EVOLUTION EQUATIONS AND
CONTROL THEORY
Volume **2**, Number **4**, December **2013**

doi:10.3934/eect.2013.2.599

pp. **599–620**

# ON LANDAU-LIFSHITZ EQUATIONS OF NO-EXCHANGE ENERGY MODELS IN FERROMAGNETICS

Wei Deng and Baisheng Yan

Department of Mathematics
Michigan State University
East Lansing, MI 48824, USA

*Dedicated to Walter Littman with admiration and gratitude*

Abstract. In this paper, we study Landau-Lifshitz equations of ferromagnetism with a total energy that does not include a so-called exchange energy. Many problems, including existence, stability, regularity and asymptotic behaviors, have been extensively studied for such equations of models with the exchange energy. The problems turn out quite different and challenging for Landau-Lifshitz equations of no-exchange energy models because the usual methods based on certain compactness do not apply. We present a new method for the existence of global weak solution to the Landau-Lifshitz equation of no-exchange energy models based on the existence of regular solutions for smooth data and certain stability of the solutions. We also study higher time regularity, energy identity and asymptotic behaviors in some special cases for weak solutions.

## 1. Introduction and main results.

**1.1. Landau-Lifshitz theory.** The well-known Landau-Lifshitz theory of ferromagnetism models the state of magnetization vector $\mathbf{m}$ of a ferromagnetic material based on formulation of a total energy consisting of several competing energy contributions. The theory for rigid ferromagnetic bodies also assumes that, below certain critical temperature, the magnetization vector $\mathbf{m}$ has constant magnitude: $|\mathbf{m}(x)| = M_s$, where $M_s > 0$ is the *saturation magnetization.* Throughout this paper, we will assume $M_s = 1$; therefore, magnetization vector $\mathbf{m}$ is a unit director field. We refer to [4, 21, 22, 23] for more backgrounds on this theory and related mathematical developments.

Under this theory, equilibrium states (including reduction theory for thin-film limits) are studied usually through the minimization of total energy, while dynamic properties are modeled and analyzed by the associated Landau-Lifshitz equations or Landau-Lifshitz-Gilbert equations derived from the given total energy.

Both equilibrium and dynamic problems have been well studied for models of total energy including the so-called *exchange energy* of density roughly proportional to $|\nabla \mathbf{m}|^2$; see, e.g., [1, 2, 3, 5, 6, 7, 8, 12, 13, 15, 24, 25, 30]. Similar dynamic problems for the models coupled with Maxwell equations of electromagnetism have been also studied in [17, 18, 19, 20, 30]. Equilibrium problems for energies excluding

2010 *Mathematics Subject Classification.* Primary: 35M31, 35Q60; Secondary: 78A25, 82D40.
*Key words and phrases.* Landau-Lifshitz equation, no-exchange energy model, existence, stability, regularity, energy identity, asymptotics.

the exchange energy (the “no-exchange energy” models) have been studied in, e.g., [9, 11, 16, 26, 27, 31, 32]; however, few work has been done on dynamic problems for no-exchange energy models except for partial results in [10, 17, 18, 33].

1.2. **Landau-Lifshitz equations of no-exchange energy models.** In this paper, we study the Landau-Lifshitz equation of no-exchange energy models; namely, we assume the total energy is given by

$$\mathcal{E}(\mathbf{m}) = \int_\Omega \varphi(\mathbf{m})\,dx - \int_\Omega \mathbf{a}(x)\cdot\mathbf{m}\,dx + \frac{1}{2}\int_{\mathbf{R}^3} |H_{\mathbf{m}}|^2\,dx. \tag{1.1}$$

Here $\Omega$ is a bounded domain in $\mathbf{R}^3$ occupied by the material, functions $\varphi$ and $\mathbf{a}$ are given physical quantities representing, respectively, material's crystallographic anisotropy and the external applied magnetic field, and the (stray) field $H_{\mathbf{m}}$ is induced by $\mathbf{m}$ through (simplified) Maxwell equations:

$$\operatorname{curl} H_{\mathbf{m}} = 0, \quad \operatorname{div}(H_{\mathbf{m}} + \mathbf{m}\chi_\Omega) = 0 \quad \text{in } \mathbf{R}^3, \tag{1.2}$$

where $\chi_\Omega$ is the characteristic function of domain $\Omega$. From the Maxwell equation, one easily has $\int_{\mathbf{R}^3} |H_{\mathbf{m}}|^2\,dx = -\int_\Omega \mathbf{m}\cdot H_{\mathbf{m}}\,dx$ and hence one can also write the energy $\mathcal{E}(\mathbf{m})$ as

$$\mathcal{E}(\mathbf{m}) = \int_\Omega \varphi(\mathbf{m})\,dx - \int_\Omega \mathbf{a}(x)\cdot\mathbf{m}\,dx - \frac{1}{2}\int_\Omega \mathbf{m}\cdot H_{\mathbf{m}}\,dx.$$

Under the energy formulation of $\mathcal{E}(\mathbf{m})$, the associated dynamic Landau-Lifshitz equation governing the evolution of magnetization $\mathbf{m} = \mathbf{m}(x,t)$ is given by

$$\partial_t\mathbf{m} = \gamma\mathbf{m}\times\mathbf{H}_{\text{eff}} + \gamma\alpha\mathbf{m}\times(\mathbf{m}\times\mathbf{H}_{\text{eff}}) \quad \text{on } \Omega\times[0,\infty), \tag{1.3}$$

where $\gamma < 0$ is material-dependent *electron gyromagnetic ratio*, $\alpha \geq 0$ is Landau-Lifshitz phenomenological *damping parameter*, and $\mathbf{H}_{\text{eff}}$ is the total *effective magnetic field* that is given by the negative $L^2$-derivative of $\mathcal{E}$ with respect to $\mathbf{m}$ as follows:

$$\mathbf{H}_{\text{eff}} = -\frac{\partial\mathcal{E}}{\partial\mathbf{m}} = -\varphi'(\mathbf{m}) + \mathbf{a}(x) + H_{\mathbf{m}}. \tag{1.4}$$

Here and throughout the paper, we assume $\varphi(\mathbf{m})$ is a smooth function on $\mathbf{R}^3$ and $\mathbf{a}\in L^\infty(\Omega;\mathbf{R}^3)$.

The Landau-Lifshitz equation (1.3) can also be written as a *Landau-Lifshitz-Gilbert equation*:

$$\partial_t\mathbf{m} = \gamma(1+\alpha^2)\mathbf{m}\times\mathbf{H}_{\text{eff}} + \alpha\mathbf{m}\times\partial_t\mathbf{m}; \tag{1.5}$$

see [14] for further discussions. Equation (1.3) or (1.5) will be supplemented with an initial value condition:

$$\mathbf{m}(x,0) = \mathbf{m}_0(x), \quad x\in\Omega, \tag{1.6}$$

where $\mathbf{m}_0 \in L^\infty(\Omega;\mathbf{R}^3)$ is a given field.

**Definition 1.1.** By a (global) *weak solution* to Eq. (1.3) with initial condition (1.6), we mean a function $\mathbf{m}\in W^{1,\infty}_{loc}([0,\infty);L^2(\Omega;\mathbf{R}^3))\cap L^\infty((0,\infty);L^\infty(\Omega;\mathbf{R}^3))$ satisfying $\mathbf{m}(0)=\mathbf{m}_0$ in $L^2(\Omega)$ such that Eq. (1.3) holds both in $L^\infty((0,T);L^2(\Omega))$ and in the sense of distribution on $\Omega\times(0,T)$ for all $0<T<\infty$.

**Remark 1.** (a) Any weak solution $\mathbf{m}$ will satisfy

$$\partial_t(|\mathbf{m}|^2) = 2\mathbf{m}\cdot\partial_t\mathbf{m} = 0 \quad \text{in } \Omega\times(0,\infty).$$

Therefore, if initial datum $\mathbf{m}_0$ satisfies the saturation condition $|\mathbf{m}_0(x)| = 1$ a.e. on $\Omega$, then solution $\mathbf{m}$ will also satisfy the saturation condition $|\mathbf{m}(x,t)| = 1$ a.e. $x \in \Omega$ for all $t \in [0,\infty)$.

(b) The regularity condition on weak solution $\mathbf{m}$ automatically requires that $\mathbf{m} \in C([0,T]; L^2(\Omega;\mathbf{R}^3))$ for all $T > 0$.

1.3. **Quasi-stationary limits.** Initial value problem (1.3) with (1.6) can be written as a *quasi-stationary* system:

$$\begin{cases} \partial_t \mathbf{m} = F_{\mathbf{a}}(x,\mathbf{m},H_{\mathbf{m}}) \quad \text{in } \Omega\times(0,\infty), \\ \operatorname{curl} H_{\mathbf{m}} = 0,\ \operatorname{div}(H_{\mathbf{m}} + \mathbf{m}\chi_\Omega) = 0 \quad \text{in } \mathbf{R}^3 \text{ for all } t\in[0,\infty), \\ \mathbf{m}(x,0) = \mathbf{m}_0(x) \quad \text{on } \Omega, \end{cases} \tag{1.7}$$

where $F_{\mathbf{a}}(x,\mathbf{m},H)$, specifying the dependence on applied field $\mathbf{a}$, is the *Landau-Lifshitz interaction* function given by

$$F_{\mathbf{a}}(x,\mathbf{m},H) = \mathbb{L}(\mathbf{m}, -\varphi'(\mathbf{m}) + \mathbf{a}(x) + H),$$

with $\mathbb{L}(\mathbf{m},\mathbf{n})$ linear in $\mathbf{n}$ and defined by

$$\mathbb{L}(\mathbf{m},\mathbf{n}) = \gamma\mathbf{m}\times\mathbf{n} + \gamma\alpha\mathbf{m}\times(\mathbf{m}\times\mathbf{n}), \quad \mathbf{m},\mathbf{n}\in\mathbf{R}^3. \tag{1.8}$$

Existence of global weak solution to system (1.7) has been established in [10, 17] using the *quasi-stationary limit* of certain Landau-Lifshitz-Maxwell systems as electric permittivity tends to zero. The method in [10] uses a simple Landau-Lifshitz-Maxwell system given by

$$\begin{cases} \epsilon\partial_t E - \operatorname{curl} H = 0, \\ \partial_t(H + M\chi_\Omega) + \operatorname{curl} E = 0 \quad \text{in } \mathbf{R}^3\times(0,\infty), \\ \partial_t M = F_{\mathbf{a}}(x,M,H) \quad \text{in } \Omega\times(0,\infty), \\ (E,H)|_{t=0} = (E_0,H_0) \ \text{ on } \mathbf{R}^3, \quad M|_{t=0} = \mathbf{m}_0 \ \text{ on } \Omega, \end{cases} \tag{1.9}$$

where $\epsilon > 0$, and the initial data $E_0, H_0$ for electric and magnetic fields $E, H$ are any vector-fields satisfying

$$E_0, H_0 \in L^2(\mathbf{R}^3;\mathbf{R}^3), \quad \operatorname{div} E_0 = \operatorname{div}(H_0 + \mathbf{m}_0\chi_\Omega) = 0. \tag{1.10}$$

System (1.9) with $\epsilon = 1$ has been studied by Joly, Metivier and Rauch [19], where existence of global weak solutions was established. Similarly, one can show that, for any $\epsilon > 0$, system (1.9) has a global weak solution $(E^\epsilon, H^\epsilon, M^\epsilon)$. In Deng and Yan [10], we have showed that, as $\epsilon\to 0$, $M^\epsilon \to \mathbf{m}$ strongly in both $C^0([0,T];L^2(\Omega;\mathbf{R}^3))$ and $L^2(\Omega\times(0,T);\mathbf{R}^3)$ for all $0 < T < \infty$ and that the limit $\mathbf{m}$ is a global weak solution to problem (1.7).

1.4. **Main results.** In this paper, we present a different method for the existence of global weak solution to (1.7) with *any* initial data $\mathbf{m}_0 \in L^\infty(\Omega;\mathbf{R}^3)$; we do not assume the saturation condition here. Our method is based on the existence of solutions to (1.7) for smooth $\mathbf{a}$ and $\mathbf{m}_0$ and a certain stability for solutions. We also study the higher time regularity and the asymptotic behaviors of solutions in some special cases.

We organize our plans of the paper and summarize the main results as follows.

1.4.1. *Finite time local $L^2$-stability.* Our main stability result is stated as follows and will be proved in Section 2 (see Theorem 2.2).

**Theorem 1.2.** *Let $0 < R, T < \infty$ be given. Then there exist constants $C = C(R,T) > 0$, $c = c(R,T) > 0$ and $\rho = \rho(R,T) > 0$ such that, for any weak solution $\mathbf{m}^k$ to the system (1.7) with applied field $\mathbf{a}^k$ and initial datum $\mathbf{m}^k(0) = \mathbf{m}_0^k$ satisfying $\|\mathbf{a}^k\|_{L^\infty} + \|\mathbf{m}_0^k\|_{L^\infty} \le R$ for $k = 1, 2$, if $\mu = \max\{\|\mathbf{m}_0^1 - \mathbf{m}_0^2\|_{L^2}, \|\mathbf{a}^1 - \mathbf{a}^2\|_{L^2}\} \le c$, then one has, for all $t \in [0,T]$,*

$$\|\mathbf{m}^1(t) - \mathbf{m}^2(t)\|_{L^2(\Omega)} \le C\mu^\rho. \tag{1.11}$$

This stability result also implies the uniqueness of weak solution to system (1.7).

1.4.2. *Existence of global weak solutions.* Based on the previous stability theorem, in Section 3, we present a new method for the existence of global solution to (1.7) with general applied fields $\mathbf{a}$ and initial data $\mathbf{m}_0$.

First, we show the existence of global solution to (1.7) for smooth fields $\mathbf{a}$ and initial data $\mathbf{m}_0 \in H^2(\Omega;\mathbf{R}^3)$. Define $\mathbf{f}(\mathbf{m}) = F_{\mathbf{a}}(x, \mathbf{m}, H_{\mathbf{m}})$. We show $\mathbf{f}\colon H^2(\Omega;\mathbf{R}^3) \to H^2(\Omega;\mathbf{R}^3)$ and is *locally Lipschitz*; the proof uses a critical estimate that $H_{\mathbf{m}} \in H^2(\Omega;\mathbf{R}^3)$ for all $\mathbf{m} \in H^2(\Omega;\mathbf{R}^3)$ (see, e.g., [8, 19]). By the abstract ODE theory in Banach spaces, problem (1.7) has a local solution if $\mathbf{m}_0 \in H^2(\Omega;\mathbf{R}^3)$. Then a no-blowup result (Theorem 3.4) shows that the local solution is in fact global on $t \in [0,\infty)$. The proof of the no-blowup result, Theorem 3.4, is given in Section 4.

We remark that in the special case when $\varphi = 0$ and $\mathbf{a} = 0$ (thus $\mathbf{H}_{\text{eff}} = H_{\mathbf{m}}$), for smooth initial data $\mathbf{m}_0 \in H^2(\Omega)$ with $\frac{\partial \mathbf{m}_0}{\partial \nu}|_{\partial\Omega} = 0$, Carbou and Fabrie [8] also established the global existence through a singular perturbation method, by including $\kappa\Delta\mathbf{m}$ in $\mathbf{H}_{\text{eff}}$ and letting $\kappa \to 0$.

Once we have obtained the global existence for smooth data $\mathbf{a}$ and $\mathbf{m}_0$, we use approximation and the stability result Theorem 1.2 to establish the existence for general data.

1.4.3. *Higher time regularity.* In Section 5, we study the higher time regularity for the simple Landau-Lifshitz equation

$$\mathbf{m}_t = \gamma\mathbf{m} \times H_{\mathbf{m}} + \alpha\gamma\mathbf{m} \times (\mathbf{m} \times H_{\mathbf{m}}) \quad \text{in } \Omega \times (0,\infty), \tag{1.12}$$

where $H_{\mathbf{m}}$ is given as above.

**Theorem 1.3.** *For any $T > 0$ and initial datum $\mathbf{m}_0 \in H^2(\Omega)$, the regular solution $\mathbf{m}$ to (1.12) satisfies, for all $p = 0, 1, 2, \cdots$*

$$\sup_{t\in[0,T]} \|\partial_t^{p+1}\mathbf{m}\|_{H^2(\Omega)} \le C < \infty,$$

*where $C$ is a constant only depending on $T, p, \|\mathbf{m}_0\|_{H^2(\Omega)}$.*

By similar methods, this result is also valid for the general equation (1.3) with smooth applied field $\mathbf{a}$ and anisotropy energy density $\varphi$.

1.4.4. *Energy identity and weak $\omega$-limit sets.* In Section 6, we first prove an energy identity for the global weak solutions to the Landau-Lifshitz equation (1.3).

**Theorem 1.4.** *The global weak solution $\mathbf{m}$ to (1.7) with bounded initial data satisfies the energy identity*

$$\mathcal{E}(\mathbf{m}(t)) - \mathcal{E}(\mathbf{m}(s)) = \gamma\alpha \int_s^t \int_\Omega |\mathbf{m} \times \mathbf{H}_{\text{eff}}|^2 \, dx d\tau \quad \forall\, 0 \le s \le t < \infty. \tag{1.13}$$

*Furthermore, if $\gamma\alpha < 0$, then $\mathbf{m}_t \in L^2((0,\infty); L^2(\Omega;\mathbf{R}^3))$.*

Therefore, the global-in-time regularity for weak solutions (even for regular solutions) is that

$$\mathbf{m} \in L^\infty((0,\infty); L^\infty(\Omega; \mathbf{R}^3)) \quad \text{with} \quad \mathbf{m}_t \in L^2((0,\infty); L^2(\Omega; \mathbf{R}^3)).$$

But this regularity is not enough to have strong convergence as $t \to \infty$; it would be enough if one has $\mathbf{m}_t \in L^1((0,\infty); L^2(\Omega; \mathbf{R}^3))$ (see [20]). Therefore, it is quite challenging to study the asymptotic behaviors for even the regular solutions. The solution orbits for general initial data may not have strong $\omega$-limit points; we thus study the *weak $\omega$-limit set*:

$$\omega^*(\mathbf{m}_0) = \{\tilde{\mathbf{m}} \mid \exists\, t_j \uparrow \infty \text{ such that } \mathbf{m}(t_j) \rightharpoonup \tilde{\mathbf{m}} \text{ weakly in } L^2(\Omega; \mathbf{R}^3)\}. \tag{1.14}$$

We then prove the following estimate of $\omega^*(\mathbf{m}_0)$ for the so-called *soft-case*, where there is no anisotropy energy ($\varphi = 0$).

**Theorem 1.5.** *Let $\gamma\alpha < 0$, $\varphi = 0$ and $\mathbf{a} \in L^\infty(\Omega; \mathbf{R}^3)$. Then, for any $\mathbf{m}_0 \in L^\infty(\Omega; \mathbf{R}^3)$ with $|\mathbf{m}_0(x)| = 1$ a.e. on $\Omega$, it follows that*

$$\omega^*(\mathbf{m}_0) \subseteq \{\tilde{\mathbf{m}} \in L^\infty(\Omega; \mathbf{R}^3) \mid |\tilde{\mathbf{m}}|^2 + 2|\tilde{\mathbf{m}} \times (\mathbf{a} + H_{\tilde{\mathbf{m}}})| \le 1 \text{ a.e. on } \Omega\}. \tag{1.15}$$

For more results on a further special case when $\mathbf{a} = 0$, see [32, 33].

1.4.5. *A special dynamics on $\mathbf{R}^3$.* Finally, in Section 7, we study a special case of (1.7) when applied field $\mathbf{a}(x) = \mathbf{a}$ is constant, domain $\Omega$ is an ellipsoid, and initial datum $\mathbf{m}_0$ is a constant unit vector. In this case, it is well-known that the magnetostatic stray field $H_{\mathbf{m}}$ induced by any *constant* field $\mathbf{m}$ has constant value on ellipsoid domain $\Omega$ (see, e.g., [27]). Hence, problem (1.7) reduces to an ODE system on $\mathbf{R}^3$:

$$\begin{cases} \dot{\mathbf{m}} = \Phi(\mathbf{m}), & t > 0, \\ \mathbf{m}(0) = \mathbf{m}_0, \end{cases} \tag{1.16}$$

for some smooth function $\Phi\colon \mathbf{R}^3 \to \mathbf{R}^3$; see (7.4) below. The dynamics of system (1.16) will be studied by the classical ODE theory using an explicit Lyapunov function.

## 2. Finite-time local $L^2$-Stability.

2.1. **Helmholtz decompositions.** In order to study the field $H_{\mathbf{m}}$, we review the standard orthogonal (Helmholtz) decomposition:

$$L^2(\mathbf{R}^3; \mathbf{R}^3) = L^2_\parallel(\mathbf{R}^3; \mathbf{R}^3) \oplus L^2_\perp(\mathbf{R}^3; \mathbf{R}^3),$$

where $L^2_\parallel(\mathbf{R}^3; \mathbf{R}^3), L^2_\perp(\mathbf{R}^3; \mathbf{R}^3)$ are the subspaces of *curl-free* or *divergence-free* functions in the sense of distributions, respectively. This decomposition can be explicitly given in terms of the Fourier transform $\hat{\mathbf{m}}$ of vector-field $\mathbf{m} \in L^2(\mathbf{R}^3; \mathbf{R}^3)$: $\mathbf{m} = \mathbf{m}_\parallel + \mathbf{m}_\perp$, where

$$\hat{\mathbf{m}}_\parallel = (\xi \cdot \hat{\mathbf{m}})\xi/|\xi|^2, \quad \hat{\mathbf{m}}_\perp = \hat{\mathbf{m}} - (\xi \cdot \hat{\mathbf{m}})\xi/|\xi|^2 = -\xi \times (\xi \times \hat{\mathbf{m}})/|\xi|^2.$$

The projection operator $P_\parallel(f) = f_\parallel$ also extends to a bounded linear operator on $L^p(\mathbf{R}^3; \mathbf{R}^3)$ for all $1 < p < \infty$, with operator norm bounded by $C_0 p$ when $p \ge 2$, where $C_0$ is an abstract constant independent of $p \ge 2$ (see Stein [28]).

With this projection operator, we see easily that the magnetostatic stray field $H_{\mathbf{m}}$ is given by $H_{\mathbf{m}} = -P_\parallel(\mathbf{m}\chi_\Omega)$.

2.2. **Decomposition of $H_{\mathbf{m}}$.** The following lemma enables us to split $H_{\mathbf{m}}$ into two parts: one is bounded in $L^\infty$, the other bounded in $L^2(\Omega)$; see also [10, Lemma 5.2] and [19, Lemma 6.2].

**Lemma 2.1.** *Let $\mathbf{m} \in L^\infty(\Omega; \mathbf{R}^3)$ and $H_{\mathbf{m}} = -P_{\|}(\mathbf{m}\chi_\Omega)$. Then, for all $\lambda \geq e$, $H_{\mathbf{m}} = H^\lambda + (H_{\mathbf{m}} - H^\lambda)$ on $\mathbf{R}^3$, where $H^\lambda$ is a function such that*

$$\|H^\lambda\|_{L^\infty} \leq C \ln \lambda, \quad \|H_{\mathbf{m}} - H^\lambda\|_{L^2} \leq C|\Omega|^{\frac{1}{2}}/\lambda, \tag{2.1}$$

*with constant $C = C_0'\|\mathbf{m}\|_{L^\infty}$ for an absolute constant $C_0'$.*

*Proof.* For the convenience of the reader, we include a proof of this result. Define $H^\lambda = H_{\mathbf{m}}\chi_{\{|H_{\mathbf{m}}(x)| \leq C \ln \lambda\}}$, where $C > 0$ is a constant to be selected later. Since $H_{\mathbf{m}} = -\tilde{\mathbf{m}}_{\|}$ with $\tilde{\mathbf{m}} = \mathbf{m}\chi_\Omega$, we have, for all $p \geq 2$,

$$\begin{aligned}\|H_{\mathbf{m}} - H^\lambda\|_{L^2}^2 &= \int_{|\tilde{\mathbf{m}}_{\|}| > C \ln \lambda} |\tilde{\mathbf{m}}_{\|}|^2 \, dx \leq \|\tilde{\mathbf{m}}_{\|}\|_{L^p}^2 |\{x : |\tilde{\mathbf{m}}_{\|}| > C \ln \lambda\}|^{\frac{p-2}{p}} \\ &\leq \|\tilde{\mathbf{m}}_{\|}\|_{L^p}^2 \frac{1}{(C \ln \lambda)^{p-2}} \|\tilde{\mathbf{m}}_{\|}\|_{L^p}^{p-2} = \|\tilde{\mathbf{m}}_{\|}\|_{L^p}^p \frac{1}{(C \ln \lambda)^{p-2}}.\end{aligned}$$

The boundedness of $P_{\|}$ on $L^p(\mathbf{R}^3; \mathbf{R}^3)$ yields that, for all $p \geq 2$,

$$\|\tilde{\mathbf{m}}_{\|}\|_{L^p} \leq C_0 p \|\tilde{\mathbf{m}}\|_{L^p} \leq C_0 p \|\mathbf{m}\|_{L^\infty(\Omega)} |\Omega|^{1/p},$$

where $C_0$ is independent of $p \geq 2$ (see [28]). Hence,

$$\|H_{\mathbf{m}} - H^\lambda\|_{L^2}^2 \leq |\Omega|(C_1 p)^p/(C \ln \lambda)^{p-2},$$

where $C_1 = C_0 \|\mathbf{m}\|_{L^\infty}$. We now select $C = 4eC_1$ and $p = 4 \ln \lambda \geq 4$ to obtain

$$\|H_{\mathbf{m}} - H^\lambda\|_{L^2}^2 \leq |\Omega|(C_1 p)^p/(C \ln \lambda)^{p-2} = |\Omega|(C \ln \lambda)^2/\lambda^4;$$

so, $\|H_{\mathbf{m}} - H^\lambda\|_{L^2} \leq C|\Omega|^{\frac{1}{2}}(\ln \lambda)/\lambda^2 \leq C|\Omega|^{\frac{1}{2}}/\lambda$, using $\ln \lambda \leq \lambda$ for $\lambda \geq e$. This proves (2.1). ☐

2.3. **Proof of Theorem 1.2.** We now prove our main stability result, Theorem 1.2.

Assume $\mathbf{m}^k$ $(k = 1, 2)$ is any weak solution to the problem (1.7) with given applied field $\mathbf{a}^k$ and initial datum $\mathbf{m}_0^k$ satisfying

$$\|\mathbf{a}^k\|_{L^\infty} + \|\mathbf{m}_0^k\|_{L^\infty} \leq R \quad \text{for } k = 1, 2, \tag{2.2}$$

where $R > 0$ is a given constant. Then, Theorem 1.2 will be proved once we prove the following result.

**Theorem 2.2.** *Given any $0 < T < \infty$, there exist constants $C = C(R, T) > 0$, $c = c(R, T) > 0$ and $\rho = \rho(R, T) > 0$ such that, if $\mu = \max\{\|\mathbf{m}_0^1 - \mathbf{m}_0^2\|_{L^2}, \|\mathbf{a}^1 - \mathbf{a}^2\|_{L^2}\} \leq c$, then one has, for all $t \in [0, T]$,*

$$\|\mathbf{m}^1(t) - \mathbf{m}^2(t)\|_{L^2(\Omega)} \leq C\, \mu^\rho. \tag{2.3}$$

*Proof.* **Step 1.** Let $\delta\mathbf{m} = \mathbf{m}^1(t) - \mathbf{m}^2(t)$ and $\delta F = F_{\mathbf{a}^1}(x, \mathbf{m}^1, H_1) - F_{\mathbf{a}^2}(x, \mathbf{m}^2, H_2)$, where $H_k = H_{\mathbf{m}^k}$ for $k = 1, 2$. Then $\partial_t(\delta\mathbf{m}) = \delta F$ and hence

$$\partial_t(\|\delta\mathbf{m}(t)\|_{L^2}) \leq \|\partial_t(\delta\mathbf{m}(t))\|_{L^2} = \|\delta F(t)\|_{L^2}.$$

So we have

$$\|\delta\mathbf{m}(t)\|_{L^2} - \|\delta\mathbf{m}_0\|_{L^2} \leq \int_0^t \|\delta F(s)\|_{L^2} \, ds. \tag{2.4}$$

**Step 2.** The function $\mathbb{L}(\mathbf{m},\mathbf{n})$ defined by (1.8) above can be written as

$$\mathbb{L}(\mathbf{m},\mathbf{n}) = \mathbb{B}(\mathbf{m})\cdot\mathbf{n}, \tag{2.5}$$

where $\mathbb{B}(\mathbf{m})$ is a $3\times 3$-matrix for each $\mathbf{m}\in\mathbf{R}^3$; note that each element of $\mathbb{B}(\mathbf{m})$ is a quadratic function of $\mathbf{m}$. Given any $\mathbf{m}^k,\mathbf{n}^k\in\mathbf{R}^3$ $(k=1,2)$, letting $\delta\mathbf{m}=\mathbf{m}^1-\mathbf{m}^2, \delta\mathbf{n}=\mathbf{n}^1-\mathbf{n}^2$, by virtue of $\mathbb{L}(\mathbf{m}^1,\mathbf{n}^1)-\mathbb{L}(\mathbf{m}^2,\mathbf{n}^2)=[\mathbb{L}(\mathbf{m}^1,\mathbf{n}^1)-\mathbb{L}(\mathbf{m}^2,\mathbf{n}^1)]+\mathbb{L}(\mathbf{m}^2,\mathbf{n}^1-\mathbf{n}^2)$, one can write

$$\mathbb{L}(\mathbf{m}^1,\mathbf{n}^1)-\mathbb{L}(\mathbf{m}^2,\mathbf{n}^2)=\mathbb{A}(\mathbf{m}^1,\mathbf{m}^2,\mathbf{n}^1)\cdot\delta\mathbf{m}+\mathbb{B}(\mathbf{m}^2)\cdot\delta\mathbf{n}, \tag{2.6}$$

where $\mathbb{A}(\mathbf{m}^1,\mathbf{m}^2,\mathbf{n}^1)$ is a matrix function given by

$$\mathbb{A}(\mathbf{m}^1,\mathbf{m}^2,\mathbf{n}^1)=\int_0^1\frac{\partial\mathbb{L}}{\partial\mathbf{m}}(t\mathbf{m}^1+(1-t)\mathbf{m}^2,\mathbf{n}^1)\,dt. \tag{2.7}$$

**Step 3.** By Remark 1 above, it follows that $\|\mathbf{m}^k(t)\|_{L^\infty}\le R$ $(k=1,2)$ for all $t\ge 0$. From $F_{\mathbf{a}^k}(x,\mathbf{m}^k,H_k)=-\mathbb{L}(\mathbf{m}^k,\varphi'(\mathbf{m}^k))+\mathbb{L}(\mathbf{m}^k,\mathbf{a}^k(x))+\mathbb{L}(\mathbf{m}^k,H_k)$, by (2.2) and (2.6), we obtain the following point-wise estimate for $\delta F$:

$$|\delta F|\le A|\delta H|+B(|H_1|+1)|\delta\mathbf{m}|+D|\delta\mathbf{a}|, \tag{2.8}$$

where $\delta H=H_1-H_2=H_{\delta\mathbf{m}}$, $\delta\mathbf{a}=\mathbf{a}^1(x)-\mathbf{a}^2(x)$, and $A=A(R), B=B(R), D=D(R)$ are constants depending only on $R$. We apply Lemma 2.1 to function $H_1(t)=-P_\|(\mathbf{m}^1(t)\chi_\Omega)$. For any $\lambda\ge e$, let $H_1=H_1^\lambda+(H-H_1^\lambda)$, where $H_1^\lambda$ is given in Lemma 2.1 with constant $C=C_0'\|\mathbf{m}^1(t)\|_{L^\infty}\le C_0'R$. So, by (2.8), we have the $L^2(\Omega)$-norm estimate:

$$\begin{aligned}\|\delta F\|_{L^2}\le&A\|\delta H\|_{L^2}+B(C\ln\lambda+1)\|\delta\mathbf{m}\|_{L^2}\\&+B\frac{C|\Omega|^{\frac12}}{\lambda}\|\delta\mathbf{m}\|_{L^\infty}+D\|\delta\mathbf{a}\|_{L^2}\\\le&(A'+B'\ln\lambda)\|\delta\mathbf{m}\|_{L^2}+\frac{C'}{\lambda}+D\|\delta\mathbf{a}\|_{L^2},\end{aligned} \tag{2.9}$$

using $\|\delta H\|_{L^2}\le\|H_{\delta\mathbf{m}}\|_{L^2(\mathbf{R}^3)}\le\|\delta\mathbf{m}\|_{L^2}$, where constants $A',B',C'$ depend on $R$.
**Step 4.** From (2.4) and (2.9), it follows that

$$\begin{aligned}\|\delta\mathbf{m}(t)\|_{L^2}-\|\delta\mathbf{m}_0\|_{L^2}&\le\int_0^t\|\delta F(s)\|_{L^2}\,ds\\&\le\int_0^t\left((A'+B'\ln\lambda)\|\delta\mathbf{m}(s)\|_{L^2}+\frac{C'}{\lambda}+D\|\mathbf{a}\|_{L^2}\right)ds\\&=\frac{C't}{\lambda}+\|\delta\mathbf{a}\|_{L^2}Dt+(A'+B'\ln\lambda)\int_0^t\|\delta\mathbf{m}(s)\|_{L^2}\,ds.\end{aligned}$$

From this, a Gronwall inequality yields

$$\begin{aligned}\|\delta\mathbf{m}(t)\|_{L^2}&\le\left(\|\delta\mathbf{m}_0\|_{L^2}+\frac{C't}{\lambda}+\|\delta\mathbf{a}\|_{L^2}Dt\right)e^{A't+B't\ln\lambda}\\&\le\left(\|\delta\mathbf{m}_0\|_{L^2}+DT\|\delta\mathbf{a}\|_{L^2}+\frac{C't}{\lambda}\right)e^{A't}\lambda^{B't}\quad\forall\ 0\le t\le T.\end{aligned} \tag{2.10}$$

**Step 5.** We consider two cases.
**Case 1.** Assume both $\delta\mathbf{m}_0=0$ and $\delta\mathbf{a}=0$. Then, by (2.10),

$$\|\delta\mathbf{m}(t)\|_{L^2}\le C'te^{A't}\lambda^{B't-1}. \tag{2.11}$$

Let $t_0=\frac{1}{B'+1}$. If $0\le t\le t_0$, then $B't-1<0$ and hence, by (2.11) with $\lambda\to\infty$, we have $\delta\mathbf{m}(t)=0$ for all $t\in[0,t_0]$. With $\mathbf{m}^k(t_0)$ as initial datum at time $t_0$, we obtain $\delta\mathbf{m}(t)=0$ on $[t_0,2t_0]$; eventually, we have $\delta\mathbf{m}(t)=0$ for all $t\geq 0$; hence (2.3) holds. This also shows the uniqueness of the weak solution to the system.

**Case 2.** Assume $0<\|\delta\mathbf{m}_0\|_{L^2}+DT\|\delta\mathbf{a}\|_{L^2}\leq 1/e<1$. In this case, setting $\lambda=(\|\delta\mathbf{m}_0\|_{L^2}+DT\|\delta\mathbf{a}\|_{L^2})^{-1}\geq e$ in (2.10), we obtain

$$\|\delta\mathbf{m}(t)\|_{L^2}\leq(1+C't)e^{A't}(\|\delta\mathbf{m}_0\|_{L^2}+DT\|\delta\mathbf{a}\|_{L^2})^{1-B't}.$$

Let $t_1=\frac{1}{2(B'+1)}$ and $C_1=(1+C't_1)e^{A't_1}>1$. Then $1-B't\geq\frac{1}{2}$ for all $0\leq t\leq t_1$; hence

$$\|\delta\mathbf{m}(t)\|_{L^2}\leq C_1(\|\delta\mathbf{m}_0\|_{L^2}+DT\|\delta\mathbf{a}\|_{L^2})^{\frac{1}{2}}\quad\forall\ 0\leq t\leq t_1.$$

Adding $DT\|\delta\mathbf{a}\|_{L^2}$ to both sides, we obtain

$$\|\delta\mathbf{m}(t)\|_{L^2}+DT\|\delta\mathbf{a}\|_{L^2}\leq C_2\left(\|\delta\mathbf{m}_0\|_{L^2}+DT\|\delta\mathbf{a}\|_{L^2}\right)^{\frac{1}{2}}\quad\forall\ 0\leq t\leq t_1,\tag{2.12}$$

where $C_2=C_1+1$ depends only on $R$.

**Step 6.** Combining Cases 1 and 2 in Step 5 above, with the constants $t_1=t_1(R)$ and $C_2=C_2(R)>1$ determined above, we have that, if $\|\delta\mathbf{m}_0\|_{L^2}+DT\,\|\delta\mathbf{a}\|_{L^2}\leq 1/e$, then

$$\|\delta\mathbf{m}(t)\|_{L^2}+DT\|\delta\mathbf{a}\|_{L^2}\leq C_2\left(\|\delta\mathbf{m}_0\|_{L^2}+DT\|\delta\mathbf{a}\|_{L^2}\right)^{\frac{1}{2}}\quad\forall\ 0\leq t\leq t_1.\tag{2.13}$$

Assume

$$C_2(\|\delta\mathbf{m}_0\|_{L^2}+DT\|\delta\mathbf{a}\|_{L^2})^{\frac{1}{2}}\leq 1/e.\tag{2.14}$$

Then, by (2.13), $\|\delta\mathbf{m}(t_1)\|_{L^2}+DT\|\delta\mathbf{a}\|_{L^2}\leq 1/e$. With $\mathbf{m}^k(t_1)$ as initial datum at time $t_1$, we apply (2.13) again to obtain

$$\begin{aligned}\|\delta\mathbf{m}(t_1+t)\|_{L^2}+DT\|\delta\mathbf{a}\|_{L^2}\leq&C_2(\|\delta\mathbf{m}(t_1)\|_{L^2}+DT\|\delta\mathbf{a}\|_{L^2})^{\frac{1}{2}}\\ \leq&C_2^{1+\frac{1}{2}}(\|\delta\mathbf{m}_0\|_{L^2}+DT\|\delta\mathbf{a}\|_{L^2})^{\frac{1}{4}}\quad\forall\ 0\leq t\leq t_1.\end{aligned}$$

We have thus proved that, if (2.14) holds then

$$\|\delta\mathbf{m}(t)\|_{L^2}+DT\|\delta\mathbf{a}\|_{L^2}\leq C_2^{1+\frac{1}{2}}(\|\delta\mathbf{m}_0\|_{L^2}+DT\|\delta\mathbf{a}\|_{L^2})^{\frac{1}{4}}\quad\forall\ 0\leq t\leq 2t_1.$$

By induction, we obtain that, for $k=1,2,\cdots$, if

$$C_2^{1+\frac{1}{2}+\cdots+\frac{1}{2^{k-1}}}\left(\|\delta\mathbf{m}_0\|_{L^2}+DT\|\delta\mathbf{a}\|_{L^2}\right)^{\frac{1}{2^k}}\leq 1/e,\tag{2.15}$$

then

$$\begin{aligned}\|\delta\mathbf{m}(t)\|_{L^2}+DT\|\delta\mathbf{a}\|_{L^2}\quad&\leq\quad C_2^{1+\frac{1}{2}+\cdots+\frac{1}{2^k}}\left(\|\delta\mathbf{m}_0\|_{L^2}+DT\|\delta\mathbf{a}\|_{L^2}\right)^{\frac{1}{2^{k+1}}}\\ &\leq\quad C_2^2\left(\|\delta\mathbf{m}_0\|_{L^2}+DT\|\delta\mathbf{a}\|_{L^2}\right)^{\frac{1}{2^{k+1}}}\end{aligned}\tag{2.16}$$

for all $0\leq t\leq 2^kt_1$.

**Step 7.** In this step, we complete the proof of the theorem. Let $k$ be the integer such that $2^{k-1}t_1<T\leq 2^kt_1$. Define

$$\rho=\rho(R,T)=1/(2^{k+1}),\quad c=c(R,T)=(C_2^2e)^{-2^k}/(1+DT).$$

Assume $\mu=\max\{\|\delta\mathbf{m}_0\|_{L^2},\|\delta\mathbf{a}\|_{L^2}\}\leq c$. Then

$$\|\delta\mathbf{m}_0\|_{L^2}+DT\|\delta\mathbf{a}\|_{L^2}\leq(1+DT)\mu\leq(C_2^2e)^{-2^k},$$

from which it is easily seen that (2.15) holds; so, by (2.16),

$$\|\delta\mathbf{m}(t)\|_{L^2}+DT\|\delta\mathbf{a}\|_{L^2}\leq C_2^2(\|\delta\mathbf{m}_0\|_{L^2}+DT\|\delta\mathbf{a}\|_{L^2})^{\rho}\quad\forall\ 0\leq t\leq T.$$

Therefore,

$$\|\delta\mathbf{m}(t)\|_{L^2}\leq C_2^2\,(1+DT)^{\rho}\,\mu^{\rho}\quad\forall\ t\in[0,T];$$

this proves (2.3) with constant $C=C_2^2(1+DT)^{\rho}$. $\square$

**Remark 2.** Theorem 1.2 generalizes our previous result [10, Theorem 5.1] to the case of different applied fields $\mathbf{a}(x)$. A similar stability result including the different anisotropy functions $\varphi(\mathbf{m})$ can also be proved.

## 3. Existence of global weak solutions.

In this section, we present a proof for the existence of global weak solution to (1.7) based on the stability theorem proved above. To this end, we introduce a nonlinear function

$$\mathcal{F}(\mathbf{m}) = F_{\mathbf{a}}(x, \mathbf{m}, H_{\mathbf{m}}) = -\mathbb{L}(\mathbf{m}, \varphi'(\mathbf{m})) + \mathbb{L}(\mathbf{m}, \mathbf{a}(x)) + \mathbb{L}(\mathbf{m}, H_{\mathbf{m}}) \tag{3.1}$$

for $\mathbf{m} \in L^\infty(\Omega; \mathbf{R}^3)$, where $H_{\mathbf{m}}$ is defined by (1.2) and $\mathbb{L}$ is defined by (1.8). As before, we always assume the anisotropy function $\varphi\colon \mathbf{R}^3 \to \mathbf{R}^3$ is smooth.

### 3.1. Properties of map $\mathcal{F}$ for smooth applied fields.

In this subsection, we assume the applied field $\mathbf{a}$ belongs to $C^\infty(\bar{\Omega}; \mathbf{R}^3)$ and show that, in this case, map $\mathcal{F}\colon H^2(\Omega; \mathbf{R}^3) \to H^2(\Omega; \mathbf{R}^3)$ and is locally Lipschitz. We need some estimates.

**Lemma 3.1.** *Let $\Omega \subset \mathbf{R}^3$ be a bounded domain with smooth boundary. Then the following estimates hold on $H^2(\Omega; \mathbf{R}^3)$ :*

$$\|\mathbf{m}\|_{L^\infty(\Omega)} + \|\mathbf{m}\|_{W^{1,p}(\Omega)} \le C_0 \,\|\mathbf{m}\|_{H^2(\Omega)} \quad \forall\; 1 \le p \le 6, \tag{3.2}$$

$$\|H_{\mathbf{m}}\|_{H^2(\Omega)} \le C_1 \,\|\mathbf{m}\|_{H^2(\Omega)}. \tag{3.3}$$

*Proof.* We omit the proof, but only mention that (3.2) is a simple consequence of the well-known embeddings: $H^2(\Omega) \subset W^{1,6}(\Omega) \subset C^{\frac{1}{2}}(\bar{\Omega}) \subset L^\infty(\Omega)$ for bounded smooth domain $\Omega \subset \mathbf{R}^3$, and that estimate (3.3) has been, e.g., proved in [8]. Finally, we remark that, from (3.2) and (3.3), it follows that, with constant $C_2 = C_0 C_1$,

$$\|H_{\mathbf{m}}\|_{L^\infty(\Omega)} \le C_2 \,\|\mathbf{m}\|_{H^2(\Omega)} \quad \forall\; \mathbf{m} \in H^2(\Omega; \mathbf{R}^3). \tag{3.4}$$

$\square$

The main result of the subsection is the following local Lipschitz property of $\mathcal{F}$ on $H^2(\Omega; \mathbf{R}^3)$.

**Proposition 3.2.** *$\mathcal{F}$ maps space $H^2(\Omega; \mathbf{R}^3)$ into itself and is locally Lipschitz on $H^2(\Omega; \mathbf{R}^3)$.*

*Proof.* Since $\mathcal{F}(0) = 0$, the self-mapping property of $\mathcal{F}$ will follow from the local Lipschitz property of $\mathcal{F}$ on $H^2(\Omega; \mathbf{R}^3)$.

To prove the local Lipschitz property of $\mathcal{F}$, given any two functions $\mathbf{m}^1, \mathbf{m}^2 \in H^2(\Omega; \mathbf{R}^3)$ satisfying

$$\max\{\|\mathbf{m}^1\|_{H^2(\Omega)},\, \|\mathbf{m}^2\|_{H^2(\Omega)}\} \le R, \tag{3.5}$$

where $R < \infty$ is a constant, we need to show that

$$\|\mathcal{F}(\mathbf{m}^1) - \mathcal{F}(\mathbf{m}^2)\|_{H^2(\Omega)} \le L \,\|\mathbf{m}^1 - \mathbf{m}^2\|_{H^2(\Omega)} \tag{3.6}$$

for a (local Lipschitz) constant $L = L(R) < \infty$ depending on $R$.

By (3.1), we write $\mathcal{F}(\mathbf{m}^1) - \mathcal{F}(\mathbf{m}^2) = I_1 + I_2$, where

$$I_1 = \mathbb{L}(\mathbf{m}^1, \mathbf{a} - \varphi'(\mathbf{m}^1)) - \mathbb{L}(\mathbf{m}^2, \mathbf{a} - \varphi'(\mathbf{m}^2))$$

and $I_2 = \mathbb{L}(\mathbf{m}^1, H_{\mathbf{m}^1}) - \mathbb{L}(\mathbf{m}^2, H_{\mathbf{m}^2})$. Let $\delta\mathbf{m} = \mathbf{m}^1 - \mathbf{m}^2$. Then, by (2.6),

$$I_2 = \mathbb{A}(\mathbf{m}^1, \mathbf{m}^2, H_{\mathbf{m}^1}) \cdot \delta\mathbf{m} + \mathbb{B}(\mathbf{m}^2) \cdot H_{\delta\mathbf{m}}, \tag{3.7}$$

where $\mathbb{A}, \mathbb{B}$ are functions defined in *Step 2* of the proof of Theorem 2.2 above. We also write $I_1$ as

$$I_1 = \int_0^1 \frac{d}{dt}\mathbb{L}(\mathbf{m}^2 + t\delta\mathbf{m}, \mathbf{a} - \varphi'(\mathbf{m}^2 + t\delta\mathbf{m}))\, dt = \mathbb{C}(\mathbf{m}^1, \mathbf{m}^2, \mathbf{a}) \cdot \delta\mathbf{m}, \tag{3.8}$$

where $\mathbb{C}(\mathbf{m}^1, \mathbf{m}^2, \mathbf{a})$ is certain smooth function of $(\mathbf{m}^1, \mathbf{m}^2, \mathbf{a}) \in \mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{R}^3$. Note that $\mathbb{C}$ is linear in $\mathbf{a}$. We aim to show

$$\|I_k\|_{H^2(\Omega)} \le L(R)\, \|\delta\mathbf{m}\|_{H^2(\Omega)} \quad (k = 1, 2)$$

for some constant $L(R)$ depending on $R$. By (3.5), (3.4) and Lemma 3.1, it follows that, for $k = 1, 2$,

$$\|\mathbf{m}^k\|_{L^\infty(\Omega)} + \|H_{\mathbf{m}^k}\|_{L^\infty(\Omega)} + \|H_{\mathbf{m}^k}\|_{H^2(\Omega)} + \|\nabla\mathbf{m}^k\|_{L^4(\Omega)} \le C_3 R. \tag{3.9}$$

We proceed in two steps.

**Step 1. Estimation of $I_1$.** Clearly, by (3.8) and (3.9),

$$\|I_1\|_{L^2(\Omega)} \le \|\mathbb{C}(\mathbf{m}^1, \mathbf{m}^2, \mathbf{a})\|_{L^\infty} \|\delta\mathbf{m}\|_{L^2} \le L(R)\, \|\delta\mathbf{m}\|_{L^2(\Omega)}.$$

We estimate the $H^2$-norm. Denote by $\partial_j$ the first partial derivative with respect to $x_j$ and by $\partial^2_{ij}$ the second partial derivative with respect to $x_j$ and $x_i$ $(i, j = 1, 2, 3)$. Note that

$$\partial_j(I_1) = \partial_j(\mathbb{C}(\mathbf{m}^1, \mathbf{m}^2, \mathbf{a})) \cdot \delta\mathbf{m} + \mathbb{C}(\mathbf{m}^1, \mathbf{m}^2, \mathbf{a}) \cdot (\delta\mathbf{m})_{x_j}$$

and

$$\begin{aligned}\partial^2_{ij}(I_1) = {} & \partial^2_{ij}(\mathbb{C}(\mathbf{m}^1, \mathbf{m}^2, \mathbf{a})) \cdot \delta\mathbf{m} + \partial_j(\mathbb{C}(\mathbf{m}^1, \mathbf{m}^2, \mathbf{a})) \cdot (\delta\mathbf{m})_{x_i} \\ & + \partial_i(\mathbb{C}(\mathbf{m}^1, \mathbf{m}^2, \mathbf{a})) \cdot (\delta\mathbf{m})_{x_j} + \mathbb{C}(\mathbf{m}^1, \mathbf{m}^2, \mathbf{a}) \cdot (\delta\mathbf{m})_{x_i x_j}.\end{aligned}$$

Since $\partial_j(\mathbb{C}(\mathbf{m}^1, \mathbf{m}^2, \mathbf{a})) = (\partial_{\mathbf{m}^1}\mathbb{C}) \cdot \mathbf{m}^1_{x_j} + (\partial_{\mathbf{m}^2}\mathbb{C}) \cdot \mathbf{m}^2_{x_j} + (\partial_{\mathbf{a}}\mathbb{C}) \cdot \mathbf{a}_{x_j}$ has $L^2$-norm controlled by $R$, we have

$$\begin{aligned}\|\partial_j(I_1)\|_{L^2} &\le \|\partial_j(\mathbb{C}(\mathbf{m}^1, \mathbf{m}^2, \mathbf{a}))\|_{L^2} \|\delta\mathbf{m}\|_{L^\infty} + \|\mathbb{C}(\mathbf{m}^1, \mathbf{m}^2, \mathbf{a})\|_{L^\infty} \|(\delta\mathbf{m})_{x_j}\|_{L^2} \\ &\le L(R)\, \|\delta\mathbf{m}\|_{H^2(\Omega)}.\end{aligned}$$

Similarly, $\partial^2_{ij}(\mathbb{C}(\mathbf{m}^1, \mathbf{m}^2, \mathbf{a}))$ contains terms up to second derivatives of $\mathbf{a}$ and terms like $(\partial^2_{\mathbf{m}^p\mathbf{m}^q}\mathbb{C}) \cdot \mathbf{m}^k_{x_i} \cdot \mathbf{m}^l_{x_j}$ and $(\partial_{\mathbf{m}^p}\mathbb{C}) \cdot \mathbf{m}^q_{x_{i'} x_{j'}}$, with certain choices of $p, q, k, l \in \{1, 2\}$ and $i', j' \in \{i, j\}$. Hence $\|\partial^2_{ij}(\mathbb{C}(\mathbf{m}^1, \mathbf{m}^2, \mathbf{a}))\|_{L^2}$ is bounded by the quantity

$$C(R)\left(\||\nabla\mathbf{m}^1|^2\|_{L^2} + \||\nabla\mathbf{m}^2|^2\|_{L^2} + \|\nabla^2\mathbf{m}^1\|_{L^2} + \|\nabla^2\mathbf{m}^2\|_{L^2} + \|\mathbf{a}\|_{H^2}\right),$$

which, due to $\||\nabla\mathbf{m}|^2\|_{L^2} = \|\nabla\mathbf{m}\|^2_{L^4} \le C\, \|\mathbf{m}\|^2_{H^2}$, is in fact bounded by another constant $C(R)$. From this, similar to the term $\partial_j(I_1)$, the $L^2$-norm of the first or fourth term of $\partial^2_{ij}(I_1)$ is bounded by $L(R)\|\delta\mathbf{m}\|_{H^2(\Omega)}$. The second and third terms of $\partial^2_{ij}(I_1)$ can be estimated as follows:

$$\begin{aligned}&\|\partial_j(\mathbb{C}(\mathbf{m}^1, \mathbf{m}^2, \mathbf{a})) \cdot (\delta\mathbf{m})_{x_i} + \partial_i(\mathbb{C}(\mathbf{m}^1, \mathbf{m}^2, \mathbf{a})) \cdot (\delta\mathbf{m})_{x_j}\|_{L^2} \\ &\quad \le 2\|\nabla(\mathbb{C}(\mathbf{m}^1, \mathbf{m}^2, \mathbf{a}))\|_{L^4} \cdot \|\nabla(\delta\mathbf{m})\|_{L^4} \\ &\quad \le C(R)\, (\|\mathbf{m}^1\|_{W^{1,4}} + \|\mathbf{m}^2\|_{W^{1,4}} + \|\nabla\mathbf{a}\|_{L^4}) \cdot \|\delta\mathbf{m}\|_{W^{1,4}} \\ &\quad \le L(R)\, \|\delta\mathbf{m}\|_{H^2(\Omega)}.\end{aligned}$$

This proves $\|I_1\|_{H^2(\Omega)} \le L(R)\|\delta\mathbf{m}\|_{H^2(\Omega)}$.

**Step 2. Estimation of $I_2$.** We write $I_2 = I_{21} + I_{22}$ with

$$I_{21} = \mathbb{A}(\mathbf{m}^1, \mathbf{m}^2, H_{\mathbf{m}^1}) \cdot \delta\mathbf{m}, \quad I_{22} = \mathbb{B}(\mathbf{m}^2) \cdot H_{\delta\mathbf{m}}.$$

The term $I_{21}$ is more like term $I_1$, except the constant field $\mathbf{a}$ is replaced by the field $H_{\mathbf{m}^1}$. Since $H_{\mathbf{m}^1} \in H^2(\Omega)$ and $\|H_{\mathbf{m}^1}\|_{L^\infty} + \|H_{\mathbf{m}^1}\|_{H^2(\Omega)} \le C_1\|\mathbf{m}^1\|_{H^2(\Omega)} \le C_1 R$, estimation resulting from $H_{\mathbf{m}^1}$ in $\mathbb{A}$ can be handled in a much similar way as the term $\mathbf{a}$ in $\mathbb{C}$ of $I_1$.

The term $I_{22}$ is simpler but slightly different than $I_1$ in that $H_{\delta\mathbf{m}}$ is in place of $\delta\mathbf{m}$. Nevertheless this term can also be estimated in a similar fashion as $I_1$, using the following estimate on $H_{\delta\mathbf{m}}$:

$$\|H_{\delta\mathbf{m}}\|_{L^\infty(\Omega)} + \|\nabla H_{\delta\mathbf{m}}\|_{L^4(\Omega)} + \|H_{\delta\mathbf{m}}\|_{H^2(\Omega)} \le C_5\,\|\delta\mathbf{m}\|_{H^2(\Omega)}.$$

We eventually obtain $\|I_2\|_{H^2(\Omega)} \le L(R)\|\mathbf{m}\|_{H^2(\Omega)}$. This completes the proof. □

3.2. **Existence of global solution for smooth data.** We continue to assume $\mathbf{a} \in C^\infty(\bar{\Omega};\mathbf{R}^3)$ in this subsection. Let $X = H^2(\Omega;\mathbf{R}^3)$. With $\mathcal{F}\colon X \to X$ defined above, we formulate the problem (1.7) as an abstract ODE on $X$ by

$$\begin{cases} \frac{d\mathbf{m}}{dt} = \mathcal{F}(\mathbf{m}), \\ \mathbf{m}(0) = \mathbf{m}_0. \end{cases} \tag{3.10}$$

A solution $\mathbf{m}$ to (3.10) on $[0,T]$ is a function $\mathbf{m} \in C([0,T];X)$ that satisfies

$$\mathbf{m}(t) = \mathbf{m}_0 + \int_0^t \mathcal{F}(\mathbf{m}(s))\,ds \quad \forall\ 0 \le t \le T.$$

We say $\mathbf{m}$ is a solution to (3.10) on $[0,T)$ if $\mathbf{m}$ is a solution on $[0,T']$ for all $0 < T' < T$ (in this case $T$ could be $\infty$).

**Theorem 3.3.** *Given any $\mathbf{m}_0 \in X$, (3.10) has a unique solution $\mathbf{m}$ on $[0,\infty)$. This solution is also a global weak solution to problem (1.7).*

*Proof.* Given $\mathbf{m}_0 \in X$, since $\mathcal{F}$ is locally Lipschitz on $X$, from the abstract theory, there exists $T > 0$ such that (3.10) has a unique solution $\mathbf{m}$ on $[0,T]$. Let

$$T_* = \sup\left\{T > 0 \;\middle|\; \text{(3.10) has a unique solution on } [0,T]\right\}.$$

We claim that $T_* = \infty$, which implies that (3.10) has a unique global solution $\mathbf{m}$ defined on $[0,\infty)$. Clearly, this solution is also a global weak solution to the Cauchy problem (1.7) above.

Suppose $T_* < \infty$. Then, by the elementary ODE theory, a solution $\mathbf{m}$ to (3.10) would exist on $[0,T_*)$ and satisfy

$$\lim_{t\to T_*^-} \|\mathbf{m}(t)\|_X = \infty.$$

The following theorem asserts that this finite time blowup is impossible; this completes the proof of Theorem 3.3. □

**Theorem 3.4.** *Given any $T > 0$, if $\mathbf{m}$ is a solution to (3.10) on $[0,T)$, then*

$$\sup_{t\in[0,T)} \|\mathbf{m}(t)\|_X \le C_{T,\|\mathbf{m}_0\|_X} < \infty. \tag{3.11}$$

The proof of this theorem involves lots of technical estimates and will be postponed to the next individual section.

3.3. **Existence of global weak solution for rough data.** In this subsection, we assume both applied field $\mathbf{a}$ and initial datum $\mathbf{m}_0$ are in $L^\infty(\Omega;\mathbf{R}^3)$.

Let $\mathbf{a}^\epsilon, \mathbf{m}_0^\epsilon \in C^\infty(\bar{\Omega};\mathbf{R}^3)$ be such that

$$\|\mathbf{a}^\epsilon\|_{L^\infty} + \|\mathbf{m}_0^\epsilon\|_{L^\infty} \le R \quad \forall\, \epsilon > 0, \tag{3.12}$$

$$\lim_{\epsilon\to 0^+} (\|\mathbf{a}^\epsilon - \mathbf{a}\|_{L^2} + \|\mathbf{m}_0^\epsilon - \mathbf{m}_0\|_{L^2}) = 0, \tag{3.13}$$

$$\mathbf{a}^\epsilon \to \mathbf{a},\ \mathbf{m}_0^\epsilon \to \mathbf{m}_0 \quad \text{point-wise in } \Omega. \tag{3.14}$$

Consider the Cauchy problem (1.7) with applied field $\mathbf{a}^\epsilon$ and initial datum $\mathbf{m}_0^\epsilon$. Then, by Theorem 3.3, for each $\epsilon > 0$, (1.7) has a global weak solution $\mathbf{m}^\epsilon$. Since $\mathbf{m}^\epsilon \cdot \mathcal{F}(\mathbf{m}^\epsilon) = 0$, it follows that $\partial_t(|\mathbf{m}^\epsilon(x,t)|^2) = 0$ and hence $|\mathbf{m}^\epsilon(x,t)| = |\mathbf{m}_0^\epsilon(x)|$ for a.e. $x \in \Omega$ and all $t > 0$. This implies

$$\|\mathbf{m}^\epsilon(t)\|_{L^\infty} = \|\mathbf{m}_0^\epsilon\|_{L^\infty} \le R. \tag{3.15}$$

For each $n \in \{1,2,3,\cdots\}$, our stability result (Theorems 1.2 and 2.2) implies that sequence $\{\mathbf{m}^\epsilon\}$ is Cauchy in Banach space $C([0,n];L^2(\Omega;\mathbf{R}^3))$ as $\epsilon \to 0^+$. Therefore, $\mathbf{m}^\epsilon \to \mathbf{m}$ in $C([0,n];L^2(\Omega;\mathbf{R}^3))$ as $\epsilon \to 0^+$ for some $\mathbf{m} \in C([0,n];L^2(\Omega;\mathbf{R}^3))$. (Presumably, $\mathbf{m} = \mathbf{m}_n$ depends on $n$.) Hence, by (3.13),

$$\mathbf{m}(0) = \mathbf{m}_0. \tag{3.16}$$

We also have $H_{\mathbf{m}^\epsilon} \to H_{\mathbf{m}}$ in $C([0,n];L^2(\Omega;\mathbf{R}^3))$. It follows that $\mathbf{m}^\epsilon \to \mathbf{m}$ and $H_{\mathbf{m}^\epsilon} \to H_{\mathbf{m}}$ also in $L^2(\Omega\times(0,n))$ as $\epsilon \to 0^+$. Using a subsequence, we can assume

$$\mathbf{m}^\epsilon(x,t) \to \mathbf{m}(x,t), \quad H_{\mathbf{m}^\epsilon}(x,t) \to H_{\mathbf{m}}(x,t) \quad \text{point-wise in } \Omega\times(0,n).$$

Therefore, $F_{\mathbf{a}^\epsilon}(x,\mathbf{m}^\epsilon,H_{\mathbf{m}^\epsilon}) \to F_{\mathbf{a}}(x,\mathbf{m},H_{\mathbf{m}})$ point-wise in $\Omega\times(0,n)$. This shows $\partial_t\mathbf{m} = F_{\mathbf{a}}(x,\mathbf{m},H_{\mathbf{m}})$ in the sense of distribution on $\Omega\times(0,n)$.

Note also that $F_{\mathbf{a}^\epsilon}(x,\mathbf{m}^\epsilon,H_{\mathbf{m}^\epsilon}) \in L^2(\Omega;\mathbf{R}^3)$ uniformly on $\epsilon$ and $t \in (0,n)$; this implies that $\partial_t\mathbf{m} = F_{\mathbf{a}}(x,\mathbf{m},H_{\mathbf{m}})$ holds in $L^\infty((0,n);L^2(\Omega))$ and that $\mathbf{m} \in W^{1,\infty}([0,n);L^2(\Omega;\mathbf{R}^3))$. Combining with (3.16), we have proved that $\mathbf{m} = \mathbf{m}_n$ is a weak solution to (1.7) on $\Omega\times(0,n)$. By the uniqueness of weak solutions, we have $\mathbf{m}_{n+1} = \mathbf{m}_n$ on $\Omega\times(0,n)$; therefore, the sequence $\{\mathbf{m}_n\}_1^\infty$ defines a unique function $\mathbf{m}$ by setting $\mathbf{m}(x,t) = \mathbf{m}_n(x,t)$ with $n = [t]+1$. It is easy to see that $\mathbf{m}$ is a global weak solution to (1.7).

Finally, we have proved the following theorem.

**Theorem 3.5.** *Let $\mathbf{a} \in L^\infty(\Omega;\mathbf{R}^3)$. Given any initial datum $\mathbf{m}_0 \in L^\infty(\Omega;\mathbf{R}^3)$, the problem (1.7) has a unique global weak solution.*

## 4. Proof of Theorem 3.4.

In this separate section, we give the proof of Theorem 3.4. This involves the special form of function $\mathbb{L}(\mathbf{m},\mathbf{n})$ and several estimates.

In what follows, assume $\mathbf{a} \in C^\infty(\bar{\Omega};\mathbf{R}^3)$, $0 < T < \infty$ and $\mathbf{m}$ is a solution to (3.10) on $[0,T)$ with initial datum $\mathbf{m}_0 \in H^2(\Omega;\mathbf{R}^3)$. Assume

$$\|\mathbf{m}_0\|_{L^\infty(\Omega)} = R > 0.$$

Then, similar to (3.15) above, we have

$$\|\mathbf{m}(t)\|_{L^\infty} = \|\mathbf{m}_0\|_{L^\infty} = R,\ \|\mathbf{m}(t)\|_{L^2} = \|\mathbf{m}_0\|_{L^2} \le R|\Omega|^{\frac{1}{2}} \quad \forall\, 0 \le t < T. \tag{4.1}$$

We would like to show

$$\sup_{t\in[0,T)} \|\mathbf{m}(t)\|_{H^2(\Omega)} \le C_{T,\|\mathbf{m}_0\|_{H^2}} < \infty. \tag{4.2}$$

Let

$$y(t) = 1 + \|\mathbf{m}(t)\|^2_{H^2(\Omega)} = 1 + \|\mathbf{m}(t)\|^2_{L^2} + \|\nabla \mathbf{m}(t)\|^2_{L^2} + \|\nabla^2 \mathbf{m}(t)\|^2_{L^2}.$$

The goal is to show

$$y'(t) \le C y(t)(1 + \ln y(t)) \quad \forall\, 0 < t < T, \tag{4.3}$$

where $C = C(R)$ is a constant depending on $R$. Once (4.3) is proved, one easily obtains that

$$\ln(y(t)) \le (\ln(y(0)) + 1)\, e^{CT} < \infty \quad \forall\, t \in [0, T),$$

from which (4.2) follows.

The rest of the section is devoted to proving (4.3).

4.1. **Energy estimates.** It is convenient to use the special structure of function $\mathbb{L}$ to write function $\mathcal{F}(\mathbf{m})$ as follows:

$$\mathcal{F}(\mathbf{m}) = \mathbb{B}(\mathbf{m}) \cdot \mathbf{a} - \mathbb{B}(\mathbf{m}) \cdot \varphi'(\mathbf{m}) + \mathbb{B}(\mathbf{m}) \cdot H_{\mathbf{m}},$$

where $\mathbb{B}(\mathbf{m})$ is a $3 \times 3$ matrix defined in (2.5) above, whose elements are quadratic functions of $\mathbf{m}$; hence $\mathbb{B}''(\mathbf{m}) = \mathbb{D}$ is a constant tensor. However, this special structure of $\mathbb{B}$ is not used; in fact, the following arguments are valid for arbitrary smooth functions $\mathbb{B}$.

Differentiating equation in (3.10) with respect to $x_i$ yields

$$\begin{aligned}\frac{d\mathbf{m}_{x_i}}{dt} &= \mathbb{B}'(\mathbf{m}) \cdot \mathbf{m}_{x_i} \cdot \mathbf{a} + \mathbb{B}(\mathbf{m}) \cdot \mathbf{a}_{x_i} \\ &\quad - \mathbb{B}'(\mathbf{m}) \cdot \mathbf{m}_{x_i} \cdot \varphi'(\mathbf{m}) - \mathbb{B}(\mathbf{m}) \cdot \varphi''(\mathbf{m}) \cdot \mathbf{m}_{x_i} \\ &\quad + \mathbb{B}'(\mathbf{m}) \cdot \mathbf{m}_{x_i} \cdot H_{\mathbf{m}} + \mathbb{B}(\mathbf{m}) \cdot (H_{\mathbf{m}})_{x_i}.\end{aligned} \tag{4.4}$$

Further differentiating equation (4.4) with respect to $x_j$ yields

$$\begin{aligned}\frac{d\mathbf{m}_{x_i x_j}}{dt} &= \mathbb{D} \cdot \mathbf{m}_{x_j} \cdot \mathbf{m}_{x_i} \cdot \mathbf{a} + \mathbb{B}' \cdot \mathbf{m}_{x_i x_j} \cdot \mathbf{a} + \mathbb{B}' \cdot \mathbf{m}_{x_i} \cdot \mathbf{a}_{x_j} \\ &\quad + \mathbb{B}' \cdot \mathbf{m}_{x_j} \cdot \mathbf{a}_{x_i} + \mathbb{B} \cdot \mathbf{a}_{x_i x_j} \\ &\quad - \mathbb{D} \cdot \mathbf{m}_{x_j} \cdot \mathbf{m}_{x_i} \cdot \varphi' - \mathbb{B}' \cdot \mathbf{m}_{x_i x_j} \cdot \varphi' - \mathbb{B}' \cdot \mathbf{m}_{x_i} \cdot \varphi'' \cdot \mathbf{m}_{x_j} \\ &\quad - \mathbb{B}' \cdot \mathbf{m}_{x_j} \cdot \varphi'' \cdot \mathbf{m}_{x_i} - \mathbb{B} \cdot \varphi''' \cdot \mathbf{m}_{x_j} \cdot \mathbf{m}_{x_i} - \mathbb{B} \cdot \varphi'' \cdot \mathbf{m}_{x_i x_j} \\ &\quad + \mathbb{D} \cdot \mathbf{m}_{x_j} \cdot \mathbf{m}_{x_i} \cdot H_{\mathbf{m}} + \mathbb{B}' \cdot \mathbf{m}_{x_i x_j} \cdot H_{\mathbf{m}} + \mathbb{B}' \cdot \mathbf{m}_{x_i} \cdot (H_{\mathbf{m}})_{x_j} \\ &\quad + \mathbb{B}' \cdot \mathbf{m}_{x_j} \cdot (H_{\mathbf{m}})_{x_i} + \mathbb{B} \cdot (H_{\mathbf{m}})_{x_i x_j}.\end{aligned} \tag{4.5}$$

Dot-product of (4.4) with $\mathbf{m}_{x_i}$ and of (4.5) with $\mathbf{m}_{x_i x_j}$ and integration over $x \in \Omega$ yield the following identities:

$$\frac{1}{2}\frac{d}{dt}\left(\|\mathbf{m}_{x_i}\|^2_{L^2}\right) = \int_\Omega \left(\mathbf{m}_{x_i} \cdot \frac{d\mathbf{m}_{x_i}}{dt}\right) dx, \tag{4.6}$$

$$\frac{1}{2}\frac{d}{dt}\left(\|\mathbf{m}_{x_i x_j}\|^2_{L^2}\right) = \int_\Omega \left(\mathbf{m}_{x_i x_j} \cdot \frac{d\mathbf{m}_{x_i x_j}}{dt}\right) dx. \tag{4.7}$$

The energy estimates involve estimating the right-hand sides of (4.6) and (4.7) with terms $\frac{d\mathbf{m}_{x_i}}{dt}$, $\frac{d\mathbf{m}_{x_i x_j}}{dt}$ given by the right-hand sides of (4.4) and (4.5).

4.2. **More subtle inequalities.** To handle the terms involved in the integrals on the right-hand sides of (4.6) and (4.7), more subtle inequalities are needed.

**Lemma 4.1.** *Let $\Omega \subset \mathbf{R}^3$ be a bounded domain with smooth boundary. Then*

$$\begin{aligned} &\|\nabla \mathbf{n}\|_{L^4} \le C_6 \, \|\mathbf{n}\|_{L^\infty}^{\frac{1}{2}} \|\mathbf{n}\|_{H^2}^{\frac{1}{2}}, \\ &\|H_{\mathbf{n}}\|_{L^\infty(\Omega)} \le C_{\|\mathbf{n}\|_{L^\infty}} \, (1 + \ln^+(\|\mathbf{n}\|_{H^2})), \end{aligned} \quad \forall\ \mathbf{n} \in H^2(\Omega; \mathbf{R}^3), \tag{4.8}$$

*where* $\ln^+ t = \max\{\ln t, 0\}$ *for* $t > 0$ *and* $C_{\|\mathbf{n}\|_{L^\infty}} < \infty$ *depends on* $\|\mathbf{n}\|_{L^\infty(\Omega)}$.

*Proof.* The first inequality of (4.8) is a consequence of the well-known Gagliardo-Nirenberg inequality:

$$\|\nabla^j f\|_{L^q(\mathbf{R}^n)} \le C \, \|f\|_{L^r(\mathbf{R}^n)}^{1-\theta} \|\nabla^l f\|_{L^p(\mathbf{R}^n)}^{\theta},$$

where $\theta = j/l \in (0,1)$ and $1/q = \theta/p + (1-\theta)/r$, $1 \le p, r \le \infty$. Here $j = 1, l = 2, p = 2, q = 4, r = \infty$ and $\theta = 1/2$. While the second inequality of (4.8) is a Judovic-type inequality proved, e.g., in [19, Lemma 7.2]. ☐

The following result is an immediate consequence of this lemma and (4.1).

**Proposition 4.2.** *For the solution* $\mathbf{m}(t)$*, with* $y(t)$ *defined above, it follows that*

$$\begin{aligned} &\|\nabla \mathbf{m}(t)\|_{L^4(\Omega)}^4 \le C_7 \, y(t), \\ &\|H_{\mathbf{m}}(t)\|_{L^\infty(\Omega)} \le C_8 \, (1 + \ln y(t)), \end{aligned} \quad \forall\ 0 \le t < T, \tag{4.9}$$

*where* $C_7, C_8$ *are constants depending on* $R = \|\mathbf{m}_0\|_{L^\infty}$.

4.3. **Energy estimates (continued) and proof of (4.3).** First of all, the integral on right-hand side of (4.6) is bounded by

$$C(R) \int_\Omega \big( |\nabla \mathbf{m}|^2 + |\nabla \mathbf{m}| + |\nabla \mathbf{m}|^2 |H_{\mathbf{m}}| + |\nabla \mathbf{m}||\nabla (H_{\mathbf{m}})| \big) \, dx.$$

The third term is bounded by $C(R) \|H_{\mathbf{m}}\|_{L^\infty(\Omega)} \|\nabla \mathbf{m}\|_{L^2}^2$ and hence, by (4.9b), is bounded by $C(R) y(t)(1 + \ln y(t))$, while all the other terms are bounded by $C(R)\|\mathbf{m}\|_{H^2}^2$ and hence by $C(R)y(t)$. Therefore,

$$\frac{d}{dt} \big( \|\mathbf{m}_{x_i}\|_{L^2}^2 \big) \le C(R) y(t)(1 + \ln y(t)), \quad \forall\ 0 < t < T. \tag{4.10}$$

Similarly, the integrand of the right-hand side of (4.7) is bounded by constant $C(R)$ times

$$|\nabla \mathbf{m}|^2 |\nabla^2 \mathbf{m}| + |\nabla^2 \mathbf{m}|^2 + |\nabla \mathbf{m}||\nabla^2 \mathbf{m}| + |\nabla^2 \mathbf{m}| + |\nabla^2 \mathbf{m}||\nabla^2 (H_{\mathbf{m}})|$$
$$+ |\nabla \mathbf{m}|^2 |H_{\mathbf{m}}||\nabla^2 \mathbf{m}| + |H_{\mathbf{m}}||\nabla^2 \mathbf{m}|^2 + |\nabla \mathbf{m}||\nabla^2 \mathbf{m}||\nabla (H_{\mathbf{m}})|.$$

Integrals of terms in the first group can all be bounded by $Cy(t)$. Integrals of the first two terms in the second group can be bounded by constant times

$$\|H_{\mathbf{m}}\|_{L^\infty(\Omega)} (\|\nabla \mathbf{m}\|_{L^4}^4 + \|\nabla^2 \mathbf{m}\|_{L^2}^2),$$

which, by (4.9a-b), is bounded by $Cy(t)(1 + \ln y(t))$. Finally, the integral of the last term in the second group can be estimated as follows:

$$\begin{aligned} \int_\Omega |\nabla \mathbf{m}||\nabla^2 \mathbf{m}||\nabla (H_{\mathbf{m}})| dx &\le \| |\nabla \mathbf{m}| \cdot |\nabla (H_{\mathbf{m}})| \|_{L^2(\Omega)} \| |\nabla^2 \mathbf{m}| \|_{L^2(\Omega)} \\ &\le \|\nabla \mathbf{m}\|_{L^4(\Omega)} \|\nabla (H_{\mathbf{m}})\|_{L^4(\Omega)} \| |\nabla^2 \mathbf{m}| \|_{L^2(\Omega)}, \end{aligned}$$

which, by using Lemma 4.1, is bounded by

$$\begin{aligned}&\le C\|\mathbf{m}\|^{\frac{1}{2}}_{H^2(\Omega)}\|H_{\mathbf{m}}\|^{\frac{1}{2}}_{L^\infty(\Omega)}\|H_{\mathbf{m}}\|^{\frac{1}{2}}_{H^2(\Omega)}\|\mathbf{m}\|_{H^2(\Omega)}\\ &\le C\|\mathbf{m}\|^{\frac{1}{2}}_{H^2(\Omega)}\|H_{\mathbf{m}}\|^{\frac{1}{2}}_{L^\infty(\Omega)}\|\mathbf{m}\|^{\frac{1}{2}}_{H^2(\Omega)}\|\mathbf{m}\|_{H^2(\Omega)} = C\|\mathbf{m}\|^{2}_{H^2(\Omega)}\|H_{\mathbf{m}}\|^{\frac{1}{2}}_{L^\infty(\Omega)}\\ &\le C\, y(t)\cdot(1+\ln y(t))^{\frac{1}{2}} \le C\, y(t)(1+\ln y(t)).\end{aligned}$$

Therefore, by (4.7), we have obtained that

$$\frac{d}{dt}\left(\|\mathbf{m}_{x_ix_j}\|^2_{L^2}\right) \le C(R)y(t)(1+\ln y(t)), \quad \forall\ 0<t<T. \tag{4.11}$$

Summing up $i,j=1,2,3$ in (4.10) and (4.11) and using (3.15), we obtain (4.3).

**Remark 3.** By the local Lipschitz property of $\mathcal{F}(\mathbf{m})$, from (4.2), one easily obtains

$$\sup_{t\in[0,T)}\|\mathbf{m}_t\|_{H^2(\Omega)} \le C_{T,\|\mathbf{m}_0\|_{H^2}} < \infty. \tag{4.12}$$

In next section, we prove higher time regularity for solutions.

5. **Higher time regularity.** The higher time regularity has been studied for Landau-Lifshitz equation with exchange energy in [6]. We study a higher time regularity of weak solutions for simple Landau-Lifshitz equation

$$\begin{cases}\mathbf{m}_t = \gamma\mathbf{m}\times H_{\mathbf{m}} + \gamma\alpha\mathbf{m}\times(\mathbf{m}\times H_{\mathbf{m}}) \ \text{ in } \Omega\times(0,\infty),\\ \mathbf{m}(0)=\mathbf{m}_0,\end{cases} \tag{5.1}$$

where $\Omega$ is a bounded smooth domain in $\mathbf{R}^3$ and $\mathbf{m}_0 \in H^2(\Omega;\mathbf{R}^3)$.

**Theorem 5.1.** *For any time $T>0$, the solution $\mathbf{m}$ to (5.1) satisfies, for $p=0,1,2,\cdots$,*

$$\sup_{t\in[0,T]}\|\partial_t^{p+1}\mathbf{m}\|_{H^2(\Omega)} \le C < \infty, \tag{5.2}$$

*where $C$ is constant depending on $T,p$ and $\|\mathbf{m}_0\|_{H^2(\Omega)}$.*

*Proof.* We use induction on $p$. The case for $p=0$ is already mentioned in Remark 3 above. Let us assume (5.2) holds for all powers up to $p-1$. We consider the case for $p$. Note that $\partial_t^i(H_{\mathbf{m}}) = H_{\partial_t^i\mathbf{m}}$ and hence, by (3.4),

$$\|H_{\partial_t^i\mathbf{m}}\|_{H^2(\Omega)} \le C\|\partial_t^i\mathbf{m}\|_{H^2(\Omega)}.$$

Therefore, by the induction assumption, it follows that, for all $t\in[0,T]$,

$$\|\partial_t^i(H_{\mathbf{m}})\|_{H^2(\Omega)} \le C\|\partial_t^i\mathbf{m}\|_{H^2(\Omega)} \le C_{T,p,\|\mathbf{m}_0\|_{H^2}} < \infty \quad \forall\ 0\le i\le p. \tag{5.3}$$

Taking $p^{th}$-derivatives with respect to $t$ to equation (5.1) yields

$$\partial_t^{p+1}\mathbf{m} = \gamma\sum_{i+j=p}\partial_t^i\mathbf{m}\times\partial_t^j H_{\mathbf{m}} + \gamma\alpha\sum_{i+j+k=p}\partial_t^i\mathbf{m}\times(\partial_t^j\mathbf{m}\times\partial_t^k H_{\mathbf{m}}). \tag{5.4}$$

We need to prove $\|\partial_t^{p+1}\mathbf{m}\|_{H^2(\Omega)} \le C_{T,p,\|\mathbf{m}_0\|_{H^2}} < \infty$.

5.1. **Estimation of $\|\partial_t^{p+1}\mathbf{m}\|_{L^2(\Omega)}$.** Since, $\forall\ 0\le i\le p$,

$$\|\partial_t^i(H_{\mathbf{m}})\|_{L^\infty(\Omega)} \le C\|\partial_t^i(H_{\mathbf{m}})\|_{H^2(\Omega)} \le C_{T,p,\|\mathbf{m}_0\|_{H^2}} < \infty,$$

the $L^2$-norm of each term on the right-hand side of (5.4) can be bounded by the $L^\infty$-norms of its factors, which are in turn bounded by constant $C_{T,p,\|\mathbf{m}_0\|_{H^2}}$. Hence we have

$$\|\partial_t^{p+1}\mathbf{m}\|_{L^2(\Omega)} \le C_{T,p,\|\mathbf{m}_0\|_{H^2}}. \tag{5.5}$$

5.2. **Estimation of $\|\partial_t^{p+1}\nabla\mathbf{m}\|_{L^2(\Omega)}$.** Taking $\partial_l = \partial_{x_l}$ on equation (5.1) yields

$$\begin{aligned}\partial_l \mathbf{m}_t &= \gamma\partial_l(\mathbf{m}\times H_{\mathbf{m}}) + \gamma\alpha\partial_l(\mathbf{m}\times(\mathbf{m}\times H_{\mathbf{m}}))\\ &= \gamma\mathbf{m}_{x_l}\times H_{\mathbf{m}} + \gamma\mathbf{m}\times(H_{\mathbf{m}})_{x_l} + \gamma\alpha\mathbf{m}\times(\mathbf{m}_{x_l}\times H_{\mathbf{m}})\\ &\quad + \gamma\alpha\mathbf{m}\times(\mathbf{m}\times(H_{\mathbf{m}})_{x_l}) + \gamma\alpha\mathbf{m}_{x_l}\times(\mathbf{m}\times H_{\mathbf{m}})\end{aligned} \tag{5.6}$$

Taking $p^{th}$ derivative with respect to $t$ on Eq. (5.6) yields

$$\begin{aligned}\partial_t^{p+1}\mathbf{m}_{x_l} =& \gamma\sum_{i+j=p}\partial_t^i\mathbf{m}_{x_l}\times\partial_t^j H_{\mathbf{m}} + \gamma\sum_{i+j=p}\partial_t^i\mathbf{m}\times\partial_t^j(H_{\mathbf{m}})_{x_l}\\ &\gamma\alpha\sum_{i+j+k=p}\partial_t^i\mathbf{m}\times(\partial_t^j\mathbf{m}\times\partial_t^k(H_{\mathbf{m}})_{x_l})\\ &+\gamma\alpha\sum_{i+j+k=p}\partial_t^i\mathbf{m}\times(\partial_t^j\mathbf{m}_{x_l}\times\partial_t^k H_{\mathbf{m}})\\ &+\gamma\alpha\sum_{i+j+k=p}\partial_t^i\mathbf{m}_{x_l}\times(\partial_t^j\mathbf{m}\times\partial_t^k H_{\mathbf{m}})\end{aligned}$$

In order to estimate $\|\partial_t^{p+1}\mathbf{m}_{x_l}\|_{L^2(\Omega)}$, it is sufficient to estimate the following $L^2$-norms:

$$\begin{aligned}&\|\sum_{i+j=p}\partial_t^i\mathbf{m}_{x_l}\times\partial_t^j H_{\mathbf{m}}\|_{L^2(\Omega)}.\\ &\|\sum_{i+j=p}\partial_t^i\mathbf{m}\times\partial_t^j(H_{\mathbf{m}})_{x_l}\|_{L^2(\Omega)}.\\ &\|\sum_{i+j+k=p}\partial_t^i\mathbf{m}\times(\partial_t^j\mathbf{m}\times\partial_t^k(H_{\mathbf{m}})_{x_l})\|_{L^2(\Omega)}.\\ &\|\sum_{i+j+k=p}\partial_t^i\mathbf{m}\times(\partial_t^j\mathbf{m}_{x_l}\times\partial_t^k H_{\mathbf{m}})\|_{L^2(\Omega)}.\\ &\|\sum_{i+j+k=p}\partial_t^i\mathbf{m}_{x_l}\times(\partial_t^j\mathbf{m}\times\partial_t^k H_{\mathbf{m}})\|_{L^2(\Omega)}.\end{aligned}$$

All these norms can be estimated in the same way: For each of the individual cross-product integrands, use the $L^2$-norm of a sole factor with $x_l$-derivative and use the $L^\infty$-norms for the other factor or factors. All these norms can be bounded by constant $C_{T,p,\|\mathbf{m}_0\|_{H^2}} < \infty$. Finally, summing up $l = 1,2,3$, we have proved

$$\|\partial_t^{p+1}\nabla\mathbf{m}\|_{L^2(\Omega)} \le C_{T,p,\|\mathbf{m}_0\|_{H^2}} < \infty. \tag{5.7}$$

5.3. **Estimation of $\|\partial_t^{p+1}\triangle\mathbf{m}\|_{L^2(\Omega)}$.** Differentiating (5.6) with respect to $x_l$ and summing up over $l = 1,2,3$ yields that

$$\begin{aligned}\triangle\mathbf{m}_t &= \gamma\triangle\mathbf{m}\times H_{\mathbf{m}} + \gamma\mathbf{m}\times\triangle H_{\mathbf{m}} + \gamma\sum_l\mathbf{m}_{x_l}\times(H_{\mathbf{m}})_{x_l}\\ &\quad+\gamma\alpha[\triangle\mathbf{m}\times(\mathbf{m}\times H_{\mathbf{m}}) + \mathbf{m}\times(\triangle\mathbf{m}\times H_{\mathbf{m}}) + \mathbf{m}\times(\mathbf{m}\times\triangle H_{\mathbf{m}})]\\ &\quad+\gamma\alpha\sum_l[\mathbf{m}_{x_l}\times(\mathbf{m}_{x_l}\times H_{\mathbf{m}}) + \mathbf{m}_{x_l}\times(\mathbf{m}\times(H_{\mathbf{m}})_{x_l}) + \mathbf{m}\times(\mathbf{m}_{x_l}\times(H_{\mathbf{m}})_{x_l})].\end{aligned} \tag{5.8}$$

Differentiating equation (5.8) $p$ times with respect to $t$ will yield a formula for $\partial_t^{p+1}\triangle\mathbf{m}$. To estimate $\|\partial_t^{p+1}\triangle\mathbf{m}\|_{L^2(\Omega)}$, we do not need to estimate every single term because lots of them are similar; it is sufficient to estimate the following 4 $L^2$-norms:

$$\| \sum_{i+j=p} \partial_t^i \triangle \mathbf{m} \times \partial_t^j H_{\mathbf{m}} \|_{L^2(\Omega)}. \tag{5.9}$$

$$\| \sum_{i+j=p} \partial_t^i \mathbf{m}_{x_l} \times \partial_t^j (H_{\mathbf{m}})_{x_l} \|_{L^2(\Omega)}. \tag{5.10}$$

$$\| \sum_{i+j+k=p} \partial_t^i \triangle \mathbf{m} \times (\partial_t^j \mathbf{m} \times \partial_t^k H_{\mathbf{m}}) \|_{L^2(\Omega)}. \tag{5.11}$$

$$\| \sum_{i+j+k=p} \partial_t^i \mathbf{m}_{x_l} \times (\partial_t^j \mathbf{m}_{x_l} \times \partial_t^k H_{\mathbf{m}}) \|_{L^2(\Omega)}. \tag{5.12}$$

For (5.9), we use

$$\| \partial_t^i \triangle \mathbf{m} \times \partial_t^j H_{\mathbf{m}} \|_{L^2(\Omega)} \leq \| \partial_t^j H_{\mathbf{m}} \|_{L^\infty} \| \partial_t^i \triangle \mathbf{m} \|_{L^2(\Omega)}.$$

For (5.10), we use

$$\| \partial_t^i \mathbf{m}_{x_l} \times \partial_t^j (H_{\mathbf{m}})_{x_l} \|_{L^2(\Omega)} \leq \| \partial_t^i \nabla \mathbf{m} \|_{L^4(\Omega)} \| \partial_t^j \nabla H_{\mathbf{m}} \|_{L^4(\Omega)}.$$

For (5.11), we use

$$\| \partial_t^i \triangle \mathbf{m} \times (\partial_t^j \mathbf{m} \times \partial_t^k H_{\mathbf{m}}) \|_{L^2(\Omega)} \leq \| \partial_t^j \mathbf{m} \|_{L^\infty} \| \partial_t^k H_{\mathbf{m}} \|_{L^\infty} \| \partial_t^i \triangle \mathbf{m} \|_{L^2(\Omega)}.$$

For (5.12), we use

$$\| \partial_t^i \mathbf{m}_{x_l} \times (\partial_t^j \mathbf{m}_{x_l} \times \partial_t^k H_{\mathbf{m}}) \|_{L^2} \leq \| \partial_t^k H_{\mathbf{m}} \|_{L^\infty} \| \partial_t^i \nabla \mathbf{m} \|_{L^4} \| \partial_t^j \nabla \mathbf{m} \|_{L^4}.$$

Finally, from these estimates, we obtain

$$\| \partial_t^{p+1} \triangle \mathbf{m} \|_{L^2(\Omega)} \leq C_{T,p,\|\mathbf{m}_0\|_{H^2}} < \infty. \tag{5.13}$$

Combining (5.5), (5.7) and (5.13), we have shown that

$$\| \partial_t^{p+1} \mathbf{m} \|_{H^2(\Omega)} \leq C_{T,p,\|\mathbf{m}_0\|_{H^2}} < \infty.$$

This completes the induction process and hence the proof. □

**Remark 4.** Theorem 5.1 is also valid for the general equation (3.10) with smooth functions $\varphi(\mathbf{m})$ and $\mathbf{a}(x)$; the proof should be similar.

## 6. **Energy identity and weak $\omega$-limit sets.**

We first prove an energy identity for global weak solutions to the Landau-Lifshitz equation (1.3). We write the initial value problem as

$$\begin{cases} \mathbf{m}_t = \mathbb{L}(\mathbf{m}, \mathbf{H}_{\text{eff}}) & \text{in } \Omega \times (0, \infty), \\ \mathbf{m}(0) = \mathbf{m}_0, \end{cases} \tag{6.1}$$

in terms of the Landau-Lifshitz interaction function $\mathbb{L}$ defined by (1.8), where the effective magnetic field $\mathbf{H}_{\text{eff}}$ is given by (1.4).

6.1. **The energy identity.** Let $\mathcal{E}(\mathbf{m})$ be defined by (1.1). Assume $\mathbf{a}, \mathbf{m}_0 \in L^\infty(\Omega; \mathbf{R}^3)$.

**Theorem 6.1.** *The global weak solution* $\mathbf{m}$ *to* (6.1) *satisfies the energy identity*

$$\mathcal{E}(\mathbf{m}(t)) - \mathcal{E}(\mathbf{m}(s)) = \gamma\alpha \int_s^t \int_\Omega |\mathbf{m} \times \mathbf{H}_{\text{eff}}|^2 \, dx d\tau \quad \forall\, 0 \le s \le t < \infty. \tag{6.2}$$

*Furthermore, if* $\gamma\alpha < 0$*, then* $\mathbf{m}_t \in L^2((0,\infty); L^2(\Omega; \mathbf{R}^3))$.

*Proof.* Note that

$$\mathbb{L}(\mathbf{m}, \mathbf{n}) \cdot \mathbf{n} = -\alpha\gamma |\mathbf{m} \times \mathbf{n}|^2 \quad \forall\, \mathbf{m}, \mathbf{n} \in \mathbf{R}^3. \tag{6.3}$$

By the definition of $\mathbf{H}_{\text{eff}} = -\frac{\partial \mathcal{E}}{\partial \mathbf{m}}$ in the $L^2$ sense, it follows that

$$\frac{d}{dt}(\mathcal{E}(\mathbf{m}(t))) = -\int_\Omega \mathbf{H}_{\text{eff}} \cdot \mathbf{m}_t \, dx = \gamma\alpha \int_\Omega |\mathbf{m} \times \mathbf{H}_{\text{eff}}|^2 \, dx$$

for a.e. $t \in (0, \infty)$. Hence (6.2) follows.

If $\gamma\alpha < 0$, by (6.2), one has $\int_0^\infty \int_\Omega |\mathbf{m} \times \mathbf{H}_{\text{eff}}|^2 \, dx dt \le \mathcal{E}(\mathbf{m}_0)/|\alpha\gamma| < \infty$. Also from the equation (6.1),

$$\begin{aligned} |\mathbf{m}_t|^2 = & |\mathbb{L}(\mathbf{m}, \mathbf{H}_{\text{eff}})|^2 = \gamma^2 |\mathbf{m} \times \mathbf{H}_{\text{eff}}|^2 \\ & + (\gamma\alpha)^2 |\mathbf{m}|^2 |\mathbf{m} \times \mathbf{H}_{\text{eff}}|^2 \le C\, |\mathbf{m} \times \mathbf{H}_{\text{eff}}|^2, \end{aligned}$$

where constant $C$ depends on $\|\mathbf{m}_0\|_{L^\infty}$. Hence $\mathbf{m}_t \in L^2((0,\infty); L^2(\Omega; \mathbf{R}^3))$. □

6.2. **Weak $\omega$-limit sets and estimation for the soft-case.** The stability theorem and all the regularity estimates previously established for (6.1) are for *finite time*; the only global-in-time regularity for the solutions (even for the regular solutions) is that

$$\mathbf{m} \in L^\infty((0,\infty); L^\infty(\Omega; \mathbf{R}^3)) \quad \text{with} \quad \mathbf{m}_t \in L^2((0,\infty); L^2(\Omega; \mathbf{R}^3)).$$

But this regularity is not enough to have strong convergence as $t \to \infty$; it would be enough if one has $\mathbf{m}_t \in L^1((0,\infty); L^2(\Omega; \mathbf{R}^3))$ (see [20]). Therefore, it is quite challenging to study the asymptotic behaviors of even regular solutions. The solution orbits for general initial data may not have strong $\omega$-limit points; we thus study the weak $\omega$-limit points.

Given $\mathbf{m}_0 \in L^\infty(\Omega; \mathbf{R}^3)$, let $\mathbf{m}$ be the global weak solution to the initial value problem (6.1) and define the *weak $\omega$-limit set* for $\mathbf{m}$ to be

$$\omega^*(\mathbf{m}_0) = \{\tilde{\mathbf{m}} \mid \exists\, t_j \uparrow \infty \text{ such that } \mathbf{m}(t_j) \rightharpoonup \tilde{\mathbf{m}} \text{ weakly in } L^2(\Omega; \mathbf{R}^3)\}. \tag{6.4}$$

We give an estimate of $\omega^*(\mathbf{m}_0)$ for the so-called *soft-case*, where there is no anisotropy energy ($\varphi = 0$). For more results on further special case when $\mathbf{a} = 0$, see [32, 33].

**Theorem 6.2.** *Let* $\gamma\alpha < 0$, $\varphi = 0$ *and* $\mathbf{a} \in L^\infty(\Omega; \mathbf{R}^3)$*. Then, for any* $\mathbf{m}_0 \in L^\infty(\Omega; \mathbf{R}^3)$ *with* $|\mathbf{m}_0(x)| = 1$ *a.e. on* $\Omega$*, it follows that*

$$\omega^*(\mathbf{m}_0) \subseteq \{\tilde{\mathbf{m}} \in L^\infty(\Omega; \mathbf{R}^3) \mid |\tilde{\mathbf{m}}|^2 + 2|\tilde{\mathbf{m}} \times (\mathbf{a} + H_{\tilde{\mathbf{m}}})| \le 1 \text{ a.e. on } \Omega\}. \tag{6.5}$$

*Proof.* Let $\mathbf{m}$ be the global weak solution to (6.1) with the given initial datum $\mathbf{m}_0$. Then $|\mathbf{m}(t)| = 1$ a.e. on $\Omega$ for all $t \ge 0$. Assume $\mathbf{m}(t_j) \rightharpoonup \tilde{\mathbf{m}}$ weakly in $L^2(\Omega; \mathbf{R}^3)$ for a sequence $t_j \uparrow \infty$. In the following, we show that

$$|\tilde{\mathbf{m}}|^2 + 2|\tilde{\mathbf{m}} \times (\mathbf{a} + H_{\tilde{\mathbf{m}}})| \le 1 \text{ a.e. on } \Omega. \tag{6.6}$$

Let $e(t) = \mathcal{E}(\mathbf{m}(t))$. Then, by (6.2), $e(t)$ is non-increasing and bounded and hence $e(t)$ has limit as $t \to \infty$; this again by (6.2) implies

$$e(t_j + 1) - e(t_j) = \gamma\alpha \int_{t_j}^{t_j+1} \|\mathbf{m}(t) \times (\mathbf{a} + H_{\mathbf{m}(t)})\|_{L^2}^2 \, dt \to 0.$$

Hence there exists some $s_j \in [t_j, t_j + 1]$ such that

$$\|\mathbf{m}(s_j) \times (\mathbf{a} + H_{\mathbf{m}(s_j)})\|_{L^2(\Omega)} \to 0. \tag{6.7}$$

By Theorem 6.1, $\mathbf{m}_t \in L^2((0,\infty); L^2(\Omega; \mathbf{R}^3))$; hence

$$\|\mathbf{m}(s_j) - \mathbf{m}(t_j)\|_{L^2} \le \int_{t_j}^{s_j} \|\mathbf{m}_t(t)\|_{L^2} dt \le (s_j - t_j)^{\frac{1}{2}} \left( \int_{t_j}^{s_j} \|\mathbf{m}_t\|_{L^2}^2 dt \right)^{\frac{1}{2}} \to 0,$$

which yields $\mathbf{m}(s_j) \rightharpoonup \tilde{\mathbf{m}}$ weakly in $L^2(\Omega; \mathbf{R}^3)$. Therefore, by (6.7), (6.6) follows from the following proposition with $\mathbf{m}_j = \mathbf{m}(s_j)$. This completes the proof. □

**Proposition 6.3.** *Let $\mathbf{m}_j \rightharpoonup \tilde{\mathbf{m}}$ weakly in $L^2(\Omega; \mathbf{R}^3)$ and satisfy*

$$(a) \quad |\mathbf{m}_j| = 1 \ a.e.\, \Omega; \quad (b) \quad \|\mathbf{m}_j \times (\mathbf{a} + H_{\mathbf{m}_j})\|_{L^2(\Omega)} \to 0.$$

*Then $\tilde{\mathbf{m}}$ satisfies the condition (6.6) above.*

*Proof.* This result can be proved by a similar method of [32, Theorem 1.1]. However, we present a different but direct proof based on the div-curl lemma [29].

For any $\mathbf{m} \in L^\infty(\Omega; \mathbf{R}^3)$, let $G_{\mathbf{m}} = \mathbf{m}\chi_\Omega + H_{\mathbf{m}}$. Then $\operatorname{div} G_{\mathbf{m}} = 0$ on $\mathbf{R}^3$. Denote

$$G_j = \mathbf{a} + G_{\mathbf{m}_j}, \ H_j = \mathbf{a} + H_{\mathbf{m}_j}; \quad \tilde{G} = \mathbf{a} + G_{\tilde{\mathbf{m}}}, \ \tilde{H} = \mathbf{a} + H_{\tilde{\mathbf{m}}}.$$

Then $G_j \rightharpoonup \tilde{G}$, $H_j \rightharpoonup \tilde{H}$ weakly in $L^2(\Omega; \mathbf{R}^3)$ and, by the div-curl lemma [29],

$$\int_\Omega G_j \cdot H_j \phi \, dx \to \int_\Omega \tilde{G} \cdot \tilde{H} \phi \, dx \quad \forall \ \phi \in C_0^\infty(\Omega). \tag{6.8}$$

Since $\mathbf{m}_j = G_j - H_j$ on $\Omega$, it follows that

$$\begin{aligned} |\mathbf{m}_j|^2 + 2|\mathbf{m}_j \times (\mathbf{a} + H_{\mathbf{m}_j})| =& |G_j - H_j|^2 + 2|G_j \times H_j| \\ =& |G_j|^2 + |H_j|^2 + 2|G_j \times H_j| - 2G_j \cdot H_j. \end{aligned} \tag{6.9}$$

Note that function $f(\mathbf{m}, \mathbf{n}) = |\mathbf{m}|^2 + |\mathbf{n}|^2 + 2|\mathbf{m} \times \mathbf{n}|$ is convex on $(\mathbf{m}, \mathbf{n}) \in \mathbf{R}^3 \times \mathbf{R}^3$. Hence, for all $\phi \in C_0^\infty(\Omega)$ with $\phi \ge 0$, one has

$$\liminf_{j\to\infty} \int_\Omega (|G_j|^2 + |H_j|^2 + 2|G_j \times H_j|)\phi \ge \int_\Omega (|\tilde{G}|^2 + |\tilde{H}|^2 + 2|\tilde{G} \times \tilde{H}|)\phi. \tag{6.10}$$

By assumptions $(a)$, $(b)$, from (6.8)–(6.10), it follows that

$$\int_\Omega (|\tilde{G}|^2 + |\tilde{H}|^2 + 2|\tilde{G} \times \tilde{H}| - 2\tilde{G} \cdot \tilde{H})\phi \, dx \le \int_\Omega \phi \, dx$$

for all $\phi \in C_0^\infty(\Omega)$ with $\phi \ge 0$. This implies

$$|\tilde{G}|^2 + |\tilde{H}|^2 + 2|\tilde{G} \times \tilde{H}| - 2\tilde{G} \cdot \tilde{H} \le 1 \quad a.e.\ \Omega,$$

which, exactly, is equivalent to (6.6). This completes the proof. □

7. **A special dynamics.** In this final section, we study a special case of (6.1) where applied field $\mathbf{a}(x) = \mathbf{a}$ is constant, domain $\Omega$ is an ellipsoid, and initial datum $\mathbf{m}_0$ is a constant unit vector. Therefore, in (6.1), the effective magnetic field $\mathbf{H}_{\text{eff}}$ is now given by

$$\mathbf{H}_{\text{eff}} = -\varphi'(\mathbf{m}) + \mathbf{a} + H_{\mathbf{m}}$$

as above, but with constant vector $\mathbf{a}$. In what follows, we assume ellipsoid domain $\Omega$ is given by

$$\Omega = \{x \in \mathbf{R}^3 \mid \sum_{i=1}^{3} x_i^2/a_i < 1\},$$

where $a_i > 0$ are constants.

7.1. **The associated ODE system on $\mathbf{R}^3$.** It is well-known that (see, e.g., [27]), for the ellipsoid domain $\Omega$ given as above, the magneto-static stray field $H_{\mathbf{m}}$ induced by any *constant* field $\mathbf{m}$ has constant value on $\Omega$ given by

$$H_{\mathbf{m}}|_{\Omega} = -\Lambda\mathbf{m} \quad (\forall\ \mathbf{m} \in \mathbf{R}^3), \tag{7.1}$$

where $\Lambda$ is a diagonal matrix of positive numbers. In fact, $\Lambda = \operatorname{diag}(b_1, b_2, b_3)$ with

$$b_i = \frac{1}{2}\int_0^\infty \frac{\sqrt{a_1a_2a_3}\,dt}{(a_i+t)\sqrt{(a_1+t)(a_2+t)(a_3+t)}}. \tag{7.2}$$

Note that, if $\Omega$ is a ball in $\mathbf{R}^3$, all $b_i$'s are equal to $1/3$.

Let $\mathbf{m}_0$ be a constant unit vector. Then, from (7.1) and the uniqueness of solution, problem (6.1) reduces to the following ODE system on $\mathbf{m} \in \mathbf{R}^3$:

$$\begin{cases} \dot{\mathbf{m}} = \Phi(\mathbf{m}), & t > 0, \\ \mathbf{m}(0) = \mathbf{m}_0, \end{cases} \tag{7.3}$$

where $\dot{\mathbf{m}} = \frac{d\mathbf{m}}{dt}$ and function $\Phi\colon \mathbf{R}^3 \to \mathbf{R}^3$ is defined by

$$\Phi(\mathbf{m}) = \mathbb{L}(\mathbf{m}, -\varphi'(\mathbf{m}) + \mathbf{a} - \Lambda\mathbf{m}), \quad \forall\ \mathbf{m} \in \mathbf{R}^3. \tag{7.4}$$

Since $\mathbf{m}\cdot\Phi(\mathbf{m}) = 0$, system (7.3) also preserves the length of $\mathbf{m}(t)$. Thus we have $|\mathbf{m}(t)| = 1$ for all $t \geq 0$. Moreover, $\mathbb{L}(\mathbf{m},\mathbf{n}) = 0$ if and only if $\mathbf{m} \times \mathbf{n} = 0$; hence, the equilibrium points of (7.3), that is, the solutions of $\Phi(\mathbf{m}) = 0$ on unit sphere $|\mathbf{m}| = 1$, are characterized by vectors $\mathbf{m} \in \mathbf{R}^3$ for which there is a real number $\lambda \in \mathbf{R}$ such that

$$-\varphi'(\mathbf{m}) + \mathbf{a} - \Lambda\mathbf{m} = \lambda\mathbf{m}, \quad |\mathbf{m}| = 1. \tag{7.5}$$

This condition is equivalent to $\mathbf{m}$ being a critical point of the function

$$P(\mathbf{m}) = \frac{1}{2}\Lambda\mathbf{m}\cdot\mathbf{m} - \mathbf{a}\cdot\mathbf{m} + \varphi(\mathbf{m}) \tag{7.6}$$

on unit sphere $|\mathbf{m}| = 1$.

In most cases, there will be at least two distinct equilibrium points for system (7.3); for example, all maximum or minimum points of $P$ on $|\mathbf{m}| = 1$ (always exist) are such points.

7.2. **Special dynamics.** The dynamics of system (7.3) can be studied by the classical ODE theory. For example, we have the following result.

**Theorem 7.1.** *Function $P$ defined by (7.6) is a Lyapunov function for system (7.3). Assume $\gamma\alpha < 0$. The $\omega$-limit set of (7.3) for any initial unit vector $\mathbf{m}_0 \in \mathbf{R}^3$ is contained in the set of all critical points of $P$ on unit sphere.*

*Proof.* Since $\mathbf{H}_{\text{eff}} = -\varphi'(\mathbf{m}) + \mathbf{a} - \Lambda\mathbf{m} = -\nabla P(\mathbf{m})$, $\forall\ \mathbf{m} \in \mathbf{R}^3$, it follows that for solution $\mathbf{m} = \mathbf{m}(t)$ of (7.3), by (6.3),

$$\frac{d}{dt}P(\mathbf{m}(t)) = \nabla P(\mathbf{m}) \cdot \dot{\mathbf{m}} = -\mathbf{H}_{\text{eff}} \cdot \mathbb{L}(\mathbf{m}, \mathbf{H}_{\text{eff}}) = \gamma\alpha|\mathbf{m} \times \mathbf{H}_{\text{eff}}|^2 \leq 0.$$

Hence $P$ is a Lyapunov function for system (7.3).

To show the second part of the theorem, assume $\gamma\alpha < 0$ and $\mathbf{m}(t_j) \to \bar{\mathbf{m}}$ for a sequence $t_j \uparrow \infty$. Let $p(t) = P(\mathbf{m}(t))$. Then $p(t)$ is smooth, non-increasing and has limit as $t \to \infty$. Hence $p(t_j+1) - p(t_j) = p'(s_j) \to 0$ for some $s_j \in (t_j, t_j+1)$. Since $p'(t) = \gamma\alpha|\mathbf{m} \times \mathbf{H}_{\text{eff}}|^2$ and $\gamma\alpha < 0$, this implies

$$|\mathbf{m}(s_j) \times \mathbf{H}_{\text{eff}}(s_j)| = |\mathbf{m}(s_j) \times \nabla P(\mathbf{m}(s_j))| \to 0. \tag{7.7}$$

As above, since $\dot{\mathbf{m}} \in L^2(0,\infty)$, one has

$$|\mathbf{m}(s_j) - \mathbf{m}(t_j)| \leq \int_{t_j}^{s_j} |\dot{\mathbf{m}}|dt \leq (s_j - t_j)^{\frac{1}{2}} \left( \int_{t_j}^{s_j} |\dot{\mathbf{m}}|^2 dt \right)^{\frac{1}{2}} \to 0.$$

This implies $\mathbf{m}(s_j) \to \bar{\mathbf{m}}$; hence, by (7.7), $|\bar{\mathbf{m}} \times \nabla P(\bar{\mathbf{m}})| = 0$, which proves $\bar{\mathbf{m}}$ is a critical point of $P$ on unit sphere. This completes the proof. □

Finally, we give a special result for $\mathbf{a} = 0$ and $\varphi = 0$.

**Proposition 7.2.** *Let $b_1, b_2, b_3$ be positive numbers determined by (7.2). If $b_k = \min\{b_1, b_2, b_3\}$, then $\pm\mathbf{e}_k$ are asymptotically stable equilibrium points for the system (7.3), where $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ are the standard basis vectors of $\mathbf{R}^3$.*

*Proof.* Without loss of generality, let us assume $b_1 = \min\{b_1, b_2, b_3\}$. It is trivial to see that $P$ has a strict relative minimum at $\pm\mathbf{e}_1$. According to the Lyapunov stability theorem, $\pm\mathbf{e}_1$ are asymptotically stable equilibrium points for (7.3). □

**Acknowledgments.** We would like to thank the anonymous referee for many helpful suggestions, and thank the organizers of the 2012 SIAM mini-symposium in honor of Walter Littman, to whom we send our best wishes on this occasion. The second author also gratefully acknowledges the support of Tang Aoqing visiting professorship from College of Mathematics, Jilin University, China, for several visits.

## REFERENCES

[1] F. Alouges and A. Soyeur, On global weak solutions for Landau-Lifshitz equations: Existence and Nonuniqueness, *Nonlinear Analysis, TMA,* **18** (1992), 1071–1084.

[2] J. M. Ball, A. Taheri and M. Winter, Local minimizers in micromagnetics and related problems, *Calc. Var.,* **14** (2002), 1–27.

[3] M. Bertsch, P. Podio-Guidugli and V. Valente, On the dynamics of deformable ferromagnets. I. Global weak solutions for soft ferromagnets at rest, *Ann. Mat. Pura Appl.,* **179** (2001), 331–360.

[4] W. F. Brown, *Micromagnetics,* Interscience, New York, 1963.

[5] A. Capella, C. Melcher and F. Otto, Effective dynamic in ferromagnetic thin films and the motion of Neel walls, *Nonlinearity,* **20** (2007), 2519–2537.

[6] I. Cimrak and R. V. Keer, Higher order regularity results in 3D for the Landau-Lifshitz equation with an exchange field, *Nonlinear Analysis,* **68** (2008), 1316–1331.

[7] G. Carbou and P. Fabrie, Time average in micromagnetics, *J. Diff. Equations,* **147** (1998), 383–409.

[8] G. Carbou and P. Fabrie, Regular solutions for Landau-Lifshitz equation in a bounded domain, *Diff. Int. Equations,* **14** (2001), 213–229.

[9] B. Dacorogna and I. Fonseca, A-B quasiconvexity and implicit partial differential equations, *Calc. Var. Partial Differential Equations,* **14** (2002), 115–149.

[10] W. Deng and B. Yan, Quasi-stationary limit and a degenerate Landau-Lifshitz equation of ferromagnetism, *Applied Mathematics Research Express,* **2013**(2) (2013), 277–296.

[11] A. DeSimone, Energy minimizers for large ferromagnetic bodies, *Arch. Rational Mech. Anal.,* **125** (1993), 99–143.

[12] A. DeSimone, R. V. Kohn, S. Müller and F. Otto, A reduced theory for thin-film micromagnetics, *Comm. Pure Appl. Math.,* **55** (2002), 1408–1460.

[13] A. DeSimone, R. V. Kohn, S. Müller and F. Otto, Recent analytical developments in micromagnetics, in *The Science of Hysteresis,* (eds G Bertotti and I Mayergoyz), Elsevier Academic Press, New York, **2** (2005), 269–381.

[14] M. Fabrizio, C. Giorgi and A. Morro, A thermodynamic approach to ferromagnetism and phase transitions, *Int. J. Engineering Science,* **47** (2009), 821–839.

[15] B. Guo and M. Hong, The Landau-Lifshitz equation of the ferromagnetic spin chain and harmonic maps, *Calc. Var. Partial Differential Equations,* **1** (1993), 311–334.

[16] R. James and D. Kinderlehrer, Frustration in ferromagnetic materials, *Cont. Mech. Thermodyn.,* **2** (1990), 215–239.

[17] F. Jochmann, Existence of solutions and a quasi-stationary limit for a hyperbolic system describing ferromagnetism, *SIAM J. Math. Anal.,* **34** (2002), 315–340.

[18] F. Jochmann, Aysmptotic behavior of the electromagnetic field for a micromagnetism equation without exchange energy, *SIAM J. Math. Anal.,* **37** (2005), 276–290.

[19] J. L. Joly, G. Metivier and J. Rauch, Global solutions to Maxwell equations in a ferromagnetic medium, *Ann. Henri Poincaré,* **1** (2000), 307–340.

[20] P. Joly, A. Komech and O. Vacus, On transitions to stationary states in a Maxwell-Landau-Lifshitz-Gilbert system, *SIAM J. Math. Anal.,* **31** (1999), 346–374.

[21] M. Kruzík and A. Prohl, Recent developments in the modeling, analysis, and numerics of ferromagnetism, *SIAM Review,* **48** (2006), 439–483.

[22] L. Landau and E. Lifshitz, On the theory of the dispersion of magnetic permeability of ferromagnetic bodies, *Phys. Z. Sowj.,* **8** (1935), 153–169.

[23] L. Landau, E. Lifshitz and L. Pitaevskii, *Electrodynamics of Continuous Media,* Pergamon Press, New York, 1984.

[24] C. Melcher, Thin-film limits for Landau-Lifshitz-Gilbert equations, *SIAM J. Math. Anal.,* **42** (2010) , 519–537.

[25] R. Moser, Boundary vortices for thin ferromagnetic films, *Arch. Rational Mech. Anal.,* **174** (2004), 267–300.

[26] P. Pedregal and B. Yan, On two-dimensional ferromagnetism, *Proc. R. Soc. Edinburgh,* **139A** (2009), 575–594.

[27] P. Pedregal and B. Yan, A duality method for micromagnetics, *SIAM J. Math. Anal.,* **41** (2010), 2431–2452.

[28] E. Stein, *Singular Integrals and Differentiability Properties of Functions,* Princeton Mathematical Series, No. 30 Princeton University Press, Princeton, N.J. 1970 xiv+290 pp.

[29] L. Tartar, The compensated compactness method applied to systems of conservation laws, in *Systems of Nonlinear Partial Differential Equations,* (J. M. Ball ed.), NATO ASI Series, Vol. CIII, D. Reidel, (1983), 263–285.

[30] A. Visintin, On Landau-Lifshitz equation for ferromagnetism, *Japan J. Appl. Math.,* **2** (1985), 69–84.

[31] B. Yan, Characterization of energy minimizers in micromagnetics, *J. Math. Anal. Appl.,* **374** (2011), 230–243.

[32] B. Yan, On the equilibrium set of magnetostatic energy by differential inclusion, *Calc. Var. Partial Differential Equations,* **47** (2013), 547–565.

[33] B. Yan, On stability and asymptotic behaviors for a degenerate Landau-Lifshitz equation, Preprint submitted.

Received September 2012; revised February 2013.

*E-mail address*: dengwei@msu.edu;yan@math.msu.edu