 + x_2s + 5x_1$

(Here, recall that you can think an element of $P_4$ is $p$, where $p$ is defined as

$$p(s) = a_4s^4 + a_3s^3 + a_2s^2 + a_1s + a_0, \text{ where } a_0, a_1, a_2, a_3, a_4 \in \mathbb{R}.$$)

Example 5.33. Prove the following using mathematical induction:

(Fundamental Theorem of Arithmetic) Any integer $n > 1$ has a prime factorization. That is, $n = p_1 \cdot p_2 \cdot p_3 \cdots \cdot p_k$ for some primes $p_1, \ldots, p_k$.

Example 5.34. Suppose $n$ (infinitely long) straight lines lie on a plane in such a way that no two of the lines are parallel, and no three of the lines intersect at a single point. Show that this arrangement divides the plane into $\frac{n^2 + n + 2}{2}$ regions.

Example 5.35. If $n \in \mathbb{Z}$ and $n \geq 0$, then use mathematical induction to show that

$$\sum_{i=0}^{n} i \cdot i! = (n + 1)! - 1.$$

Example 5.36. Use the method of proof by contradiction to prove the following statement:

If $A$ and $B$ are sets, then $A \cap (B \setminus A) = \emptyset$. 
5.2 Recitation Problems

Example 5.37. Which if the following are partitions of \( A = \{a, b, c, d, e, f, g\} \)? For each collection of subsets that is not a partition of \( A \), explain your answer.

(a) \( S_1 = \{\{a, c, e, g\}, \{b, f\}, \{d\}\} \).
(b) \( S_2 = \{\{a, b, c, d\}, \{e, f\}\} \).
(c) \( S_3 = \{A\} \).
(d) \( S_4 = \{\{a\}, \emptyset, \{b, c, d\}, \{e, f, g\}\} \).
(e) \( S_5 = \{\{a, c, d\}, \{b, g\}, \{e\}, \{b, f\}\} \).

Example 5.38. Give an example of a partition of \( \mathbb{N} \) into three subsets.

Example 5.39. Which of the following sets are closed? Justify your answer.

(i) \( A = \{x \in \mathbb{R} : |x - 1| > 3\} \)
(ii) \( B = [-1, 0) \cup (0, 1) \)
(iii) \( C = \{x \in \mathbb{R} : x^2 > 0\} \)
(iv) \( D = \{0, 1, 2, \ldots, 100\} \)
(v) \( E = \bigcap_{k=1}^{10} \left[0, \frac{1}{k}\right] \)

Example 5.40. Let \( x \) be a positive real number. Prove that if \( x - \frac{2}{x} > 1 \), then \( x > 2 \) by
(a) a direct proof,  (b) a proof by contrapositive and  (c) a proof by contradiction.

Example 5.41. Suppose \( a, b \in \mathbb{R} \). If \( a \) is rational and \( ab \) is irrational, then \( b \) is irrational.

Example 5.42. Suppose \( A \) and \( B \) are open sets. Prove that \( A \cap B \) is an open set.

Example 5.43. Write down the negation of what it means for a sequence to converge, and then prove that \( \{\sin \left(\frac{\pi n}{4}\right)\}_{n \in \mathbb{N}} \) does not converge.

Example 5.44. Using the formal definition of the limit of a sequence, prove \( \lim_{n \to \infty} \frac{5}{n^2} = 0 \).

Example 5.45. Use the formal definition of a limit of a sequence to prove that
\[
\lim_{n \to \infty} \frac{2n - 3}{1 - 5n} = -\frac{2}{5}.
\]

Example 5.46. Find the additive inverse, in the vector space, of the following:

1. In \( \mathbb{P}_3 \), of the element \( -3 - 2x + x^2 \).
2. In the space of \( 2 \times 2 \) matrices, of the element \( \begin{pmatrix} 1 & -1 \\ 0 & 3 \end{pmatrix} \).
3. In \( \{ae^x + be^{-x} \mid a, b \in \mathbb{R}\} \), the space of functions of the real variable \( x \), the element \( 3e^x - 2e^{-x} \).

You may assume that these vector spaces are defined over the scalar field \( \mathbb{R} \) and that in each case, natural definitions of addition and scalar multiplication hold.
Example 5.47. Show that the set of linear polynomials \( P_1 = \{ a_0 + a_1 x \mid a_0, a_1 \in \mathbb{R} \} \) under the usual polynomial addition and scalar multiplication operations is a vector space under the scalar field \( \mathbb{R} \).

Example 5.48. Define the function \( T : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) by the transformation
\[
T((x_0, x_1)) = (x_0, 0),
\]
where \((x_0, x_1) \in \mathbb{R}^2 \). Is \( T \) a linear function? Justify your answer.

Example 5.49. Prove by induction that
\[
\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.
\]

Example 5.50. Prove the following: If \( n \in \mathbb{N} \), then
\[
\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n-1)!}.
\]

Example 5.51. Use the method of proof by contradiction to prove the following statement:
(You should also think about how a direct or contrapositive proof would work. You will find that a proof by contradiction is easier.)

Suppose \( a, b, c \in \mathbb{Z} \). If \( a^2 + b^2 = c^2 \), then \( a \) or \( b \) is even.

Example 5.52. Define the set \( V = \{ p \in \mathbb{P}_3 : \int_{-1}^{1} p(s)ds = 0 \} \). Sketch the graph of at least 3 different elements of \( V \). Is \( V \) a vector space over the scalar field \( \mathbb{R} \)? Justify your answer!

Example 5.53. For each of the following \( L \), answer “yes” or “no”, and briefly justify your answer:

(i) Is \( L : \mathbb{R} \rightarrow \mathbb{R} \), with \( L(x) = \cos(x) \), a linear function?

(ii) Is \( L : \mathbb{R} \rightarrow \mathbb{R} \), with \( L(x) = \frac{-1}{10} x \), a linear function?

(iii) Is \( L : \mathbb{R} \rightarrow \mathbb{R} \), with \( L(x) = 100x - 7.99 \), a linear function?

Example 5.54. Define the set \( S = \{ x \in \mathbb{R}^2 : x_1 + x_2 = 0, \text{ and } x_1 - 2x_2 = 0 \} \).

(i) Write down a matrix, \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \), with \( a, b, c, d \in \mathbb{R} \) such that
\[
S = \{ x \in \mathbb{R}^2 : Ax = 0 \}.
\]

(ii) Justify that the set \( S \) is actually a vector space over \( \mathbb{R} \). Use the vector addition and scalar multiplication that is naturally inherited from that of the ambient space, \( \mathbb{R}^2 \). You may either check the properties directly or you may invoke a result from a previous HW/example question. It is your choice.

Example 5.55. Use mathematical induction to prove the following proposition concerning the Fibonacci sequence:
\[
\sum_{k=1}^{n} F_k^2 = F_n F_{n+1}.
\]
(Here \( F_n \) denotes the \( n \)th term of the Fibonacci sequence. Recall that the Fibonacci Sequence is entirely determined by the rules \( F_1 = 1, F_2 = 1, \text{ and } F_n = F_{n-1} + F_{n-2} \).)
Example 5.56. Use mathematical induction to prove the following:

For any integer $n \geq 0$, it follows that $3 \mid (n^3 + 5n + 6)$.

Example 5.57. Use the method of proof by contradiction to prove the following:

$$\forall x \in [\pi/2, \pi], \quad \sin x - \cos x \geq 1.$$
6 Week 6

6.1 In Class Problems

Example 6.1. Compute $|E|$, where

$$E = \{n \in \mathbb{N} : n \leq 3076 \text{ and } n \text{ is divisible by 19} \}.$$  

Example 6.2. Illustrate the division algorithm for:

(i) $a = 17, b = 125$

(ii) $a = -17, b = 125$

(iii) $a = 8, b = 96$

Example 6.3. Use the Euclidean Algorithm to find the greatest common divisor for each of the following pairs of integers:

- $(a)$ 51 and 288
- $(b)$ 357 and 629
- $(c)$ 180 and 252

Example 6.4. Prove that the square of every odd integer is of the form $4k + 1$, where $k \in \mathbb{Z}$ (that is, for each odd integer $a \in \mathbb{Z}$, there exists $k \in \mathbb{Z}$ such that $a^2 = 4k + 1$).

Example 6.5. Prove that if $a$ divides $b$ and $c$ divides $d$, then $ac$ divides $bd$.

Example 6.6. Prove that if $n \in \mathbb{N}$, then $4^{2n} + 10n - 1$ is divisible by 25.

Example 6.7. Answer true or false and give a complete justification. If $p$ is prime, then $p^2 + 1$ is prime.

Example 6.8. Prove that if $x \equiv 1 \mod 4$ and $y \equiv 1 \mod 4$, then $xy \equiv 1 \mod 4$.

Example 6.9. Prove that if $x \not\equiv 0 \mod 5$ and $y \not\equiv 0 \mod 5$, then $xy \not\equiv 0 \mod 5$.

Example 6.10. Prove the following: Let $a, b, c, m, n \in \mathbb{Z}$, where $m, n \geq 2$. If $a \equiv b \pmod{m}$ and $a \equiv c \pmod{n}$, then $b \equiv c \pmod{d}$, where $d = \gcd(m, n)$.

Example 6.11. Let $a, b \in \mathbb{Z}$, where not both $a$ and $b$ are 0. Prove that if $d = \gcd(a, b)$, $a = a_1d$ and $b = b_1d$, then $\gcd(a_1, b_1) = 1$.

Example 6.12. Prove or disprove and provide a complete justification:

$$\gcd(ac, bc) = c \gcd(a, b).$$

Example 6.13. Answer true or false and give a complete justification. If $p$ is prime and $p$ divides $ab$, then $p$ divides $a$ or $p$ divides $b$. 

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6.2 Recitation Problems

Example 6.14. Compute $|E|$, where

$$E = \{n \in \mathbb{N} : n \leq 3076 \text{ and } n \text{ is not divisible by } 17\}.$$

Example 6.15. Illustrate the division algorithm for:

(i) $a = 22, b = -17$

(ii) $a = 15, b = 0$

Example 6.16. Determine integers $x$ and $y$ such that

(a) $\gcd(51, 288) = 51x + 288y$

(b) $\gcd(357, 629) = 357x + 629y$

(c) $\gcd(180, 252) = 180x + 252y$

Example 6.17. Prove that if $p$ is prime and $p > 2$, then $p$ is odd.

Example 6.18. Answer true or false, and completely justify. If $a$ divides $bc$, then $a$ divides $b$ and $a$ divides $c$.

Example 6.19. Prove that if $n \in \mathbb{N}$, then $n^5 - n$ is divisible by 30.

Example 6.20. Prove that if $p$ is prime and $p > 2$, then $p \equiv 1 \mod 4$ or $p \equiv 3 \mod 4$.

Example 6.21. Answer true or false and give a complete justification.

$$\gcd(a, b) = \gcd(a, a - 2b).$$
7 Week 7

7.1 In Class Problems

Example 7.1. A relation $\sim$ is defined on $\mathbb{N}$ by $a \sim b$ if $a^2 + b^2$ is even. Prove that $\sim$ is an equivalence relation. Determine the distinct equivalence classes.

Example 7.2. The relation $\sim$ on $\mathbb{Z}$ defined by $a \sim b$ if $a^2 \equiv b^2 \mod 4$ is known to be an equivalence relation. Determine the distinct equivalence classes.

Example 7.3. Define a relation $\sim$ on $\mathbb{R}$ by $x \sim y$ means $|x| + |y| = |x + y|$.

Is $\sim$ an equivalence relation? Justify your answer.

Example 7.4. Let $\sim$ be an equivalence relation on $A = \{a, b, c, d, e, f, g\}$ such that $a \sim c$, $c \sim d$, $d \sim g$ and $b \sim f$. If there are three distinct equivalence classes resulting from $\sim$, then determine these equivalence classes and determine all elements of the equivalence relation.

Example 7.5. Prove that $5 \mid (3^{3n+1} + 2^{n+1})$ for every positive integer $n$.

Example 7.6. For $(a, b), (c, d) \in \mathbb{R}^2$ define $(a, b) \sim (c, d)$ to mean that $2a - b = 2c - d$. Show that $\sim$ is an equivalence relation on $\mathbb{R}^2$.

Example 7.7. Define a relation $\sim$ on $\mathbb{Z}$ as $x \sim y$ if and only if $4 \mid (x + 3y)$. Prove $\sim$ is an equivalence relation. Describe its equivalence classes.

Example 7.8. Let $X = \mathbb{R}^2$, the $x$-$y$ plane. Define $(x_1, y_1) \sim (x_2, y_2)$ to mean

$x_1^2 + y_1^2 = x_2^2 + y_2^2$.

Is $\sim$ an equivalence relation? Justify your answer. Describe the equivalence classes of $\sim$.

Example 7.9. Do the following calculations in $\mathbb{Z}_9$ (see page 238 of the text for a description of this notation), in each case expressing your answer as $[a]$ with $0 \leq a \leq 8$.


Example 7.10. Prove that the following statement is true: If $a \equiv b \mod m$, then $a^n \equiv b^n \mod m$ for every positive integer $n$. Make two different proofs.

(i) Use induction.

(ii) Make a direct proof using the binomial theorem for the expansion of powers of a binomial, $(a + b)^n$, where $a$ and $b$ are real numbers. You can either ask the instructor to write the theorem on the board or you can look up the binomial theorem on wikipedia.

Example 7.11. Let $a, b, c, d \in \mathbb{Z}$ with $a, c \neq 0$. Prove that if $a \mid b$ and $c \mid d$, then $ac \mid (ad + bc)$.
Example 7.12. Let \( \mathbb{P} \) be the set of all polynomials with real coefficients. Define a relation \( \sim \) on \( \mathbb{P} \) as follows. Given \( f, g \in \mathbb{P} \), let \( f \sim g \) mean that \( f \) and \( g \) have the same degree. Thus \( (x^2 + 3x - 4) \sim (3x^2 - 2) \) and \( (x^3 + 3x^2 - 4) \not\sim (3x^2 - 2) \), for example, where \( x \in \mathbb{R} \). Is \( \sim \) an equivalence relation? Justify your answer. Now write down the equivalence class \([3x^2 + 2]\).

Example 7.13. Let \( \sim \) be an equivalence relation on a nonempty set \( A \) and let \( a \) and \( b \) be elements of \( A \). Then, prove that
\[
[a] = [b] \text{ if and only if } a \sim b.
\]
Use the above result to show the following:
Let \( \sim \) be an equivalence relation defined on a nonempty set \( A \). Show that the set
\[
\{[a] : a \in A\}
\]
of equivalence classes resulting from \( \sim \) is a partition of \( A \).

Example 7.14. Prove that if \( a_1, a_2, \ldots, a_n \) are \( n \geq 2 \) integers such that \( a_i \equiv 1 \mod 3 \) for every integer \( i \) \((1 \leq i \leq n)\), then \( a_1a_2 \ldots a_n \equiv 1 \mod 3 \).

7.2 Recitation Problems

Example 7.15. Prove the following: Let \( d \) be a nonzero integer. If \( a_1, a_2, \ldots, a_n \) and \( x_1, x_2, \ldots, x_n \) are collections of integers, for some \( n \geq 2 \), such that \( d \mid a_i \) for all \( i \) \((1 \leq i \leq n)\), then \( d \mid \sum_{i=1}^{n} a_ix_i \).

Example 7.16. Define a relation \( \sim \) on \( \mathbb{Z} \times \mathbb{N} \) by \( (j,k) \sim (m,n) \) if and only if \( jn = km \). Show that \( \sim \) is an equivalence relation. Describe its equivalence classes.

Example 7.17. Define a relation \( \sim \) on \( \mathbb{Z} \times \mathbb{N} \) by \( (j,k) \sim (m,n) \) if and only if \( jn = km \). Show that \( \sim \) is an equivalence relation.

Example 7.18. Suppose \([a], [b] \in \mathbb{Z}_5\) and \([a] \cdot [b] = [0]\). Is it necessarily true that either \([a] = [0]\) or \([b] = [0]\)? Justify your answer.

Example 7.19. Suppose that \( I \) is a fixed interval of \( \mathbb{R} \), and that \( S \) is the set of differentiable functions from \( I \) into \( \mathbb{R} \). Consider the equivalence relation associated with the derivative operator \( D \) on \( S \), so that \( D(f) = f' \). For \( f \in S \), give a simple description of \([f]\).

Example 7.20. Let \( \sim \) be the relation defined on \( \mathbb{Z} \) by \( a \sim b \) if \( a^2 \equiv b^2 \mod 5 \). Prove that \( \sim \) is an equivalence relation and determine the distinct equivalence classes.