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MTH 849 – Partial Differential Equations

(based on L.C. Evans's textbook)

by

Baisheng Yan

Department of Mathematics
Michigan State University
yan@math.msu.edu

Contents

Single First Order Equations

1.1. Transport Equation

$$u_t + B \cdot Du(x,t) = 0,$$

where $B = (b_1, \dots, b_n)$ is a constant vector in $\mathbb{R}^n$.

(a) ($n = 1$) Find all solutions to $u_t + cu_x = 0$.

Geometric method: (u_x, u_t) is perpendicular to $(c, 1)$. The directional derivative along the direction $(c, 1)$ is zero, hence the function along the straight line $x = ct + d$ is constant. i.e., $u(x,t) = f(d) = f(x - ct)$. Here $x - ct = d$ is called a characteristic line.

Coordinate method: Change variable. $x_1 = x - ct$, $x_2 = cx + t$, then $u_x = u_{x_1} + u_{x_2}c$, $u_t = u_{x_1}(-c) + u_{x_2}$, hence $u_t + cu_x = u_{x_2}(1 + c^2) = 0$; i.e., $u(x,t) = f(x_1) = f(x - ct)$.

(b) (general n) Let us consider the initial value problem

$$\begin{cases} u_t + B \cdot Du(x,t) = 0 & \text{in } \mathbb{R}^n \times (0, \infty), \\ u(x,0) = g(x). \end{cases}$$

As in (a), given a point (x,t), the line through (x,t) with the direction $(B, 1)$ is represented parametrically by $(x + Bs, t + s)$, $\quad s \in \mathbb{R}$. This line hits the plane $t = 0$ when $s = -t$ at $(x - Bt, 0)$. Since

$$\frac{d}{ds} u(x + Bs, t + s) = Du \cdot B + u_t = 0$$

and hence u is constant on the line, we have

$$u(x,t) = u(x - Bt, 0) = g(x - Bt).$$

If $g \in C^1$, then u is a classical solution. But if g is not in C^1, u is not a classical solution, but it is a weak solution as we will see later.

(c) Non-homogeneous problem

$$\begin{cases} u_t + B \cdot Du(x,t) = f(x,t) & \text{in } \mathbb{R}^n \times (0, \infty) \\ u(x,0) = g(x). \end{cases}$$

This problem can be solved as in part (b). Let $z(s) = u(x + Bs, t + s)$. Then $\dot{z} = f(x + Bs, t + s)$. Hence

$$u(x,t) = z(0) = z(-t) + \int_{-t}^{0} z'(s)ds = g(x - Bt) + \int_0^t f(x + (s-t)B, s)ds.$$

1.2. Linear first order equation

$$\begin{cases} a(x,t)u_t(x,t) + b(x,t)u_x(x,t) = c(x,t)u + d(x,t) & \text{in } \mathbb{R} \times (0,\infty) \\ u(x,0) = g(x). \end{cases}$$

The idea is to find a curve $(x(s), t(s))$ so that the values of $z(s) = u(x(s), t(s))$ on this curve can be calculated easily. Note that

$$\frac{d}{ds} z(s) = u_x \dot{x} + u_t \dot{t}.$$

We see that if $x(s)$, $t(s)$ satisfy

$$\begin{cases} \dot{x} = a(x,t), \\ \dot{t} = b(x,t), \end{cases} \tag{1.1}$$

which is called the **characteristic ODEs** for the PDE, then we would have

$$\frac{d}{ds} z(s) = c(x,t)z(s) + d(x,t). \tag{1.2}$$

This linear ODE in z along with the ODE (1.1) can be solved easily if $x(0) = \tau$, $t(0) = 0$ and $z(0) = g(\tau)$ are given as initial data at $s = 0$, with $\tau \in \mathbb{R}$ being a parameter. Let $x(\tau, s), t(\tau, s)$ and $z(\tau, s)$ be the solutions.

Finally, if we can solve (τ, s) in terms of (x,t) from $x(\tau, s)$ and $t(\tau, s)$, then we plug into $z(\tau, s)$ and obtain the solution defined by $u(x,t) = z(\tau, s)$.

Example 1.1. Solve the initial value problem

$$u_t + xu_x = u, \quad u(x,0) = x^2.$$

Solution: The characteristic ODEs with the initial data are given by

$$\begin{cases} \dot{x} = x, & x(0) = \tau, \\ \dot{t} = 1, & t(0) = 0, \\ \dot{z} = z, & z(0) = \tau^2. \end{cases}$$

Hence

$$x(\tau, s) = e^s \tau, \quad t(\tau, s) = s, \quad z(\tau, s) = e^s \tau^2.$$

Solve (τ, s) in terms of $(x(\tau, s), t(\tau, s))$. From the expression, we see that

$$s = t, \quad \tau = xe^{-t}.$$

Therefore,

$$u(x,t) = z(\tau, s) = \tau^2 e^s = e^{-t} x^2.$$

1.3. The Cauchy problem for general first order equations

Consider a general first-order PDE

$$F(Du, u, x) = 0, \quad x = (x_1, \dots, x_n) \in \Omega \subset \mathbb{R}^n,$$

where $F = F(p, z, x)$ is smooth in $p \in \mathbb{R}^n, z \in \mathbb{R}, x \in \mathbb{R}^n$. Let

$$D_x F = (F_{x_1}, \dots, F_{x_n}),\ D_z F = F_z,\ D_p F = (F_{p_1}, \dots, F_{p_n}).$$

The **Cauchy problem** is to find a solution $u(x)$ such that the hypersurface $z = u(x)$ in the xz-space passes a prescribed $(n-1)$-dimensional manifold S in xz-space.

Assume the projection of S onto the x-space is a smooth $(n-1)$-dimensional surface Γ. We are then concerned with finding a solution u in some domain Ω in $\mathbb{R}^n$ containing Γ with a given **Cauchy data**:

$$u(x) = g(x) \quad \text{for } x \in \Gamma.$$

Let Γ be parameterized by a parameter $y \in D \subset \mathbb{R}^{n-1}$ with $x = f(y)$. Then the Cauchy data is given by

$$u(f(y)) = g(f(y)) = h(y) \quad y \in D.$$

1.3.1. Derivation of characteristic ODE.. The method we will use is motivated from the transport equation and linear equation, and is called the **method of characteristics**. The idea is the following: To calculate $u(x)$ for some fixed point $x \in \Omega$, we need to find a curve connecting this x with a point $x_0 \in \Gamma$, along which we can compute u easily.

How do we choose the curve so that all this will work?

Suppose that u is a smooth (C^2) solution and $x(s)$ is our curve with $x(0) \in \Gamma$ defined on an interval I containing 0 as interior point. Let $z(s) = u(x(s))$ and $p(s) = Du(x(s))$; namely $p_i(s) = u_{x_i}(x(s))$ for $i = 1, 2, \cdots, n$. Hence

$$F(p(s), z(s), x(s)) = 0. \tag{1.3}$$

We now attempt to choose the function $x(s)$ so that we can compute $z(s)$ and $p(s)$. First, we differentiate $p_i(s) = u_{x_i}(x(s))$ with respect to s to get

$$\dot{p}_i(s) = \sum_{j=1}^{n} u_{x_i x_j}(x(s))\dot{x}_j(s), \quad (i = 1, \dots, n). \tag{1.4}$$

(Here and below, the "$\cdot$" means "$\frac{d}{ds}$".) This expression is not very promising since it involves the second derivatives of u. However, we can differentiate the equation $F(Du, u, x) = 0$ with respect to x_i to obtain

$$\sum_{j=1}^{n} \frac{\partial F}{\partial p_j}(Du, u, x) u_{x_i x_j} + \frac{\partial F}{\partial z}(Du, u, x) u_{x_i} + \frac{\partial F}{\partial x_i}(Du, u, x) = 0. \tag{1.5}$$

Evaluating this identity along the curve $x = x(s)$ we obtain

$$\sum_{j=1}^{n} \frac{\partial F}{\partial p_j} u_{x_i x_j}(x(s)) + \frac{\partial F}{\partial z} u_{x_i}(x(s)) + \frac{\partial F}{\partial x_i} = 0, \tag{1.6}$$

where F_{p_j}, F_z and F_{x_i} are evaluated at $(p(s), z(s), x(s))$. Suppose $x(s)$ satisfies

$$\dot{x}_j(s) = \frac{\partial F}{\partial p_j}(p(s), z(s), x(s)) \quad \forall\ j = 1, 2, \cdots, n. \tag{1.7}$$

Of course, when the solution $u(x)$ is known, this is just an ODE for $x(\tau)$, for $p(s) = Du(x(s))$ and $z(s) = u(x(s))$. So we can solve (1.7) to get $x(s)$ if u is known in C^2.

Combining (1.4)-(1.7) yields

$$\dot{p}_i(s) = -\frac{\partial F}{\partial z}(p(s), z(s), x(s))p_i(s) - \frac{\partial F}{\partial x_i}(p(s), z(s), x(s)). \tag{1.8}$$

Finally we differentiate $z(s) = u(x(s))$ to get

$$\dot{z}(s) = \sum_{j=1}^{n} p_j(s)\dot{x}_j(s) = \sum_{j=1}^{n} p_j(s)\frac{\partial F}{\partial p_j}(p(s), z(s), x(s)). \tag{1.9}$$

We can rewrite equations (1.7)-(1.9) into a vector form:

$$\begin{aligned} \dot{x}(s) &= D_pF(p(s), z(s), x(s)), \\ \dot{z}(s) &= D_pF(p(s), z(s), x(s)) \cdot p(s), \\ \dot{p}(s) &= -D_xF(p(s), z(s), x(s)) - D_zF(p(s), z(s), x(s))p(s). \end{aligned} \tag{1.10}$$

Definition 1.1. This system of $2n+1$ first-order ODEs together with $F(p(s), z(s), x(s)) = 0$ are called the **characteristic equations** for $F(Du, u, x) = 0$. The solution $(p(s), z(s), x(s))$ is called the **full characteristics** and its projection $x(s)$ is called the **projected characteristics**.

What we have just demonstrated is the following theorem.

Theorem 1.1. *Let $u \in C^2(\Omega)$ solve $F(Du, u, x) = 0$ in Ω. Assume $x(s)$ solves* (1.7)*, where $p(s) = Du(x(s)), z(s) = u(x(s))$. Then $p(s)$ and $z(s)$ solve the equations in* (1.10)*, for those s where $x(s) \in \Omega$.*

Remark: The characteristic equations are useful because they form a closed system of ODEs for $(p(s), z(s), x(s))$ as $2n + 1$ unknown functions. If we know the initial data $(p(0), z(0), x(0))$ when $s = 0$, then we can find the values of $(p(s), z(s), x(s))$ for at least small s. Since $z(s) = u(x(s))$, we will know the value of $u(x)$ at $x = x(s)$. As we will choose the initial data $(z(0), x(0))$ according to the Cauchy data and thus

$$z(0) = h(y), \quad x(0) = f(y),$$

we therefore need to find the initial data for $p(0)$; this will be an important part of the method of solving the equation using the characteristics. We will accomplish this after some examples.

1.3.2. Special cases. (a) **Linear equations** $B(x)Du(x) + c(x)u = d(x)$. In this case,

$$F(p, z, x) = B(x)p + c(x)z - d(x),$$

and hence $D_pF = B(x)$, $D_zF = c(x)$. So (1.10), together with $F(p, z, x) = 0$, becomes

$$\dot{x}(s) = D_pF(p, z, x) = B(x),$$

$$\dot{z}(s) = B(x)p = -c(x)z(s).$$

One can solve the first set of ODEs, then solve z. In this case, p is not needed.

Example 1.2. Solve

$$\begin{cases} xu_y - yu_x = u, & x > 0, y > 0, \\ u(x, 0) = g(x), & x > 0. \end{cases}$$

(**Answer:** $u(x,y) = g((x^2+y^2)^{1/2})e^{\arctan(y/x)}$.)

(b) **Quasilinear equations** $B(x,u)Du + c(x,u) = 0$. In this case,

$$F(p,z,x) = B(x,z)p + c(x,z), \quad D_pF = B(x,z).$$

Hence (1.10), together with $F(p,z,x) = 0$, becomes

$$\begin{cases} \dot{x} = B(x,z), \\ \dot{z} = B(x,z)p = -c(x,z), \end{cases}$$

which are autonomous system for x, z. Once again, p is not needed.

Example 1.3. Solve

$$\begin{cases} u_x + u_y = u^2, \quad y > 0, \\ u(x,0) = g(x). \end{cases}$$

(**Answer:** $u(x,y) = \frac{g(x-y)}{1-yg(x-y)}$. This solution makes sense only if $1 - yg(x-y) \neq 0$.)

(c) **Fully nonlinear problem.**

Example 1.4.

$$\begin{cases} u_{x_1}u_{x_2} = u, \quad x_1 > 0, \\ u(0,x_2) = x_2^2. \end{cases}$$

Solution: In this case, $F(p,z,x) = p_1p_2 - z$. The characteristic equations are

$$\dot{x_1} = F_{p_1} = p_2, \quad \dot{x_2} = F_{p_2} = p_1,$$

$$\dot{z} = p_1F_{p_1} + p_2F_{p_2} = p_1p_2 + p_2p_1 = 2p_1p_2 = 2z,$$

$$\dot{p_1} = -F_{x_1} - F_zp_1 = p_1, \quad \dot{p_2} = -F_{x_2} - F_zp_2 = p_2.$$

The initial data with parameter y are given by

$$x_1(0) = 0, \quad x_2(0) = y, \quad z(0) = y^2,$$

and

$$p_1(0) = q_1(y), \quad p_2(0) = q_2(y),$$

where, from $F(p(0), z(0), x(0)) = 0$,

$$q_1(y)q_2(y) = y^2.$$

Since $u(0,y) = y^2$, differentiating with respect to y, we have

$$p_2(0) = u_{x_2}(y,0) = 2y = q_2(y), \quad \text{so} \quad p_1(0) = q_1(y) = \frac{1}{2}y.$$

Using these initial data we can solve the characteristic ODEs to obtain

$$z = Z(y,s) = y^2e^{2s}, \quad p_1 = p_1(y,s) = \frac{1}{2}ye^s, \quad p_2 = p_2(y,s) = 2ye^s,$$

$$x_1 = x_1(y,s) = \int_0^s p_2(y,s)\,ds = 2y(e^s - 1),$$

$$x_2 = x_2(y,s) = y + \int_0^s p_1(y,s)ds = \frac{1}{2}y(e^s + 1).$$

This implies

$$u(x_1(y,s), x_2(y,s)) = Z(y,s) = y^2e^{2s}.$$

Given $x = (x_1, x_2)$, we solve $x_1(y, s) = x_1$ and $x_2(y, s) = x_2$ for (y, s) and then obtain the value $u(x_1, x_2)$. Note that

$$x_1 + 4x_2 = 4ye^s,$$

and hence we must have

$$u(x_1, x_2) = Z(y, s) = y^2 e^{2s} = \frac{1}{16}(x_1 + 4x_2)^2.$$

(Check this is a true solution!)

1.3.3. The Cauchy problem. We now discuss the *method of characteristics* to solve the Cauchy problem:

$$F(Du, u, x) = 0 \quad \text{in } \Omega; \quad u(x) = g(x) \quad \text{on } \Gamma.$$

First of all, we assume that Γ is parameterized by $x = f(y)$ with parameter $y \in D \subset \mathbb{R}^{n-1}$, where D is a domain and f is a smooth function on D. Let

$$u(f(y)) = g(f(y)) := h(y), \quad y \in D \subset \mathbb{R}^{n-1}.$$

Fix $y_0 \in D$. Let $x_0 = f(y_0), z_0 = h(y_0) = u(x_0)$. We assume that $p_0 \in \mathbb{R}^n$ is given such that

$$F(p_0, z_0, x_0) = F(p_0, h(y_0), f(y_0)) = 0. \tag{1.11}$$

In order that $p_0 = (p_0^1, \cdots, p_0^n)$ may equal $Du(x_0)$, it is necessary that

$$h_{y_j}(y_0) = \sum_{i=1}^n f^i_{y_j}(y_0) p_0^i \quad \forall\, j = 1, 2, \cdots, n-1, \tag{1.12}$$

where $(f^1, f^2, \cdots, f^n) = f$.

Given $y_0 \in D$, we say that a vector p_0 is *admissible at* y_0 (or that (p_0, y_0) is admissible) if (1.11) and (1.12) are satisfied. Note that an admissible p_0 may or may not exist; even when it exists, it may not be unique.

Conditions (1.11) and (1.12) can be written in terms of a map $\mathcal{F}(p, y)$ from $\mathbb{R}^n \times D$ to $\mathbb{R}^n$ defined by $\mathcal{F} = (\mathcal{F}_1, \cdots, \mathcal{F}_n)$, where

$$\mathcal{F}_j(p, y) = \sum_{i=1}^n f^i_{y_j}(y) p^i - h_{y_j}(y) \quad (j = 1, 2, \cdots, n-1);$$

$$\mathcal{F}_n(p, y) = F(p, h(y), f(y)), \quad y \in D, \quad p \in \mathbb{R}^n.$$

Note that p_0 is admissible at y_0 if and only if $\mathcal{F}(p_0, y_0) = 0$.

We say that an admissible (p_0, y_0) is **non-characteristic** if

$$\det \frac{\partial \mathcal{F}}{\partial p}(p, y)|_{(p_0, y_0)} = \det \begin{pmatrix} \frac{\partial f^1(y_0)}{\partial y_1} & \cdots & \frac{\partial f^n(y_0)}{\partial y_1} \\ \vdots & \vdots & \vdots \\ \frac{\partial f^1(y_0)}{\partial y_{n-1}} & \cdots & \frac{\partial f^n(y_0)}{\partial y_{n-1}} \\ F_{p_1}(p_0, z_0, x_0) & \cdots & F_{p_n}(p_0, z_0, x_0) \end{pmatrix} \neq 0.$$

In this case, by the implicit function theorem, there exists a smooth function $q(y)$ for y near y_0 such that $q(y_0) = p_0$ and

$$\mathcal{F}(q(y), y) = 0; \quad \text{that is, } p = q(y) \text{ is admissible at } y. \tag{1.13}$$

In what follows, we assume (p_0, y_0) is admissible and non-characteristic. Let $q(y)$ be determined by (1.13) in a neighborhood J of y_0 in $D \subset \mathbb{R}^{n-1}$.

Let $(p(s), z(s), x(s))$ be the solution to Equation (1.10) with initial data

$$p(0) = q(y), \quad z(0) = h(y), \quad x(0) = f(y) \quad \forall\, y \in J.$$

Since these solutions depend on parameter $y \in J$, we denote them by $p = P(y, s)$, $z = Z(y, s)$ and $x = X(y, s)$, where

$$P(y, s) = (p^1(y, s), p^2(y, s), \cdots, p^n(y, s)),$$
$$X(y, s) = (x^1(y, s), x^2(y, s), \cdots, x^n(y, s)),$$

to display the dependence on y. By continuous dependence of ODE, P, Z, X are C^2 in (y, s).

Lemma 1.5. *Let (p_0, y_0) be admissible and non-characteristic. Let $x_0 = f(y_0)$. Then, there exists an open interval I containing 0, a neighborhood J of y_0, and a neighborhood V of x_0 such that for each $x \in V$ there exist unique $s = s(x) \in I$, $y = y(x) \in J$ such that $x = X(y, s)$. Moreover, $s(x), y(x)$ are C^2 in $x \in V$.*

Proof. We have $X(y_0, 0) = f(y_0) = x_0$. The inverse function theorem gives the answer. In fact, using $\frac{\partial X}{\partial y}|_{s=0} = \frac{\partial f}{\partial y}$ and $\frac{\partial X}{\partial s}|_{s=0} = F_p(p_0, z_0, x_0)$, we find that the Jacobian determinant

$$\det \frac{\partial X(y, s)}{\partial(y, s)}|_{y=y_0, s=0} = \det \begin{pmatrix} \frac{\partial f^1(y_0)}{\partial y_1} & \dots & \frac{\partial f^1(y_0)}{\partial y_{n-1}} & F_{p_1}(p_0, z_0, x_0) \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial f^n(y_0)}{\partial y_1} & \dots & \frac{\partial f^n(y_0)}{\partial y_{n-1}} & F_{p_n}(p_0, z_0, x_0) \end{pmatrix} \neq 0$$

from the noncharacteristic condition. Note that since $X(y, s)$ is C^2 we also have $y(x), s(x)$ are C^2. □

Finally we define

$$u(x) = Z(y(x), s(x)) \quad \forall\, x \in V.$$

Then, we have the following:

Theorem 1.2. *Let (p_0, y_0) be admissible and non-characteristic. Then the function u defined above solves the equation $F(Du, u, x) = 0$ on V with the Cauchy data $u(x) = g(x)$ on $\Gamma \cap V$.*

Proof. (1) If $x \in \Gamma \cap V$ then $x = f(y) = X(y, 0)$ for some $y \in J$. So $s(x) = 0$, $y(x) = y$ and hence $u(x) = Z(y(x), s(x)) = h(y(x)) = h(y) = g(f(y)) = g(x)$. So Cauchy data follow.

(2) The function

$$f(y, s) = F(P(y, s), Z(y, s), X(y, s)) = 0$$

for all $s \in I$ and $y \in J$. In fact, $f(y, 0) = 0$ since $(q(y), y)$ is admissible, and

$$f_s(y, s) = F_p \cdot P_s + F_z Z_s + F_x \cdot X_s$$
$$= F_p \cdot (-F_z P - F_x) + F_z F_p \cdot P + F_x \cdot F_p = 0$$

from the characteristic ODEs.

(3) Let $p(x) = P(y(x), s(x))$ for $x \in V$. From the definition of $u(x)$ and Step (2), we have

$$F(p(x), u(x), x) = 0 \quad \forall\, x \in V.$$

To finish the proof, it remains to show that $p(x) = Du(x)$. Note that

$$Z_s(y, s) = \sum_{j=1}^{n} p^j(y, s)\dot{x}^j(y, s),$$

and, for $i = 1, \dots, n-1$, we claim

$$\frac{\partial Z}{\partial y_i}(y,s) = \sum_{j=1}^n p^j(y,s)\frac{\partial x^j}{\partial y_i}(y,s).$$

The proof comes from the following argument. Fixed $i = 1, 2, \cdots, n-1$, let

$$r(s) = \frac{\partial Z}{\partial y_i}(y,s) - \sum_{j=1}^n p^j(y,s)\frac{\partial x^j}{\partial y_i}(y,s).$$

Then $r(0) = h_{y_i}(y) - \sum_{j=1}^n q_j(y)\frac{\partial f^j}{\partial x_i}(y) = 0$ by the choice of $q(y) = P(y,0)$, and

$$\begin{aligned} \dot r(s) &= \frac{\partial^2 Z}{\partial y_i \partial s} - \sum_{j=1}^n p^j \frac{\partial^2 x^j}{\partial y_i \partial s} - \sum_{j=1}^n \frac{\partial p^j}{\partial s}\frac{\partial x^j}{\partial y_i} \\ &= \sum_{j=1}^n \left(\frac{\partial p^j}{\partial y_i}\frac{\partial x^j}{\partial s} - \frac{\partial p^j}{\partial s}\frac{\partial x^j}{\partial y_i} \right). \end{aligned}$$

In the last equality we have used the fact

$$\frac{\partial Z}{\partial s} = \sum_{j=1}^n p^j \frac{\partial x^j}{\partial s},$$

and hence

$$\frac{\partial^2 Z}{\partial y_i \partial s} = \sum_{j=1}^n p^j \frac{\partial^2 x^j}{\partial y_i \partial s} + \sum_{j=1}^n \frac{\partial p^j}{\partial y_i}\frac{\partial x^j}{\partial s}.$$

Finally from $F(P(y,s), Z(y,s), X(y,s)) = 0$ and differentiating with respect to y_i, we have, for each $i = 1, 2, \cdots, n-1$,

$$\sum_{j=1}^n \left(\frac{\partial F}{\partial x_j}\frac{\partial x^j}{\partial y_i} + \frac{\partial F}{\partial p_j}\frac{\partial p^j}{\partial y_i} \right) = -\frac{\partial F}{\partial z}\frac{\partial Z}{\partial y_i}.$$

Hence

$$\begin{aligned} \dot r(s) &= \sum_{j=1}^n \left(\frac{\partial p^j}{\partial y_i}\frac{\partial x^j}{\partial s} - \frac{\partial p^j}{\partial s}\frac{\partial x^j}{\partial y_i} \right) \\ &= \sum_{j=1}^n \left[\frac{\partial p^j}{\partial y_i}\frac{\partial F}{\partial p_j} + \left(\frac{\partial F}{\partial x_j} + \frac{\partial F}{\partial z} p^j \right) \frac{\partial x^j}{\partial y_i} \right] \\ &= -\frac{\partial F}{\partial z}(P(y,s), Z(y,s), X(y,s)) r(s). \end{aligned} \tag{1.14}$$

Consequently $r(s) \equiv 0$, for $r(s)$ solves a linear ODE with $r(0) = 0$.

(4) Finally, using $u(x) = Z(y(x), s(x))$ and $x = X(y(x), s(x))$ for all $x \in V$,

$$\begin{aligned} u_{x_j}(x) &= Z_s \frac{\partial s}{\partial x_j} + \sum_{i=1}^{n-1} Z_{y_i} \frac{\partial y^i}{\partial x_j} \\ &= \sum_{k=1}^{n} p^k \dot{x}^k \frac{\partial s}{\partial x_j} + \sum_{i=1}^{n-1} \left(\sum_{k=1}^{n} p^k \frac{\partial x^k}{\partial y_i} \right) \frac{\partial y^i}{\partial x_j} \\ &= \sum_{k=1}^{n} p^k \left(\dot{x}^k \frac{\partial s}{\partial x_j} + \sum_{i=1}^{n-1} \frac{\partial x^k}{\partial y_i} \frac{\partial y^i}{\partial x_j} \right) \\ &= \sum_{k=1}^{n} p^k \frac{\partial x_k}{\partial x_j} = \sum_{k=1}^{n} p^k \delta_{kj} = p^j(x). \end{aligned} \tag{1.15}$$

So $Du(x) = p(x)$ on V. □

Example 1.6. What happens if Γ is a projected characteristic surface? Let us look at two examples in this case.

(a)

$$\begin{cases} xu_x + yu_y = u, \\ u(\tau, \tau) = \tau & \forall\ \tau. \end{cases}$$

In this case, $\Gamma = \{(\tau, \tau)\}$ is a projected characteristic curve. We can see easily that $u = \alpha x + (1 - \alpha) y$ is a solution for any $\alpha \in \mathbb{R}$.

(b)

$$\begin{cases} xu_x + yu_y = u, \\ u(\tau, \tau) = 1 & \forall\ \tau. \end{cases}$$

In this case, $\Gamma = \{(\tau, \tau)\}$ is a projected characteristic curve; however, the problem does not have a solution. In fact, if u were a solution, along the line $x = y$, we would have

$$0 = \frac{d}{dx} u(x, x) = u_x(x, x) + u_y(x, x) = \frac{u(x, x)}{x} = \frac{1}{x} \neq 0.$$

(c)

$$\begin{cases} xu_x + yu_y = u, \\ u(x, 0) = g(x) & \forall\ x \in \mathbb{R}. \end{cases}$$

In this case, $\Gamma = \{(\tau, 0)\}$ is a projected characteristic curve; the problem can not have a solution unless $g(x) = kx$ for some constant k. In fact, if u is a solution, then, along the line $y = 0$, we would have

$$g'(x) = \frac{d}{dx} u(x, 0) = u_x(x, 0) = \frac{u(x, 0)}{x} = \frac{g(x)}{x}.$$

Hence $g(x) = kx$. In the case $g(x) = kx$ the solution $u = kx$, obtained by eliminating parameters in the characteristic solutions.

EXAMPLE 1.7. Solve

$$\begin{cases} \sum_{j=1}^n x_j u_{x_j} = \alpha u \\ u(x_1, \ldots, x_{n-1}, 1) = h(x_1, \ldots, x_{n-1}). \end{cases}$$

Solution: The characteristic ODEs are

$$\dot{x}_j = x_j, \quad j = 1, 2, \ldots, n,$$

$$\dot{z} = \alpha z$$

with the initial data $x_j(0) = y_j$ for $j = 1, \ldots, n-1$, $x_n(0) = 1$ and $z(0) = h(y_1, \ldots, y_n)$. So

$$x_j(y, s) = y_j e^s, \; j = 1, \ldots, n-1; \quad x_n(y, s) = e^s; \quad z(y, s) = e^{\alpha s} h(y).$$

Solve (y, s) in terms of x, we have

$$e^s = x_n, \; y_j = x_j / x_n.$$

Therefore,

$$u(x) = z(y, s) = e^{\alpha s} h(y) = x_n^\alpha h(\frac{x_1}{x_n}, \ldots, \frac{x_{n-1}}{x_n}).$$

If $\alpha < 0$, the only nice solution in a neighborhood of zero is identically zero since $u(tx_0) = t^\alpha u(x_0)$.

EXAMPLE 1.8. Solve

$$\begin{cases} u_t + B(u) \cdot Du = 0, \\ u(x, 0) = g(x), \end{cases} \tag{1.16}$$

where $u = u(x, t)$ $(x \in \mathbb{R}^n, \; t \in \mathbb{R})$, $Du = (u_{x_1}, \cdots, u_{x_n})$, $B\colon \mathbb{R} \to \mathbb{R}^n$ and $g\colon \mathbb{R}^n \to \mathbb{R}$ are given smooth functions.

In this case, let $q = (p, p_{n+1})$ and $F(q, z, x) = p_{n+1} + B(z) \cdot p$, and so the characteristic ODEs are

$$\begin{cases} \dot{x} = B(z), & x(0) = y \\ \dot{t} = 1, & t(0) = 0 \\ \dot{z} = 0, & z(0) = g(y), \end{cases}$$

where $y \in \mathbb{R}^n$ is the parameter. The solution is given by

$$z(y, s) = g(y), \quad x(y, s) = y + sB(g(y)), \quad t(y, s) = s.$$

The projected characteristic curve C_y in the xt-space passing through the point $(y, 0)$ is the line $(x, t) = (y + tB(h(y)), t)$ along which u is a constant $g(y)$. Hence u satisfies the implicit equation

$$u(x, t) = g(x - tB(u(x, t))).$$

Furthermore, two distinct projected characteristics C_{y_1} and C_{y_2} intersect at a point (x, t) if and only if

$$y_1 - y_2 = t(B(g(y_2)) - B(g(y_1))). \tag{1.17}$$

At the intersection point (x, t), $g(y_1) \neq g(y_2)$; hence the characteristic solution u becomes singular (undefined). Therefore, $u(x, t)$ becomes singular for some $t > 0$ if and only if there exist $y_1 \neq y_2$ such that (1.17) holds.

When $n = 1$, $u(x, t)$ becomes singular for some $t > 0$ unless $B(g(y))$ is a *nondecreasing function* of y. In fact, if $B(g(y))$ is not nondecreasing in y, then there exist $y_1 < y_2$ satisfying

$B(g(y_1)) > B(g(y_2))$ and hence two characteristics C_{y_1} and C_{y_2} intersect at a point (x,t) with

$$t = -\frac{y_2 - y_1}{B(g(y_2)) - B(g(y_1))} > 0.$$

If u were regular solution at (x,t), then $u(x,t)$ would be equal to $g(y_1)$ and $g(y_2)$ by the analysis above. However, since $B(g(y_1)) > B(g(y_2))$, $g(y_1) \neq g(y_2)$, which gives a contradiction.

The same argument also shows that the problem cannot have a regular solution $u(x,t)$ defined on *whole* $\mathbb{R}^2$ unless $B(g(y))$ is a constant function.

1.3.4. Weak solutions. For certain first-order PDEs, it is possible to define **weak solutions,** that are not necessarily in C^1 or even continuous.

For example, for the divergence form equation

$$\begin{cases} \frac{\partial R(u)}{\partial y} + \frac{\partial S(u)}{\partial x} = 0, \quad x \in \mathbb{R},\ y > 0, \\ u(x,0) = g(x), \end{cases} \tag{1.18}$$

we say that u is a **weak solution** (or an **integral solution**) if

$$\int_0^\infty \int_{-\infty}^\infty (R(u)v_y + S(u)v_x)\, dxdy = \int_{-\infty}^\infty g(x,0)v(x,0)\, dx \tag{1.19}$$

holds for all smooth functions $v\colon \mathbb{R} \times [0,\infty) \to \mathbb{R}$ with compact support.

Suppose R, S are smooth functions. Suppose V is a subdomain of $\mathbb{R} \times (0,\infty)$ and C is a smooth curve $x = \zeta(y)$ which separates V into two regions $V_l = V \cap \{x < \zeta(y)\}$ and $V_r = V \cap \{x > \zeta(y)\}$. Let u be a weak solution to (1.18) such that u is smoothly extended onto both $\bar{V}_l$ and $\bar{V}_r$.

First we choose a test function v with compact support in V_l. Then (1.19) becomes

$$0 = \iint_{V_l} [R(u)v_y + S(u)v_x]\, dxdy = -\iint_{V_l} [(R(u))_y + (S(u))_x]\, v\, dxdy.$$

This implies

$$(R(u))_y + (S(u))_x = 0 \quad \text{in } V_l. \tag{1.20}$$

Similarly,

$$(R(u))_y + (S(u))_x = 0 \quad \text{in } V_r. \tag{1.21}$$

Now select a test function v with compact support in V. Again (1.19) becomes

$$0 = \int_0^\infty \int_{-\infty}^\infty [R(u)v_y + S(u)v_x]\, dxdy$$

$$= \iint_{V_l} [R(u)v_y + S(u)v_x]\, dxdy + \iint_{V_l} [R(u)v_y + S(u)v_x]\, dxdy.$$

Now since v has compact support in V, we have, by integration by parts and (1.20),

$$\iint_{V_l} [R(u)v_y + S(u)v_x]\, dxdy = -\iint_{V_l} [(R(u))_y + (S(u))_x] v\, dxdy$$

$$+ \int_C [R(u^-)\nu_y^- + S(u^-)\nu_x^-] v\, dl = \int_C [R(u^-)\nu_y^- + S(u^-)\nu_x^-] v\, dl,$$

where $\nu^- = (\nu_x^-, \nu_y^-)$ is the unit out-normal of domain V_l, and u^- is the limit of u from V_l to the curve C. Likewise,

$$\iint_{V_r} [R(u)v_y + S(u)v_x]\, dxdy = \int_C [R(u^+)\nu_y^+ + S(u^+)\nu_x^+] v\, dl,$$

where $\nu^+ = (\nu_x^+, \nu_y^+)$ is the unit out-normal of domain V_r, and u^+ is the limit of u from V_r to the curve C. Note that $\nu^- = -\nu^+$. Therefore, we deduce that

$$\int_C [(R(u^+) - R(u^-))\nu_y^- + (S(u^+) - S(u^-))\nu_x^-] v\, dl = 0.$$

This equation holds for all test functions v as above, and so

$$(R(u^+) - R(u^-))\nu_y^- + (S(u^+) - S(u^-))\nu_x^- = 0 \quad \text{on } C.$$

Since the domain V_l is inside $x < \zeta(y)$, the unit out-normal ν^- on C is given by

$$\nu^- = (\nu_x^-, \nu_y^-) = \frac{(1, -\zeta'(y))}{\sqrt{1 + (\zeta'(y))^2}}.$$

Therefore, we obtain the **Rankine-Hugoniot condition**:

$$\frac{d\zeta}{dy}[R(u^+) - R(u^-)] = [S(u^+) - S(u^-)]. \tag{1.22}$$

This condition sometimes is written as

$$\zeta'[[R(u)]] = [[S(u)]],$$

with $[[A]]$ standing for the **jump** of A across curve C.

In fact, if u is smooth up to $\bar{V}_l$ and $\bar{V}_r$ and satisfies (1.20), (1.21) and (1.22), then u is a weak solution to (1.18) in V (defined by (1.19) for all test functions v with compact support in V).

Example 1.9. Consider the Burgers' equation $u_y + uu_x = 0$ as a divergence-form equation with $R(u) = u$, $S(u) = u^2/2$:

$$u_y + \left(\frac{u^2}{2}\right)_x = 0.$$

(Different R, S lead to different results.) We study the weak solution of this equation with intial data $u(x, 0) = h(x)$, where

$$h(x) = \begin{cases} 1, & \text{if } x \le 0, \\ 1 - x, & \text{if } 0 \le x \le 1, \\ 0, & \text{if } x \ge 1. \end{cases}$$

The classical solution $u(x, y)$ (by the characteristics method) is well-defined for $0 < y < 1$ and is given by

$$u(x, y) = \begin{cases} 1 & \text{if } x \le y,\ 0 \le y < 1, \\ \frac{1-x}{1-y} & \text{if } y \le x \le 1,\ 0 \le y < 1, \\ 0 & \text{if } x \ge 1,\ 0 \le y < 1. \end{cases}$$

How should we define u for $y \ge 1$? We set $\zeta(y) = (1 + y)/2$, and define

$$u(x, y) = \begin{cases} 1 & \text{if } x < \zeta(y),\ y \ge 1, \\ 0 & \text{if } x > \zeta(y),\ y \ge 1. \end{cases}$$

It can be checked that this u is a weak solution containing a **shock** (where u is discontinuous). (Note that $[[R(u)]] = -1, [[S(u)]] = -\frac{1}{2}$ and $\zeta' = \frac{1}{2}$.)

We now consider the initial data

$$h(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ 1 & \text{if } x > 0. \end{cases}$$

Then the method of characteristics this time does not lead to any ambiguity in defining u for all $y > 0$, but does fail to define solution in the domain $0 < x < y$. Define

$$u_1(x,y) = \begin{cases} 0 & \text{if } x < y/2 \\ 1 & \text{if } x > y/2. \end{cases}$$

It is easy to check that the shock condition (Rankine-Hugoniot) is satisfied, and u is a weak solution. (Note that $[[R(u_1)]] = 1, [[S(u_1)]] = \frac{1}{2}$ and $\zeta' = \frac{1}{2}$.) However, we can define another such solution as follows:

$$u_2(x,y) = \begin{cases} 1 & \text{if } x > y \\ x/y & \text{if } 0 < x < y \\ 0 & \text{if } x < 0. \end{cases}$$

The function u_2 is also a weak solution, called a **rarefaction wave**. (Note that u_2 is continuous across both rays $x = 0$ $(y > 0)$ and $x = y$ $(y > 0)$ and is a smooth solution on each of the three regions between the two rays.) So there are two different weak solutions for this Cauchy problem; therefore, the sense of weak solutions so defined does not give the uniqueness of solution.

Can we find a further condition which ensures that the weak solution is unique and physical? The answer is yes, it is usually called the entropy condition. We omit the details in this course.

1.4. Introduction to Hamilton-Jacobi equations

We study the initial value problem for the Hamilton-Jacobi equation

$$\begin{cases} u_t + H(Du, x) = 0, \\ u(x,0) = g(x). \end{cases}$$

The equation can be written as $F(Du, u_t, u, x, t) = u_t + H(Du, x) = 0$ with $F(q, z, y) = p_{n+1} + H(p, x)$, where $q = (p, p_{n+1})$ and $y = (x, t)$. So

$$F_q = (H_p(p,x), 1), \quad F_y = (H_x(p,x), 0), \quad F_z = 0.$$

The characteristic equations are

$$\dot{p} = -H_x(p,x), \quad \dot{z} = H_p(p,x) \cdot p - H(p,x), \quad \dot{t} = 1, \quad \dot{x} = H_p(p,x).$$

In particular, the equations

$$\begin{cases} \dot{p} = -H_x(p,x), \\ \dot{x} = H_p(p,x) \end{cases}$$

are the so-called **Hamilton's equations** in ODEs.

The initial data for the characteristic ODEs at $s = 0$ are given by

$$x(0) = y, \ t(0) = 0, \ z(0) = g(y), \ p(0) = h(y),$$

where $h(y)$ is some function satisfying the admissible conditions. So $t = t(y,s) = s$ and hence $x = x(y,t)$, $p = p(y,t)$ and $z = z(y,t)$.

The quantities $L = \dot{z}$, $v = \dot{x}$ and $p = p(y,t)$ are related by

$$v = H_p(p,x), \quad L = H_p(p,x)\cdot p - H(p,x). \tag{1.23}$$

Suppose from (1.23) we can eliminate p to have the relationship $L = L(v,x)$. Then $\dot{z} = L(\dot{x},x)$. So the solution u is obtained from $\dot{x}(s)$ and $x(s)$ by

$$u(x,t) = z(y,t) = g(y) + \int_0^t L(\dot{x}(s),x(s))ds \quad \text{with } x(0) = y \text{ and } x(t) = x. \tag{1.24}$$

This is the *action functional* in classical mechanics.

1.4.1. The calculus of variations. Assume $L(v,x)$, $v,x \in \mathbb{R}^n$, is a given smooth function, called the **Lagrangian**.

Fix two points $x,y \in \mathbb{R}^n$ and a time

$$\mathcal{A} := \{w\colon [0,t] \to \mathbb{R}^n \mid w(0) = y,\ \ w(t) = x\}.$$

Define the *action functional*

$$I[w] = \int_0^t L(\dot{w}(s),w(s))\,ds \quad \forall\ w \in \mathcal{A}.$$

A basic problem in the calculus of variations is to find a curve $x(s) \in \mathcal{A}$ minimizing the functional $I[w]$; namely,

$$I[x] \le I[w] \quad \forall\ w \in \mathcal{A}.$$

We assume such a minimizer x exists in $\mathcal{A}$.

Theorem 1.3. *The minimizer x satisfies the* **Euler-Lagrange equation***:*

$$-\frac{d}{ds}(L_v(\dot{x}(s),x(s))) + L_x(\dot{x}(s),x(s)) = 0 \quad (0 \le s \le t).$$

Proof. 1. Let $y(s) = (y^1(s),\cdots,y^n(s))$ be a smooth function satisfying $y(0) = y(t) = 0$. Define $i(\tau) = I[x+\tau y]$. Then $i(\tau)$ is smooth in τ and has minimum at $\tau = 0$. So $i'(0) = 0$.

2. We can explicitly compute $i'(\tau)$ to obtain

$$i'(\tau) = \int_0^t (L_v(\dot{x}+\tau\dot{y},x+\tau y)\cdot\dot{y} + L_x(\dot{x}+\tau\dot{y},x+\tau y)\cdot y)\,ds.$$

So $i'(0) = 0$ implies

$$\int_0^t (L_v(\dot{x},x)\cdot\dot{y} + L_x(\dot{x},x)\cdot y)\,ds = 0.$$

We use integration by parts in the first integral to obtain

$$\int_0^t \left(-\frac{d}{ds}L_v(\dot{x},x) + L_x(\dot{x},x)\right)\cdot y\,ds = 0.$$

This identity holds for all smooth functions $y(s)$ satisfying $y(0) = y(t) = 0$, and so for all $0 \le s \le t$,

$$-\frac{d}{ds}L_v(\dot{x},x) + L_x(\dot{x},x) = 0.$$

□

We now set the *generalized momentum* to be

$$p(s) := L_v(\dot{x}(s), x(s)) \quad (0 \le s \le t).$$

We make the following assumption:

For all $p, x \in \mathbb{R}^n$, the equation $p = L_v(v,x)$ can be uniquely solved for v as a smooth function of (p,x), $v = v(p,x)$.

We then define the **Hamiltonian** $H(p,x)$ associated with Lagrangian $L(v,x)$ by

$$H(p,x) = p \cdot v(p,x) - L(v(p,x),x) \quad (p, x \in \mathbb{R}^n).$$

EXAMPLE 1.10. Let $L(v,x) = \frac{1}{2}m|v|^2 - \phi(x)$. Then $L_v(v,x) = mv$. Given $p \in \mathbb{R}^n$, equation $p = L_v(v,x)$ can be uniquely solved for v to have $v = \frac{1}{m}p$. Hence the Hamiltonian

$$H(p,x) = p \cdot \frac{1}{m}p - \frac{1}{2}m|\frac{1}{m}p|^2 + \phi(x) = \frac{1}{2m}|p|^2 + \phi(x).$$

In this example, L is total action and H is total energy density. The corresponding Euler-Lagrange equation is the **Newton's law**:

$$m\ddot{x}(s) = F(x(s)), \quad F(x) = -D\phi(x).$$

Theorem 1.4. *The minimizer $x(s)$ and momentum $p(s)$ satisfy* **Hamilton's equations***:*

$$\dot{x}(s) = H_p(p(s), x(s)), \quad \dot{p}(s) = -H_x(p(s), x(s))$$

for $0 \le s \le t$. Furthermore

$$H(p(s), x(s)) = \textit{constant}.$$

Proof. By assumption,

$$p = L_v(v(p,x),x) \quad (p, x \in \mathbb{R}^n).$$

Since $p(s) = L_v(\dot{x}(s), x(s))$, we have

$$\dot{x}(s) = v(p(s), x(s)) \quad (0 \le s \le t).$$

We compute by the definition of $H(p,x)$, for each $i = 1, 2, \cdots, n$,

$$H_{x_i}(p,x) = p \cdot v_{x_i}(p,x) - L_v(v(p,x),x) \cdot v_{x_i}(p,x) - L_{x_i}(v(p,x),x) = -L_{x_i}(v(p,x),x),$$

and

$$H_{p_i}(p,x) = v^i(p,x) + p \cdot v_{p_i}(p,x) - L_v(v(p,x),x) \cdot v_{p_i}(p,x) = v^i(v(p,x),x).$$

Thus

$$\dot{x}(s) = v(p(s), x(s)) = H_p(p(s), x(s)).$$

By definition of $p(s)$ and the Euler-Lagrange equation,

$$\dot{p}(s) = L_x(\dot{x}(s), x(s)) = L_x(v(p(s), x(s)), x(s)) = -H_x(p(s), x(s)).$$

Finally, observe

$$\frac{d}{ds}H(p(s), x(s)) = H_p \cdot \dot{p}(s) + H_x \cdot \dot{x}(s) = -H_pH_x + H_xH_p = 0.$$

So $H(p(s), x(s))$ is constant on $0 \le s \le t$. □

1.4.2. Legendre-Fenchel transform and the Hopf-Lax formula. We now study the simple case where Hamiltonian $H = H(p)$ depends only on $p \in \mathbb{R}^n$. We also assume $H\colon \mathbb{R}^n \to \mathbb{R}$ is convex and satisfies the *super-linear growth* condition:

$$\lim_{|p|\to\infty} \frac{H(p)}{|p|} = \infty.$$

The set of all such convex functions on $\mathbb{R}^n$ is denoted by $\mathcal{C}(\mathbb{R}^n)$.

Given any function $L\colon \mathbb{R}^n \to \mathbb{R}$, we define the **Legender-Fenchel transform** of L to be

$$L^*(p) = \sup_{q\in\mathbb{R}^n} [q \cdot p - L(q)] \quad (p \in \mathbb{R}^n).$$

Theorem 1.5. *For each $L \in \mathcal{C}(\mathbb{R}^n)$, $L^* \in \mathcal{C}(\mathbb{R}^n)$. Moreover, $(L^*)^* = L$.*

Thus, if $L \in \mathcal{C}(\mathbb{R}^n)$ and $H = L^*$ then $L = H^*$. In this case we say H, L are **dual convex functions**. Moreover, if L is differentiable at q then, for any $p \in \mathbb{R}^n$, the following conditions are equivalent:

$$p \cdot q = L(q) + L^*(p), \quad p = L_q(q).$$

Likewise, if $H = L^*$ is differentiable at p then, for any $q \in \mathbb{R}^n$, the following conditions are equivalent:

$$p \cdot q = L(q) + H(p), \quad q = H_p(p).$$

(For nonsmooth convex functions, similar duality properties also hold; see Problem 11.)

Therefore, if H is smooth then, from

$$q = H_p(p), \quad L = H_p(p) \cdot p - H(p),$$

we can eliminate p to have $L = L(q) = H^*(q)$.

In order to solve the Hamilton-Jacobi equation, (1.24) suggests us to study the following function:

$$u(x,t) := \inf \left\{ \int_0^t L(\dot{w}(s))\, ds + g(w(0)) \;\Big|\; w \in C^1([0,t];\mathbb{R}^n),\;\; w(t) = x \right\},$$

where we assume that $g\colon \mathbb{R}^n \to \mathbb{R}$ is a **Lipschitz continuous** function; that is,

$$\mathrm{Lip}(g) = \sup_{x,y\in\mathbb{R}^n,\ x\neq y} \frac{|g(x) - g(y)|}{|x-y|} < \infty.$$

The following Hopf-Lax formula is useful because it transforms this infinite-dimensional minimization problem to a finite-dimensional one and provides a good weak solution to the Hamilton-Jacobi equation.

Theorem 1.6. *(Hopf-Lax formula) For $x \in \mathbb{R}^n,\ t > 0$,*

$$u(x,t) = \min_{y\in\mathbb{R}^n} \left\{ tL\left(\frac{x-y}{t}\right) + g(y) \right\},$$

and u is Lipschitz continuous on $\mathbb{R}^n \times [0,\infty)$ satisfying $u(x,0) = g(x)$ for all $x \in \mathbb{R}^n$. Furthermore, if u is differentiable at some point (x,t), $t > 0$, then at this point (x,t), the equation

$$u_t(x,t) + H(Du(x,t)) = 0$$

is satisfied.

Proof. Step 1. We first show that

$$u(x,t) = \inf_{y\in\mathbb{R}^n} \left\{ tL\left(\frac{x-y}{t}\right) + g(y) \right\}$$

and that the infimum is a minimum follows by the growth condition of L. If $w \in C^1([0,t];\mathbb{R}^n)$ with $w(t) = x$, then by **Jensen's inequality**,

$$\frac{1}{t}\int_0^t L(\dot{w}(s))ds \geq L\left(\frac{1}{t}\int_0^t \dot{w}(s)ds\right) = L(\frac{x-y}{t}),$$

where $y = w(0)$. Hence

$$\int_0^t L(\dot{w}(s))ds + g(w(0)) \geq tL(\frac{x-y}{t}) + g(y) \geq \inf_{y\in\mathbb{R}^n} \left\{ tL\left(\frac{x-y}{t}\right) + g(y) \right\}.$$

This proves

$$u(x,t) \geq \inf_{y\in\mathbb{R}^n} \left\{ tL\left(\frac{x-y}{t}\right) + g(y) \right\}.$$

On the other hand, given $y \in \mathbb{R}^n$, let $w(s) = y + \frac{s}{t}(x-y)$ for $s \in [0,t]$. Use this w we have

$$u(x,t) \leq \int_0^t L(\dot{w}(s))ds + g(w(0)) = tL\left(\frac{x-y}{t}\right) + g(y).$$

Hence

$$u(x,t) \leq \inf_{y\in\mathbb{R}^n} \left\{ tL\left(\frac{x-y}{t}\right) + g(y) \right\}.$$

Step 2. We prove that u is Lipschitz on $\mathbb{R}^n \times [0,\infty)$. First, fix $t > 0$ and $x, \hat{x} \in \mathbb{R}^n$. Choose $y \in \mathbb{R}^n$ such that

$$u(x,t) = tL(\frac{x-y}{t}) + g(y).$$

Then

$$\begin{aligned} u(\hat{x},t) - u(x,t) &= \min_z [tL(\frac{\hat{x}-z}{t}) + g(z)] - tL(\frac{x-y}{t}) - g(y) \\ (\text{choosing } z = \hat{x} - x + y) \quad &\leq g(\hat{x} - x + y) - g(y) \leq \mathrm{Lip}(g)|x - \hat{x}|. \end{aligned}$$

This proves $|u(x,t) - u(\hat{x},t)| \leq \mathrm{Lip}(g)|x - \hat{x}|$.

To establish the Lipschitz condition on t, we first prove the folllowing result, which will also be used for checking the Hamilton-Jacobi equation

Lemma 1.11. *For each $x \in \mathbb{R}^n$ and $0 \leq s < t$,*

$$u(x,t) = \min_{y\in\mathbb{R}^n} \left\{ (t-s)L\left(\frac{x-y}{t-s}\right) + u(y,s) \right\}.$$

In other words, to compute $u(x,t)$, we can compute u at an earlier time s and then use $u(y,s)$ as initial data (at initial time s) to solve $u(x,t)$ on $[s,t]$.

Proof. 1. Fix $y \in \mathbb{R}^n$. Choose $z \in \mathbb{R}^n$ such that $u(y,s) = sL(\frac{y-z}{s}) + g(z)$. Since L is convex,

$$tL(\frac{x-z}{t}) \leq (t-s)L(\frac{x-y}{t-s}) + sL(\frac{y-z}{s}).$$

Hence

$$\begin{aligned} u(x,t) &\leq tL(\frac{x-z}{t}) + g(z) \leq (t-s)L(\frac{x-y}{t-s}) + sL(\frac{y-z}{s}) + g(z) \\ &= (t-s)L(\frac{x-y}{t-s}) + u(y,s). \end{aligned}$$

This proves

$$u(x,t) \le \inf_{y\in\mathbb{R}^n} \left\{ (t-s)L(\frac{x-y}{t-s}) + u(y,s) \right\}.$$

2. Let $y \in \mathbb{R}^n$ be such that $u(x,t) = tL(\frac{x-y}{t}) + g(y)$. Choose z such that $\frac{x-z}{t-s} = \frac{x-y}{t}$; namely, $z = \frac{s}{t}x + (1-\frac{s}{t})y$. So, in this case,

$$\frac{x-z}{t-s} = \frac{x-y}{t} = \frac{z-y}{s}.$$

Hence

$$\begin{aligned} u(z,s) + (t-s)L(\frac{x-z}{t-s}) &\le sL(\frac{z-y}{s}) + g(y) + (t-s)L(\frac{x-z}{t-s}) \\ &= tL(\frac{x-y}{t}) + g(y) = u(x,t). \end{aligned}$$

This proves

$$u(x,t) \ge \inf_{y\in\mathbb{R}^n} \left\{ (t-s)L(\frac{x-y}{t-s}) + u(y,s) \right\}.$$

Finally the infimum is a minimum as seen from the above. $\square$

Continuation of Step 2. From the lemma,

$$u(x,t) \le (t-s)L(0) + u(x,s) \quad \forall\, x\in\mathbb{R}^n, \quad t > s \ge 0.$$

Furthermore,

$$\begin{aligned} u(x,t) - u(x,s) &= \min_{y\in\mathbb{R}^n} \left\{ (t-s)L\left(\frac{x-y}{t-s}\right) + u(y,s) - u(x,s) \right\} \\ &\ge \min_{y\in\mathbb{R}^n} \left\{ (t-s)L\left(\frac{x-y}{t-s}\right) - \mathrm{Lip}(g)|x-y| \right\} \\ &= (t-s)\min_{z\in\mathbb{R}^n} \{L(z) - \mathrm{Lip}(g)|z|\} \quad \text{(choosing } y = x-(t-s)z) \\ &= -(t-s)\max_{z\in\mathbb{R}^n}\max_{w\in B_R(0)} \{w\cdot z - L(z)\} \quad (R = \mathrm{Lip}(g)) \\ &= -(t-s)\max_{w\in B_R(0)}\max_{z\in\mathbb{R}^n} \{w\cdot z - L(z)\} = -(t-s)\max_{w\in B_R(0)} L^*(w). \end{aligned}$$

Therefore,

$$-|t-s|\max_{w\in B_R(0)} L^*(w) \le u(x,t) - u(x,s) \le L(0)|t-s|.$$

Hence

$$|u(x,t) - u(x,s)| \le C|t-s| \quad \forall\, x\in\mathbb{R}^n,\ \ t > s \ge 0,$$

where $C = \max\{|L(0)|, \max_{w\in B_R(0)} |L^*(w)|\}$.

Finally, given $\hat{t} > 0, t > 0$ and $\hat{x}, x \in \mathbb{R}^n$, we have

$$|u(x,t) - u(\hat{x},\hat{t})| \le |u(x,t) - u(\hat{x},t)| + |u(\hat{x},t) - u(\hat{x},\hat{t})| \le C(|x-\hat{x}| + |t-\hat{t}|),$$

where

$$C = \max\{\mathrm{Lip}(g), |L(0)|, \max_{w\in B_R(0)} |L^*(w)|\}.$$

Step 3. Assume u is differentiable at a point (x_0,t_0), $t_0 > 0$. We show that

$$u_t(x_0,t_0) + H(Du(x_0,t_0)) = 0.$$

Let $p_0 = Du(x_0,t_0)$. Let $\epsilon > 0$ and, given any $q\in\mathbb{R}^n$, let

$$\phi(\epsilon) = u(x_0,t_0) - u(x_0-\epsilon q, t_0-\epsilon).$$

Then $\phi'(0) = u_t(x_0, t_0) + q \cdot p_0$. On the other hand, by the lemma above

$$u(x_0, t_0) \le \epsilon L(q) + u(x_0 - \epsilon q, t_0 - \epsilon)$$

and hence $\frac{\phi(\epsilon)}{\epsilon} \le L(q)$. This shows that $\phi'(0) \le L(q)$, or

$$q \cdot p_0 - L(q) \le -u_t(x_0, t_0) \quad \forall\, q \in \mathbb{R}^n.$$

So, $H(p_0) \le -u_t(x_0, t_0)$ and this proves that $u_t + H(Du) \le 0$ at (x_0, t_0).

To prove the other direction, let $y \in \mathbb{R}^n$ be such that $u(x_0, t_0) = t_0 L(\frac{x_0 - y}{t_0}) + g(y)$. For $0 < \epsilon < t_0$, let $z = z_\epsilon$ be such that $\frac{x_0-y}{t_0} = \frac{z-y}{t_0-\epsilon}$; that is, $z_\epsilon = y + (1 - \frac{\epsilon}{t_0})(x_0 - y)$. Then

$$u(x_0, t_0) - u(z_\epsilon, t_0 - \epsilon) \ge t_0 L(\frac{x_0 - y}{t_0}) + g(y) - \left[(t_0 - \epsilon) L(\frac{z_\epsilon - y}{t_0 - \epsilon}) + g(y)\right] = \epsilon L(\frac{x_0 - y}{t_0}).$$

Define

$$\phi(\epsilon) = u(x_0, t_0) - u(z_\epsilon, t_0 - \epsilon) = u(x_0, t_0) - u\left(y + (1 - \frac{\epsilon}{t_0})(x_0 - y), t_0 - \epsilon\right).$$

Then $\frac{\phi(\epsilon)}{\epsilon} \ge L(\frac{x_0-y}{t_0})$ for all $\epsilon > 0$. So $\phi'(0) \ge L(\frac{x_0-y}{t_0})$, which reads

$$\frac{x_0 - y}{t_0} \cdot Du(x_0, t_0) + u_t(x_0, t_0) \ge L(\frac{x_0 - y}{t_0}).$$

Hence

$$H(Du(x_0, t_0)) \ge \frac{x_0 - y}{t_0} \cdot Du(x_0, t_0) - L(\frac{x_0 - y}{t_0}) \ge -u_t(x_0, t_0).$$

We have finally proved that $u_t + H(Du) = 0$ at (x_0, t_0). □

Example 1.12. (a) Find the Hopf-Lax solution to the problem

$$u_t + \frac{1}{2}|Du|^2 = 0, \quad u(x, 0) = |x|.$$

Solution: In this case, $H(p) = \frac{1}{2}|p|^2$ and hence $L(q) = \frac{1}{2}|q|^2$. (**Exercise!**) So

$$u(x, t) = \min_{y \in \mathbb{R}^n} \left\{ \frac{|x - y|^2}{2t} + |y| \right\}.$$

Given $x \in \mathbb{R}^n$, $t > 0$, the function $f(y) = \frac{|x-y|^2}{2t} + |y|$ has a critical point at $y = 0$ and possible critical points at $y_0 \ne 0$ if $f'(y_0) = 0$. Solving $f'(y) = 0$ leads to $x = y + \frac{y}{|y|} t$, which has no nonzero solution y if $x = 0$; in this case, $u(0, t) = f(0) = 0$. If $x \ne 0$ then $f'(y) = 0$ has nonzero solution

$$y_0 = (|x| - t)\frac{x}{|x|}.$$

Comparing $f(y_0)$ and $f(0)$, we have $u(x, t) = f(y_0)$ if $|x| > t$ and $u(x, t) = f(0)$ if $|x| \le t$. Hence

$$u(x, t) = \begin{cases} |x| - \frac{t}{2} & \text{if } |x| \ge t, \\ \frac{|x|^2}{2t} & \text{if } |x| \le t. \end{cases}$$

(b) Find the Hopf-Lax solution to the problem

$$u_t + \frac{1}{2}|Du|^2 = 0, \quad u(x, 0) = -|x|.$$

Solution: In this case,

$$u(x, t) = \min_{y \in \mathbb{R}^n} \left\{ \frac{|x - y|^2}{2t} - |y| \right\}.$$

Given $x \in \mathbb{R}^n$, $t > 0$, the function $f(y) = \frac{|x-y|^2}{2t} - |y|$ has a critical point at $y = 0$ and possible critical points at $y_0 \neq 0$ if $f'(y_0) = 0$. Solving $f'(y) = 0$ leads to $x = y - \frac{y}{|y|}t$, which has nonzero solutions y_0 with $|y_0| = t$ if $x = 0$. If $x \neq 0$ then $f'(y) = 0$ has nonzero solution

$$y_0 = (|x| + t)\frac{x}{|x|}.$$

Comparing $f(y_0)$ and $f(0)$ in all cases, we have $u(x,t) = f(y_0)$ for all x, t. Hence

$$u(x,t) = -|x| - \frac{t}{2} \quad (x \in \mathbb{R}^n, \quad t > 0).$$

EXAMPLE 1.13. (**Problem 6 in the Text.**)

Given a smooth vector field $\mathbf{b}(x) = (b^1(x), \cdots, b^n(x))$ on $\mathbb{R}^n$, let $x(s) = X(s,x,t)$ solve

$$\begin{cases} \dot{x} = \mathbf{b}(x) \quad (s \in \mathbb{R}), \\ x(t) = x. \end{cases}$$

(a) Define the Jacobian

$$J(s,x,t) := \det D_x X(s,x,t).$$

Show the *Euler formula*:

$$J_s = (\operatorname{div} \mathbf{b})(X)J.$$

(b) Demonstrate that

$$u(x,t) = g(X(0,x,t))J(0,x,t)$$

solves

$$\begin{cases} u_t + \operatorname{div}(u\mathbf{b}) = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases}$$

Proof. (a) For any smooth matrix function $M(s)$ of $s \in \mathbb{R}$, it follows that

$$\frac{d}{ds}(\det M(s)) = \operatorname{tr}((\operatorname{adj}M(s))\dot{M}(s)), \tag{1.25}$$

where for any $n \times n$ matrix M, adj$M$ is the *adjugate matrix* of M, satisfying

$$M(\operatorname{adj}M) = (\operatorname{adj}M)M = (\det M)I.$$

(**Exercise:** Prove (1.25)!)

Let $M(s) = (X^i_{x_j}(s,x,t))$ and $B(x) = (b^i_{x_j}(x))$. Then, by the equation,

$$\dot{M}(s) = B(X)M(s) \quad s \in \mathbb{R}.$$

Therefore

$$\begin{aligned} J_s &= \operatorname{tr}((\operatorname{adj}M(s))\dot{M}(s)) = \operatorname{tr}((\operatorname{adj}M(s))B(X)M(s)) \\ &= \operatorname{tr}(B(X)M(s)(\operatorname{adj}M(s))) = \operatorname{tr}(B(X)J) = (\operatorname{div} \mathbf{b})(X)J. \end{aligned}$$

(b) The uniqueness of $X(s,x,t)$ implies that

$$X(s,x,t) = X(s-t,x,0) := X_0(s-t,x) \quad (x \in \mathbb{R}^n, \ s, \ t \in \mathbb{R}).$$

So $X_0(s,x)$ satisfies the group property:

$$X_0(0,x) = x, \quad X_0(s, X_0(t,x)) = X_0(s+t,x).$$

Hence $J(s,x,t) = J(s-t,x,0) := J_0(s-t,x)$. By the Euler formula, J_0 is given by

$$J_0(y,s) = e^{\int_0^s (\operatorname{div} \mathbf{b})(X_0(\tau,y))d\tau}.$$

From the group property of X_0, it follows that $J_0(t+s,x) = J_0(s, X_0(t,x))J_0(t,x)$, which implies

$$J(s, X_0(-s,x))J_0(-s,x) = J_0(0,x) = 1 \quad (x \in \mathbb{R}^n,\ s \in \mathbb{R}).$$

Note that

$$u(x,t) = g(X_0(-t,x))J_0(-t,x).$$

Therefore, $u(x,0) = g(x)$ $(x \in \mathbb{R}^n)$. We want to show that the classical solution built from the characteristics method agrees with this function $u(x,t)$. The characteristic ODEs are

$$\dot{x} = \mathbf{b}(x), \quad \dot{t} = 1, \quad \dot{z} = -(\operatorname{div}\mathbf{b})(x)z.$$

The initial data are given by

$$x(0) = y, \quad t(0) = 0, \quad z(0) = g(y) \quad (y \in \mathbb{R}^n).$$

Hence

$$x = x(y,s) = X_0(y,s), \quad t = t(y,s) = s$$

and

$$z = z(y,s) = g(y)e^{-\int_0^s (\operatorname{div}\mathbf{b})(X_0(\tau,y))d\tau} = \frac{g(y)}{J_0(y,s)}.$$

Solving (y,s) in terms of $x = x(y,s)$ and $t = t(y,s)$, we have

$$s = t, \quad y = X_0(-t,x).$$

Hence our classical solution $w(x,t)$ from the characteristics method is given by

$$w(x,t) = z(y,s) = \frac{g(X_0(-t,x))}{J_0(t, X_0(-t,x))} = g(X_0(-t,x))J_0(-t,x),$$

which is exactly the function $u(x,t)$. □

Suggested exercises

Materials covered are from Chapter 3 of the textbook. So complete the arguments that are left in lectures. Also try working on the following problems related to the covered materials.

Chapter 3: Problems 4, 5, 6, 10, 11, 13, 16.

Homework # 1.

(1) (7 points) Write down an explicit formula for a function u solving the initial value problem

$$\begin{cases} u_t + b \cdot Du + cu = 0 & \text{in } \mathbf{R}^n \times (0,\infty), \\ u = g & \text{on } \mathbf{R}^n \times \{t = 0\}. \end{cases}$$

Here $c \in \mathbf{R}$, $b \in \mathbf{R}^n$ are constants, and g is a given function.

(2) (8 points) Let u be a C^1 solution of linear PDE

$$B(x) \cdot Du(x) = -u$$

on the closed unit ball Ω in $\mathbf{R}^n$. If $B(x) \cdot x > 0$ for all $x \in \partial\Omega$, show that $u \equiv 0$ on Ω.

(Hint: Show that $\max_\Omega u \leq 0$ *and* $\min_\Omega u \geq 0$*).*

(3) (15 points) Let $a(z)$ be a given smooth function of $z \in \mathbf{R}$. Consider the quasilinear PDE

$$u_y + a(u)\, u_x = 0$$

with initial data $u(x,0) = h(x)$, where h is also a given smooth function.

(a) Use the characteristics method to show that the solution is given implicitly by

$$u = h(x - a(u)y).$$

(b) Show that the solution becomes singular (undefined) for some $y > 0$, unless $a(h(z))$ is a nondecreasing function of $z \in \mathbf{R}$.

(c) Suppose u is a C^2 solution in each of two domains separated by a smooth curve $x = \xi(y)$. Assume u is continuous but u_x has a jump discontinuity across the curve, and also assume u restricted to the curve is smooth. Prove the curve $x = \xi(y)$ must be a characteristics, i.e., $d\xi/dy = a(u)$ along the curve.

(4) (10 points) Consider the equation $u_y = (u_x)^3$.

(a) Find the solution with initial data $u(x,0) = 2x^{3/2}$.

(b) Show that every C^∞ solution to the equation *on whole* $\mathbf{R}^2$ must be of the form

$$u(x,y) = ax + by + c$$

for some constants $a, b, c \in \mathbf{R}$.

(Hint: You may use the conclusion of Problem (3)(*b*) *above.)*

Homework # 2.

(1) Find the Legendre transform of L for

(a) (5 points)

$$L(q) = \frac{1}{r}|q|^r \quad (q \in \mathbf{R}^n),$$

where $1 < r < \infty$.

(b) (5 points)

$$L(q) = \frac{1}{2} q \cdot Aq + b \cdot q = \frac{1}{2}\sum_{i,j=1}^n a_{ij} q_i q_j + \sum_{i=1}^n b_i q_i \quad (q \in \mathbf{R}^n),$$

where $A = (a_{ij})$ is a $n \times n$ symmetric, positive definite $n \times n$-matrix, and $b = (b_1, \cdots, b_n) \in \mathbf{R}^n$.

(2) Let $H \colon \mathbf{R}^n \to \mathbf{R}$ be a convex function satisfying

$$\lim_{|p|\to\infty} \frac{H(p)}{|p|} = +\infty, \qquad \min_{p \in \mathbf{R}^n} H(p) = H(0) = 0.$$

(a) (5 points) Show that

$$\min_{q \in \mathbf{R}^n} H^*(q) = H^*(0) = 0.$$

(b) (10 points) Let g be Lipschitz continuous and have compact support in $\mathbf{R}^n$. If $u(x,t)$ is the Hopf-Lax solution to the Hamilton-Jacobi problem

$$\begin{cases} u_t + H(Du) = 0 & \text{in } \mathbf{R}^n \times (0,\infty) \\ u(x,0) = g(x), \end{cases}$$

show that, for each $t > 0$, $u(x,t)$ has compact support in $x \in \mathbf{R}^n$.

(3) (5 points) Let $H(p)$ be a convex function satisfying the superlinear growth at ∞, and let g_1, g_2 be given Lipschitz functions. Assume u^1, u^2 are the Hopf-Lax solutions of the initial value problem

$$\begin{cases} u^i_t + H(Du^i) = 0 & \text{in } \mathbf{R}^n \times (0,\infty), \\ u^i(x,0) = g^i(x), \end{cases} \qquad (i = 1, 2).$$

Prove the L^∞-contraction inequality:

$$\sup_{x\in\mathbf{R}^n} |u^1(x,t) - u^2(x,t)| \le \sup_{x\in\mathbf{R}^n} |g^1(x) - g^2(x)| \quad \forall\, t > 0.$$

Laplace's Equation

2.1. Green's identities

For a smooth vector field $\vec{F} = (f^1, \dots, f^n)$, we define the **divergence** of $\vec{F}$ by

$$\operatorname{div} \vec{F} = \sum_{j=1}^{n} \frac{\partial f^j}{\partial x_j} = \sum_{j=1}^{n} D_j f^j = \sum_{j=1}^{n} f^j_{x_j},$$

and for a smooth function $u(x)$ we define the **gradient** of u by

$$Du = \nabla u = (u_{x_1}, \cdots, u_{x_n}) = \operatorname{div}(\nabla u).$$

The **Laplace operator** on $\mathbb{R}^n$ is defined by

$$\Delta u = \sum_{k=1}^{n} u_{x_k x_k} = \operatorname{div}(\nabla u). \tag{2.1}$$

A C^2 function u satisfying $\Delta u = 0$ on a domain $\Omega \subseteq \mathbb{R}^n$ is called a **harmonic function** on Ω.

The **divergence theorem** for C^1-vector fields $\vec{F} = (f^1, \dots, f^n)$ on $\bar{\Omega}$ states that

$$\int_\Omega \operatorname{div} \vec{F}\, dx = \int_{\partial\Omega} \vec{F} \cdot \nu \, dS, \tag{2.2}$$

where ν is the **outer unit normal** to the boundary $\partial\Omega$.

For any $u, v \in C^2(\bar{\Omega})$, let $\vec{F} = vDu = v\nabla u$, we have $\operatorname{div} \vec{F} = \nabla u \cdot \nabla v + v\Delta u$, hence

$$\int_\Omega v\Delta u dx = -\int_\Omega \nabla u \cdot \nabla v dx + \int_{\partial\Omega} v \nabla u \cdot \nu dS = -\int_\Omega \nabla u \cdot \nabla v dx + \int_{\partial\Omega} v \frac{\partial u}{\partial \nu} dS. \tag{2.3}$$

Exchanging u and v in (2.3) we obtain

$$\int_\Omega u\Delta v dx = -\int_\Omega \nabla u \cdot \nabla v dx + \int_{\partial\Omega} u \frac{\partial v}{\partial \nu} dS. \tag{2.4}$$

Combining (2.3)-(2.4) yields

$$\int_\Omega (v\Delta u - u\Delta v) dx = \int_{\partial\Omega} (v \frac{\partial u}{\partial \nu} - u \frac{\partial v}{\partial \nu})\, dS. \tag{2.5}$$

Equation (2.3) is called **Green's first identity** and Equation (2.5) is called **Green's second identity.**

Taking $v = 1$ in (2.3), we have

$$\int_\Omega \Delta u dx = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} dS,$$

which implies that if there is a $C^2(\overline{\Omega})$ solution u to the **Neumann (boundary) problem**

$$\begin{cases} \Delta u = 0 \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu}|_{\partial\Omega} = g(x), \end{cases}$$

then $\int_{\partial\Omega} g(x) dS = 0$. The **Dirichlet (boundary) problem** for Laplace's equation is:

$$\begin{cases} \Delta u = 0 \quad \text{in } \Omega, \\ u|_{\partial\Omega} = f. \end{cases}$$

In (2.3), choose $u = v$, and we have

$$\int_\Omega u\Delta u dx = -\int_\Omega |\nabla u|^2 \, dx + \int_{\partial\Omega} u \frac{\partial u}{\partial \nu} dS. \tag{2.6}$$

Let $u_1, u_2 \in C^2(\bar\Omega)$ be solutions to the Neumann problem above and let $u = u_1 - u_2$. Then $\Delta u = 0$ in Ω and $\frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$; hence, by (2.6), $\nabla u = 0$; consequently $u \equiv$ constant, if Ω is connected. Therefore, for such domains, any two C^2-solutions of the Neumann problem differ by a constant.

Let $u_1, u_2 \in C^2(\bar\Omega)$ be solutions to the Dirichlet problem above and let $u = u_1 - u_2$. Then $\Delta u = 0$ in Ω and $u = 0$ on $\partial\Omega$; hence, by (2.6), $\nabla u = 0$; consequently $u \equiv$ constant on each connected component of Ω; however, since $u = 0$ on $\partial\Omega$, the constant must be zero. So $u_1 = u_2$ on Ω (we don't need Ω is connected in this case). Therefore, the $C^2(\bar\Omega)$-solution of the Dirichlet problem is *unique.* But later we will prove such a uniqueness result for solutions in $C^2(\Omega) \cap C(\bar\Omega)$.

2.2. Fundamental solutions and Green's function

We try to seek a harmonic function $u(x)$ that depends only on the radius $r = |x|$, i.e., $u(x) = v(r)$, $r = |x|$ (**radial function**). Computing Δu for such a function leads to an ODE for v:

$$\Delta u = v''(r) + \frac{n-1}{r} v'(r) = 0.$$

For $n = 1$, $v(r) = r$ is a solution. For $n \geq 2$, let $s(r) = v'(r)$, we have $s'(r) = -\frac{n-1}{r} s(r)$. This is a first-order linear ODE for $s(r)$ and solving it yields that $s(r) = cr^{1-n}$; consequently, we obtain a solution for $v(r)$:

$$v(r) = \begin{cases} Cr, & n = 1, \\ C \ln r, & n = 2, \\ Cr^{2-n}, & n \geq 3. \end{cases}$$

Note that $v(r)$ is well-defined for $r > 0$, but is singular at $r = 0$ when $n \geq 2$.

2.2.1. Fundamental solutions.

Definition 2.1. We call function $\Phi(x) = \phi(|x|)$ a **fundamental solution** of Laplace's equation in $\mathbb{R}^n$, where

$$\phi(r) = \begin{cases} -\frac{1}{2}r, & n = 1, \\ -\frac{1}{2\pi}\ln r, & n = 2, \\ \frac{1}{n(n-2)\alpha_n} r^{2-n}, & n \geq 3. \end{cases} \tag{2.7}$$

Here, for $n \geq 3$, α_n is the volume of the unit ball in $\mathbb{R}^n$.

Theorem 2.1. *For any $f \in C_c^2(\mathbb{R}^n)$, define*

$$u(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y)\, dy.$$

Then $u \in C^2(\mathbb{R}^n)$ and solves the **Poisson's equation**

$$-\Delta u(x) = f(x) \quad \text{for all } x \in \mathbb{R}^n. \tag{2.8}$$

Proof. We only prove this for $n \geq 2$ and leave the case of $n = 1$ (where $\Phi(x) = -\frac{1}{2}|x|$) as an exercise.

1. Let $f \in C_c^2(\mathbb{R}^n)$. Fix any bounded ball $B \subset \mathbb{R}^n$ and $x \in B$. Then

$$u(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y)\, dy = \int_{\mathbb{R}^n} \Phi(y) f(x-y)\, dy = \int_{B(0,R)} \Phi(y) f(x-y)\, dy,$$

where $B(0,R)$ is a large ball in $\mathbb{R}^n$ such that $f(x-y) = 0$ for all $x \in B$ and $y \notin B(0,R/2)$. Therefore, by differentiation under the integral,

$$u_{x_i}(x) = \int_{B(0,R)} \Phi(y) f_{x_i}(x-y)\, dy, \quad u_{x_i x_j}(x) = \int_{B(0,R)} \Phi(y) f_{x_i x_j}(x-y)\, dy.$$

This proves that $u \in C^2(B)$. Since B is arbitrary, it proves $u \in C^2(\mathbb{R}^n)$. Moreover

$$\Delta u(x) = \int_{B(0,R)} \Phi(y) \Delta_x f(x-y)\, dy = \int_{B(0,R)} \Phi(y) \Delta_y f(x-y)\, dy.$$

2. Fix $0 < \epsilon < R$. Write

$$\Delta u(x) = \int_{B(0,\epsilon)} \Phi(y) \Delta_y f(x-y)\, dy + \int_{B(0,R)\setminus B(0,\epsilon)} \Phi(y) \Delta_y f(x-y)\, dy =: I_\epsilon + J_\epsilon.$$

Now

$$|I_\epsilon| \leq C \|D^2 f\|_{L^\infty} \int_{B(0,\epsilon)} |\Phi(y)|\, dy \leq \begin{cases} C\epsilon^2 |\ln \epsilon| & (n = 2) \\ C\epsilon^2 & (n = 3). \end{cases}$$

Hence $I_\epsilon \to 0$ as $\epsilon \to 0^+$. For J_ϵ we use Green's second identity (2.5) with $\Omega = B(0,R) \setminus B(0,\epsilon)$ to have

$$\begin{aligned} J_\epsilon &= \int_\Omega \Phi(y) \Delta_y f(x-y)\, dy \\ &= \int_\Omega f(x-y) \Delta\Phi(y)\, dy + \int_{\partial\Omega} \left[\Phi(y) \frac{\partial f(x-y)}{\partial \nu_y} - f(x-y) \frac{\partial \Phi(y)}{\partial \nu_y} \right] dS_y \\ &= \int_{\partial\Omega} \left[\Phi(y) \frac{\partial f(x-y)}{\partial \nu_y} - f(x-y) \frac{\partial \Phi(y)}{\partial \nu_y} \right] dS_y \\ &= \int_{\partial B(0,\epsilon)} \left[\Phi(y) \frac{\partial f(x-y)}{\partial \nu_y} - f(x-y) \frac{\partial \Phi(y)}{\partial \nu_y} \right] dS_y, \end{aligned}$$

where $\nu_y = -\frac{y}{\epsilon}$ is the outer unit normal of $\partial\Omega$ on the sphere $\partial B(0,\epsilon)$. Now

$$\left| \int_{\partial B(0,\epsilon)} \Phi(y) \frac{\partial f(x-y)}{\partial \nu_y} \right| \le C|\phi(\epsilon)| \|Df\|_{L^\infty} \int_{\partial B(0,\epsilon)} dS \le C\epsilon^{n-1}|\phi(\epsilon)| \to 0$$

as $\epsilon \to 0^+$. Furthermore, $\nabla\Phi(y) = \phi'(|y|)\frac{y}{|y|}$; hence

$$\frac{\partial \Phi(y)}{\partial \nu_y} = \nabla\Phi(y) \cdot \nu_y = -\phi'(\epsilon) = \begin{cases} \frac{1}{2\pi}\epsilon^{-1} & (n=2) \\ \frac{1}{n\alpha_n}\epsilon^{1-n} & (n \ge 3), \end{cases} \qquad \text{for all } y \in \partial B(0,\epsilon).$$

That is, $\frac{\partial\Phi(y)}{\partial\nu_y} = \frac{1}{\int_{\partial B(0,\epsilon)} dS}$ and hence

$$\int_{\partial B(0,\epsilon)} f(x-y)\frac{\partial \Phi(y)}{\partial \nu_y} dS_y = ⨍_{\partial B(0,\epsilon)} f(x-y)\, dS_y \to f(x),$$

as $\epsilon \to 0^+$. Combining all the above, we finally prove that

$$-\Delta u(x) = f(x) \quad \forall\ x \in \mathbb{R}^n.$$

□

The reason the function $\Phi(x)$ is called a fundamental solution of Laplace's equation is as follows. The function $\Phi(x)$ formally satisfies

$$-\Delta_x \Phi(x) = \delta_0 \quad \text{on } x \in \mathbb{R}^n,$$

where δ_0 is the **Dirac measure** concentrated at 0:

$$\langle \delta_0, f \rangle = f(0) \quad \forall\ f \in C_c^\infty(\mathbb{R}^n).$$

If $u(x) = \int_{\mathbb{R}^n} \Phi(x-y)f(y)\,dy$, then, we can formally compute that (in terms of distributions)

$$\begin{aligned} -\Delta u(x) &= \int_{\mathbb{R}^n} \Delta_x \Phi(x-y) f(y)\,dy = -\int_{\mathbb{R}^n} \Delta_y \Phi(x-y) f(y)\,dy \\ &= -\int_{\mathbb{R}^n} \Delta_y \Phi(y) f(x-y)\,dy = \langle \delta_0, f(x-\cdot) \rangle = f(x) \end{aligned}$$

2.2.2. Green's function. Suppose $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$. Let $h \in C^2(\bar\Omega)$ be any harmonic function in Ω.

Given any function $u \in C^2(\bar\Omega)$, fix $x \in \Omega$ and $0 < \epsilon < \operatorname{dist}(x, \partial\Omega)$. Let $\Omega_\epsilon = \Omega \setminus B(x,\epsilon)$. Apply Green's second identity

$$\int_{\Omega_\epsilon} (u(y)\Delta v(y) - v(y)\Delta u(y))\,dy = \int_{\partial\Omega_\epsilon} \left(u(y)\frac{\partial v}{\partial \nu}(y) - v(y)\frac{\partial u}{\partial\nu}(y) \right) dS$$

to functions $u(y)$ and $v(y) = \Gamma(x,y) = \Phi(y-x) - h(y)$ on Ω_ϵ, where $\Phi(y) = \phi(|y|)$ is the fundamental solution above, and since $\Delta v(y) = 0$ on Ω_ϵ, we have

$$\begin{aligned} -\int_{\Omega_\epsilon} \Gamma(x,y)\Delta u(y)\,dy &= \int_{\partial\Omega_\epsilon} u(y)\frac{\partial\Gamma}{\partial\nu_y}(x,y)dS - \int_{\partial\Omega_\epsilon} \Gamma(x,y)\frac{\partial u}{\partial \nu_y}(y)dS \\ &= \int_{\partial\Omega} u(y)\frac{\partial\Gamma}{\partial\nu_y}(x,y)dS - \int_{\partial\Omega} \Gamma(x,y)\frac{\partial u}{\partial\nu_y}(y)dS \\ &\quad + \int_{\partial B(x,\epsilon)} u(y)\left(\frac{\partial\Phi}{\partial\nu}(y-x) - \frac{\partial h}{\partial\nu}(y)\right) dS \\ &\quad - \int_{\partial B(x,\epsilon)} (\Phi(y-x) - h(y))\frac{\partial u}{\partial \nu}(y)dS, \end{aligned} \tag{2.9}$$

where $\nu = \nu_y$ is the outer unit normal at $y \in \partial\Omega_\epsilon = \partial\Omega \cup \partial B(x,\epsilon)$. Note that $\nu_y = -\frac{y-x}{\epsilon}$ at $y \in \partial B(x,\epsilon)$. Hence $\frac{\partial \Phi}{\partial \nu_y}(y-x) = -\phi'(\epsilon) = \frac{1}{\int_{\partial B(x,\epsilon)} dS}$ for $y \in \partial B(x,\epsilon)$. So, in (2.9), letting $\epsilon \to 0^+$ and noting that

$$\int_{\partial B(x,\epsilon)} u(y)\left(\frac{\partial \Phi}{\partial \nu_y}(y-x) - \frac{\partial h}{\partial \nu_y}(y)\right) dS_y$$
$$= \fint_{\partial B(x,\epsilon)} u(y) dS_y - \int_{\partial B(x,\epsilon)} u(y)\frac{\partial h}{\partial \nu_y}(y) dS_y \to u(x)$$

and

$$\int_{\partial B(x,\epsilon)} (\Phi(y-x) - h(y))\frac{\partial u}{\partial \nu_y}(y) dS \to 0,$$

we deduce

Theorem 2.2 (Representation formula). *Let $\Gamma(x,y) = \Phi(y-x) - h(y)$, where $h \in C^2(\bar\Omega)$ is harmonic in Ω. Then, for all $u \in C^2(\bar\Omega)$,*

$$u(x) = \int_{\partial\Omega}\left[\Gamma(x,y)\frac{\partial u}{\partial \nu_y}(y) - u(y)\frac{\partial \Gamma}{\partial \nu_y}(x,y)\right] dS - \int_\Omega \Gamma(x,y)\Delta u(y)\, dy \quad (x \in \Omega). \tag{2.10}$$

This formula permits us to solve for u *if* we know the values of Δu in Ω and *both* u and $\frac{\partial u}{\partial \nu}$ on $\partial\Omega$. However, for Poisson's equation with Dirichlet boundary condition, $\partial u/\partial \nu$ is not known (and cannot be prescribed arbitrarily). We must modify this formula to remove the boundary integral term involving $\partial u/\partial \nu$.

Given $x \in \Omega$, we assume that there exists a *corrector function* $h = h^x \in C^2(\bar\Omega)$ solving the special Dirichlet problem:

$$\begin{cases} \Delta_y h^x(y) = 0 & (y \in \Omega), \\ h^x(y) = \Phi(y-x) & (y \in \partial\Omega). \end{cases} \tag{2.11}$$

Definition 2.2. We define **Green's function** for domain Ω to be

$$G(x,y) = \Phi(y,x) - h^x(y) \quad (x \in \Omega,\ y \in \bar\Omega,\ x \neq y).$$

Then $G(x,y) = 0$ for $y \in \partial\Omega$ and $x \in \Omega$; hence, by (2.10),

$$u(x) = -\int_{\partial\Omega} u(y)\frac{\partial G}{\partial \nu_y}(x,y) dS - \int_\Omega G(x,y)\Delta u(y)\, dy. \tag{2.12}$$

The function

$$K(x,y) = -\frac{\partial G}{\partial \nu_y}(x,y) \quad (x \in \Omega,\ y \in \partial\Omega)$$

is called **Poisson's kernel** for domain Ω. Given a function g on $\partial\Omega$, the function

$$K[g](x) = \int_{\partial\Omega} K(x,y) g(y) dS_y \quad (x \in \Omega)$$

is called the **Poisson integral of g with kernel** K.

Remark 2.3. A corrector function h^x, if exists for bounded domain Ω, must be unique. Here we require the corrector function h^x exist in $C^2(\bar\Omega)$, which may not be possible for general bounded domains Ω. However, for bounded domains Ω with smooth boundary, existence of h^x in $C^2(\bar\Omega)$ is guaranteed by the general existence and regularity theory and consequently for such domains the Green's function always exists and is unique; we do not discuss these issues in this course.

Theorem 2.3 (Representation by Green's function)**.** *If $u \in C^2(\bar{\Omega})$ solves the Dirichlet problem*

$$\begin{cases} -\Delta u(x) = f(x) & (x \in \Omega), \\ u(x) = g(x) & (x \in \partial\Omega), \end{cases}$$

then

$$u(x) = \int_{\partial\Omega} K(x,y)g(y)dS + \int_{\Omega} G(x,y)f(y)\,dy \quad (x \in \Omega). \tag{2.13}$$

Theorem 2.4 (Symmetry of Green's function)**.** *$G(x,y) = G(y,x)$ for all $x, y \in \Omega,\ x \neq y$.*

Proof. Fix $x, y \in \Omega$, $x \neq y$. Let

$$v(z) = G(x,z), \quad w(z) = G(y,z) \qquad (z \in U).$$

Then $\Delta v(z) = 0$ for $z \neq x$ and $\Delta w(z)$ for $z \neq y$ and $v|_{\partial\Omega} = w|_{\partial\Omega} = 0$. For sufficiently small $\epsilon > 0$, we apply Green's second identity on $\Omega_\epsilon = \Omega \setminus (B(x,\epsilon) \cup B(y,\epsilon))$ for functions $v(z)$ and $w(z)$ to obtain

$$\int_{\partial\Omega_\epsilon} \left(v(z)\frac{\partial w}{\partial\nu}(z) - w(z)\frac{\partial v}{\partial\nu}(z) \right) dS = 0.$$

This implies

$$\int_{\partial B(x,\epsilon)} \left(v(z)\frac{\partial w}{\partial\nu}(z) - w(z)\frac{\partial v}{\partial\nu}(z) \right) dS = \int_{\partial B(y,\epsilon)} \left(w(z)\frac{\partial v}{\partial\nu}(z) - v(z)\frac{\partial w}{\partial\nu}(z) \right) dS, \tag{2.14}$$

where ν denotes the *inward* unit normal vector on $\partial B(x,\epsilon) \cup \partial B(x,\epsilon)$.

We compute the limits of two terms on both sides of (2.14) as $\epsilon \to 0^+$. For the term on LHS, since $w(z)$ is smooth near $z = x$,

$$\left| \int_{\partial B(x,\epsilon)} v(z)\frac{\partial w}{\partial\nu}(z)dS \right| \leq C\epsilon^{n-1} \sup_{z \in \partial B(x,\epsilon)} |v(z)| = o(1).$$

Also, $v(z) = \Phi(x-z) - h^x(z) = \Phi(z-x) - h^x(z)$, where the corrector h^x is smooth in Ω. Hence

$$\lim_{\epsilon\to 0^+} \int_{\partial B(x,\epsilon)} w(z)\frac{\partial v}{\partial\nu}(z)dS = \lim_{\epsilon\to 0^+} \int_{\partial B(x,\epsilon)} w(z)\frac{\partial\Phi}{\partial\nu}(z-x)dS = w(x).$$

So

$$\lim_{\epsilon\to 0^+} \text{LHS of (2.14)} = -w(x).$$

Likewise

$$\lim_{\epsilon\to 0^+} \text{RHS of (2.14)} = -v(y),$$

proving $w(x) = v(y)$, which is exactly $G(y,x) = G(x,y)$. □

Remark 2.4. (1) Strong Maximum Principle below implies that $G(x,y) > 0$ for all $x, y \in \Omega,\ x \neq y$. (**Homework!**) Since $G(x,y) = 0$ for $y \in \partial\Omega$, it follows that $\frac{\partial G}{\partial\nu_y}(x,y) \leq 0$, where ν_y is outer unit normal of Ω at $y \in \partial\Omega$. (In fact, we have $\frac{\partial G}{\partial\nu_y}(x,y) < 0$ for all $x \in \Omega$ and $y \in \partial\Omega$.)

(2) Since $G(x,y)$ is harmonic in $y \in \Omega \setminus \{x\}$, by the symmetry property, we know that $G(x,y)$ is also harmonic in $x \in \Omega \setminus \{y\}$. In particular, $G(x,y)$ is harmonic in $x \in \Omega$ for all $y \in \partial\Omega$; hence Poisson's kernel $K(x,y) = -\frac{\partial G}{\partial\nu_y}(x,y)$ is harmonic in $x \in \Omega$ for all $y \in \partial\Omega$.

(3) We always have that $K(x,y) \geq 0$ for all $x \in \Omega$ and $y \in \partial\Omega$ and that, by Green's representation theorem (Theorem 2.3), with $f = 0$, $g = 1$ and $u \equiv 1$,

$$\int_{\partial\Omega} K(x,y)dS_y = 1 \quad (x \in \Omega).$$

Theorem 2.5 (Poisson integrals as solutions)**.** *Assume, for all $x^0 \in \partial\Omega$ and $\delta > 0$*

$$\lim_{x \to x^0,\, x \in \Omega} \int_{\partial\Omega \setminus B(x^0,\delta)} K(x,y)dS_y = 0. \tag{2.15}$$

Then, for each $g \in C(\partial\Omega)$ its Poisson integral $u(x) = K[g](x)$ is harmonic in Ω and satisfies

$$\lim_{x \to x^0,\, x \in \Omega} u(x) = g(x^0) \quad (x^0 \in \partial\Omega).$$

That is, $u = K[g]$ can be extended to be a $C^2(\Omega) \cap C(\bar{\Omega})$-solution to the Dirichlet problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases}$$

Proof. Let $M = \|g\|_{L^\infty}$. For $\varepsilon > 0$, let $\delta > 0$ be such that

$$|g(y) - g(x^0)| < \epsilon \quad \forall\ y \in \partial\Omega,\ |y - x^0| < \delta.$$

Then

$$\begin{aligned} |u(x) - g(x^0)| &= \left| \int_{\partial\Omega} K(x,y)(g(y) - g(x^0))\, dS_y \right| \leq \int_{\partial\Omega} K(x,y)|g(y) - g(x^0)|\, dS_y \\ &\leq \int_{B(x^0,\delta)\cap\partial\Omega} K(x,y)|g(y) - g(x^0)|\, dS_y + \int_{\partial\Omega\setminus B(x^0,\delta)} K(x,y)|g(y) - g(x^0)|\, dS_y \\ &\leq \varepsilon \int_{B(x^0,\delta)\cap\partial\Omega} K(x,y)dS_y + 2M \int_{\partial\Omega\setminus B(x^0,\delta)} K(x,y)\, dS_y \\ &\leq \varepsilon + 2M \int_{\partial\Omega\setminus B(x^0,\delta)} K(x,y)\, dS_y. \end{aligned}$$

Hence, by assumption (2.15),

$$\limsup_{x \to x^0,\, x \in \Omega} |u(x) - g(x^0)| \leq \varepsilon + 2M \limsup_{x \to x^0,\, x \in \Omega} \int_{\partial\Omega\setminus B(x^0,\delta)} K(x,y)\, dS_y = \varepsilon.$$

□

2.2.3. Green's functions for a half-space and for a ball. Green's functions for certain special domains can be explicitly found from the fundamental solution $\Phi(x)$.

Fix $x \in \Omega$ and let $b(x) \in \mathbb{R}$ and $a(x) \in \mathbb{R}^n \setminus \Omega$. Then function $\Phi(b(x)(y - a(x))$ is harmonic in Ω. For such a function to be the corrector function $h^x(y)$ we need

$$\Phi(y - x) = \Phi(b(x)(y - a(x)) \quad (y \in \partial\Omega),$$

which requires that

$$|b(x)||y - a(x)| = |y - x| \quad (x \in \Omega,\ y \in \partial\Omega).$$

Such a construction is possible when Ω is a half-space or a ball. (Although a Green's function is defined above for a bounded domain with smooth boundary, it can be similarly defined for unbounded domains or domains with nonsmooth boundaries; however, the representation formula may not be valid for such domains or valid only in certain extended sense.)

Case 1. Green's function for a half-space. Let

$$\Omega = \mathbb{R}^n_+ = \{x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n \mid x_n > 0\}.$$

If $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$, its **reflection** with respect to the hyper-plane $x_n = 0$ is the point

$$\hat{x} = (x_1, x_2, \cdots, x_{n-1}, -x_n).$$

Clearly $\hat{\hat{x}} = x$ ($x \in \mathbb{R}^n$), $\hat{x} = x$ ($x \in \partial\mathbb{R}^n_+$) and $\Phi(x) = \Phi(\hat{x})$ ($x \in \mathbb{R}^n$).

In this case, we can easily see that $h^x(y) = \Phi(y - \hat{x})$ solves (2.11). So a **Green's function for** $\mathbb{R}^n_+$ is given by

$$G(x, y) = \Phi(y - x) - \Phi(y - \hat{x}) \quad (x, y \in \mathbb{R}^n_+).$$

Then

$$\begin{aligned} \frac{\partial G}{\partial y_n}(x, y) &= \frac{\partial \Phi}{\partial y_n}(y - x) - \frac{\partial \Phi}{\partial y_n}(y - \hat{x}) \\ &= \frac{-1}{n\alpha_n}\left[\frac{y_n - x_n}{|y - x|^n} - \frac{y_n + x_n}{|y - \hat{x}|^n}\right]. \end{aligned}$$

The corresponding **Poisson's kernel of** $\mathbb{R}^n_+$ is given by

$$K(x, y) = -\frac{\partial G}{\partial \nu}(x, y) = \frac{\partial G}{\partial y_n}(x, y) = \frac{2x_n}{n\alpha_n |x - y|^n} \quad (x \in \mathbb{R}^n_+,\ y \in \partial\mathbb{R}^n_+).$$

If we write $y = (y', 0)$ with $y' \in \mathbb{R}^{n-1}$, then

$$K(x, y) = \frac{2x_n}{n\alpha_n |x - y|^n} = \frac{2x_n}{n\alpha_n (|x' - y'|^2 + x_n^2)^{n/2}} := H(x, y').$$

Hence the Poisson integral $u = K[g]$ can be written as

$$u(x) = \int_{\mathbb{R}^{n-1}} H(x, y') g(y')\, dy' = \frac{2x_n}{n\alpha_n} \int_{\mathbb{R}^{n-1}} \frac{g(y')\, dy'}{(|x' - y'|^2 + x_n^2)^{n/2}} \qquad (x \in \mathbb{R}^n_+). \tag{2.16}$$

This formula is called **Poisson's formula** for $\mathbb{R}^n_+$. We show that this formula provides a solution to the Dirichlet problem.

Theorem 2.6 (Poisson's formula for half-space)**.** *Assume $g \in C(\mathbb{R}^{n-1}) \cap L^\infty(\mathbb{R}^{n-1})$ and define u by* (2.16)*. Then*

(i) $u \in C^\infty(\mathbb{R}^n_+) \cap L^\infty(\mathbb{R}^n_+)$,

(ii) $\Delta u(x) = 0$ *in* $\mathbb{R}^n_+$,

(iii) for all $x^0 \in \partial\mathbb{R}^n_+$,

$$\lim_{x \to x^0,\ x \in \mathbb{R}^n_+} u(x) = g(x^0).$$

This theorem differs from the general theorem (Theorem 2.5), where we assumed the domain Ω is bounded. However, the proof is similar; we give a full proof below just to emphasize the main idea.

Proof. 1. It can be verified that $H(x, y')$ is harmonic in $x \in \mathbb{R}^n_+$ for each $y' \in \mathbb{R}^{n-1}$ and (very complicated)

$$\int_{\mathbb{R}^{n-1}} H(x, y') dy' = 1 \quad (x \in \mathbb{R}^n_+).$$

2. Since g is bounded, u defined above is also bounded. Since $H(x,y')$ is smooth in $x \in \mathbb{R}^n_+$, we have

$$D^\alpha u(x) = \int_{\mathbb{R}^{n-1}} D^\alpha_x H(x,y) g(y') dy'$$

is continuous in $x \in \mathbb{R}^n_+$ for all multi-indexes. This also shows that

$$\Delta u(x) = \int_{\mathbb{R}^{n-1}} \Delta_x H(x,y') g(y') dy' = 0 \quad (x \in \mathbb{R}^n_+).$$

3. Now fix $x^0 \in \partial\mathbb{R}^n_+$, $\epsilon > 0$. Choose $\delta > 0$ so small that

$$|g(y') - g(x^0)| < \epsilon \quad \text{if } |y' - x^0| < \delta.$$

Then if $|x - x^0| < \delta/2$ and $x \in \mathbb{R}^n_+$,

$$|u(x) - g(x^0)| = \left| \int_{\mathbb{R}^{n-1}} H(x,y')(g(y') - g(x^0))\, dy' \right|$$

$$\leq \int_{\mathbb{R}^{n-1} \cap B(x^0,\delta)} H(x,y')|g(y') - g(x^0)|dy' + \int_{\mathbb{R}^{n-1} \setminus B(x^0,\delta)} H(x,y')|g(y') - g(x^0)|dy'$$

$$= I + J.$$

Now that

$$I \leq \epsilon \int_{\mathbb{R}^{n-1}} H(x,y')dy' = \epsilon.$$

Furthermore, if $|x - x^0| \leq \delta/2$ and $|y' - x^0| \geq \delta$, we have

$$|y' - x^0| \leq |y' - x| + \delta/2 \leq |y - x| + \frac{1}{2}|y' - x^0|$$

and so $|y' - x| \geq \frac{1}{2}|y' - x^0|$. Thus

$$J \leq 2\|g\|_{L^\infty} \int_{\mathbb{R}^{n-1} \setminus B(x^0,\delta)} H(x,y')dy'$$

$$\leq \frac{2^{n+2}\|g\|_{L^\infty} x_n}{n\alpha_n} \int_{\mathbb{R}^{n-1} \setminus B(x^0,\delta)} |y' - x^0|^{-n} dy'$$

$$= \frac{2^{n+2}\|g\|_{L^\infty} x_n}{n\alpha_n} \left((n-1)\alpha_{n-1} \int_\delta^\infty r^{-n} r^{n-2}\, dr \right)$$

$$= 2^{n+2}\|g\|_{L^\infty} \frac{(n-1)\alpha_{n-1}}{n\alpha_n \delta} x_n \to 0,$$

as $x_n \to 0^+$ if $x \to x^0$. Combining these estimates, we deduce $|u(x) - g(x^0)| \leq 2\epsilon$ provided that $|x - x^0|$ is sufficiently small. □

Case 2. Green's function for a ball. Let

$$\Omega = B(0,1) = \{x \in \mathbb{R}^n \mid |x| < 1\}.$$

If $x \in \mathbb{R}^n \setminus \{0\}$, the point

$$\tilde{x} = \frac{x}{|x|^2}$$

is called the **inversion point** to x with respect to the unit sphere $\partial B(0,1)$. The mapping $x \mapsto \tilde{x}$ is called **inversion** with respect to $\partial B(0,1)$.

Given $x \in B(0,1)$, $x \neq 0$, we try to find the corrector function h^x in the form of

$$h^x(y) = \Phi(b(x)(y - \tilde{x})).$$

We need to have

$$|b(x)||y - \tilde{x}| = |y - x| \quad (y \in \partial B(0,1)).$$

If $y \in \partial B(0,1)$ then $|y| = 1$ and

$$|y - \tilde{x}|^2 = 1 - 2y \cdot \tilde{x} + |\tilde{x}|^2 = 1 - \frac{2y \cdot x}{|x|^2} + \frac{1}{|x|^2} = \frac{|y - x|^2}{|x|^2}.$$

So $b(x) = |x|$ will do the job. Consequently, for $x \neq 0$, the corrector function h^x is given by

$$h^x(y) = \Phi(|x|(y - \tilde{x})) \quad (y \in \bar{B}(0,1)).$$

Definition 2.5. The **Green's function for ball** $B(0,1)$ is given by

$$G(x,y) = \begin{cases} \Phi(y - x) - \Phi(|x|(y - \tilde{x})) & (x \neq 0,\ x \neq y), \\ G(y,0) = \Phi(-y) - \Phi(-|y|\tilde{y}) = \Phi(y) - \phi(1) & (x = 0,\ y \neq 0). \end{cases}$$

(Note that $G(0,y)$ cannot be given by first formula since $\tilde{0}$ is undefined, but it is found from the symmetry of G: $G(0,y) = G(y,0)$ for $y \neq 0$.)

Since $\Phi_{y_i}(y) = \phi'(|y|)\frac{y_i}{|y|} = \frac{-y_i}{n\alpha_n |y|^n}$ ($y \neq 0$ and $n \geq 2$), we deduce, if $x \neq 0$, $x \neq y$,

$$\begin{aligned} G_{y_i}(x,y) &= \Phi_{y_i}(y - x) - \Phi_{y_i}(|x|(y - \tilde{x}))|x| \\ &= \frac{1}{n\alpha_n}\left[\frac{x_i - y_i}{|y - x|^n} - \frac{|x|^2((\tilde{x})_i - y_i)}{(|x||y - \tilde{x}|)^n}\right] \\ &= \frac{1}{n\alpha_n}\left[\frac{x_i - y_i}{|y - x|^n} - \frac{x_i - |x|^2 y_i}{(|x||y - \tilde{x}|)^n}\right]. \end{aligned}$$

So, if $y \in \partial B(0,1)$, since $|x||y - \tilde{x}| = |y - x|$ and $\nu_y = y$, we have

$$\begin{aligned} \frac{\partial G}{\partial \nu_y}(x,y) = \sum_{i=1}^n G_{y_i}(x,y) y_i &= \frac{1}{n\alpha_n}\sum_{i=1}^n \left[\frac{x_i y_i - y_i^2}{|y - x|^n} - \frac{x_i y_i - |x|^2 y_i^2}{|y - x|^n}\right] \\ &= \frac{1}{n\alpha_n}\frac{|x|^2 - 1}{|y - x|^n} \quad (x \in B(0,1) \setminus \{0\}). \end{aligned}$$

The same formula is also valid if $x = 0$ and $y \in \partial B(0,1)$.

Therefore, the **Poisson's kernel for** $B(0,1)$ is given by

$$K(x,y) = -\frac{\partial G}{\partial \nu_y}(x,y) = \frac{1}{n\alpha_n}\frac{1 - |x|^2}{|y - x|^n} \quad (x \in B(0,1),\ y \in \partial B(0,1)).$$

Given $g \in C(\partial B(0,1))$, its Poisson integral $u = K[g]$ is given by

$$u(x) = \int_{\partial B(0,1)} g(y) K(x,y)\, dS_y = \frac{1 - |x|^2}{n\alpha_n}\int_{\partial B(0,1)} \frac{g(y)\, dS_y}{|y - x|^n} \quad (x \in B(0,1)). \tag{2.17}$$

By Green's representation formula, the $C^2(\bar{B}(0,1))$-solution u of the Dirichlet problem

$$\begin{cases} \Delta u = 0 & \text{in } B(0,1), \\ u = g & \text{on } \partial B(0,1), \end{cases} \tag{2.18}$$

is given by the formula (2.17).

Suppose u is a C^2 solution to the following Dirichlet problem on a closed ball $\bar{B}(a,r)$, where $a \in \mathbb{R}^n$ is the center and $r > 0$ is the radius,

$$\begin{cases} \Delta u = 0 & \text{in } B(a,r), \\ u = g & \text{on } \partial B(a,r). \end{cases}$$

Then function $\tilde{u}(x) = u(a + rx)$ solves (2.18) with $\tilde{g}(x) = g(a + rx)$ replacing g. In this way, we deduce

$$\begin{aligned} u(x) = \tilde{u}\left(\frac{x-a}{r}\right) &= \frac{1 - |\frac{x-a}{r}|^2}{n\alpha_n} \int_{\partial B(0,1)} \frac{g(a+ry)\, dS_y}{|y - \frac{x-a}{r}|^n} \\ &= \frac{r^2 - |x-a|^2}{n\alpha_n r^2} \int_{\partial B(a,r)} \frac{g(z) r^{1-n}\, dS_z}{r^{-n}|z-x|^n} \quad (z = a + ry). \end{aligned}$$

Hence, changing z back to y,

$$\begin{aligned} u(x) &= \frac{r^2 - |x-a|^2}{n\alpha_n r} \int_{\partial B(a,r)} \frac{g(y)\, dS_y}{|y-x|^n} \\ &= \int_{\partial B(a,r)} g(y) K(x,y;a,r)\, dS_y \quad (x \in B(a,r)), \end{aligned} \tag{2.19}$$

where

$$K(x,y;a,r) = \frac{1}{n\alpha_n r} \frac{r^2 - |x-a|^2}{|y-x|^n} \quad (x \in B(a,r),\ y \in \partial B(a,r))$$

is the **Poisson's kernel for ball** $B(a,r)$.

The formula (2.19) is called **Poisson's formula on ball** $B(a,r)$. This formula has a special consequence that at the center $x = a$ it implies

$$u(a) = \frac{r}{n\alpha_n} \int_{\partial B(a,r)} \frac{g(y)}{|y|^n}\, dS_y = \frac{1}{n\alpha_n r^{n-1}} \int_{\partial B(a,r)} g(y)\, dS_y = ⨍_{\partial B(a,r)} g(y)\, dS_y.$$

Therefore, if u is harmonic in Ω and $B(a,r) \subset\subset \Omega$ (this means closed ball $\bar{B}(a,r) \subset \Omega$), then

$$u(a) = ⨍_{\partial B(a,r)} u(y) dS_y. \tag{2.20}$$

This is the **mean-value property** for harmonic functions that we will give another proof of and study further in the section.

To conclude, we show that Poisson's formula (2.19) indeed gives a smooth solution in $B(a,r)$ with boundary condition g.

Theorem 2.7 (Poisson's formula for ball)**.** *Assume $g \in C(\partial B(a,r))$ and define u by* (2.19)*. Then*

(i) $u \in C^\infty(B(a,r))$,

(ii) $\Delta u = 0$ *in* $B(a,r)$,

(iii) for each $x^0 \in \partial B(a,r)$,

$$\lim_{x\to x^0,\, x\in B(a,r)} u(x) = g(x^0).$$

Proof. This theorem follows from the general theorem (Theorem 2.5 above) by verifying condition (2.15). We prove the boundary condition (iii) only.

Fix $x^0 \in \partial B(0,1)$, $\epsilon > 0$. Choose $\delta > 0$ so small that

$$|g(y) - g(x^0)| < \epsilon \quad \text{if } y \in \partial B(0,1) \text{ and } |y - x^0| < \delta.$$

Then if $|x - x^0| < \delta/2$ and $x \in B(0,1)$,

$$\begin{aligned} |u(x) - g(x^0)| &= \left| \int_{\partial B(0,1)} K(x,y)(g(y) - g(x^0))\, dS_y \right| \\ &\le \int_{B(x^0,\delta)\cap\partial B(0,1)} K(x,y)|g(y) - g(x^0)| dS_y \\ &\quad + \int_{(\partial B(0,1))\setminus B(x^0,\delta)} K(x,y)|g(y) - g(x^0)| dS_y \\ &= I + J. \end{aligned}$$

Now that

$$I \le \epsilon \int_{\partial B(0,1)} K(x,y) dS_y = \epsilon.$$

Furthermore, if $|x - x^0| \le \delta/2$ and $|y - x^0| \ge \delta$, we have

$$|y - x| \ge |y - x^0| - |x - x^0| \ge \delta/2.$$

Thus

$$\begin{aligned} J &\le 2\|g\|_{L^\infty} \int_{(\partial B(0,1))\setminus B(x^0,\delta)} K(x,y) dS_y \\ &\le \frac{2\|g\|_{L^\infty}(1 - |x|^2)}{n\alpha_n} 2^n \delta^{-n} n\alpha_n = C\delta^{-n}(1 - |x|^2), \end{aligned}$$

which goes to 0 as $x \to x^0$, since $|x| \to |x^0| = 1$. Combining these estimates, we deduce $|u(x) - g(x^0)| \le 2\epsilon$ provided that $|x - x^0|$ is sufficiently small. □

2.3. Mean-value property

For two sets V and Ω, we write $V \subset\subset \Omega$ if the closure $\bar{V}$ of V is a *compact subset* of Ω.

Theorem 2.8 (Mean-value property for harmonic functions)**.** *Let* $u \in C^2(\Omega)$ *be harmonic. Then*

$$u(x) = ⨍_{\partial B(x,r)} u(y) dS = ⨍_{B(x,r)} u(y)\, dy$$

for each ball $B(x,r) \subset\subset \Omega$.

Proof. The first equality (the **spherical mean-value property**) has been proved above by Poisson's formula in (2.20). We now give another proof. Let $B(x,r) \subset\subset \Omega$. For any $\rho \in (0,r)$, let

$$h(\rho) = ⨍_{\partial B(x,\rho)} u(y) dS_y = ⨍_{\partial B(0,1)} u(x + \rho z) dS_z.$$

Then, using Green's first identity (2.3) with $v = 1$,

$$h'(\rho) = \fint_{\partial B(0,1)} \nabla u(x+\rho z)\cdot z dS_z = \fint_{\partial B(x,\rho)} \nabla u(y)\cdot \frac{y-x}{\rho}\, dS_y$$

$$= \fint_{\partial B(x,\rho)} \nabla u(y)\cdot \nu_y\, dS_y = \fint_{\partial B(x,\rho)} \frac{\partial u(y)}{\partial \nu_y}\, dS_y = \frac{\rho}{n}\fint_{B(x,\rho)} \Delta u(y)\, dy = 0.$$

(Note that $|\partial B(0,1)| = n|B(0,1)|$.) This proves that h is constant on $(0,r)$. Hence

$$h(r) = h(0^+) = \lim_{\rho\to 0^+}\fint_{\partial B(0,1)} u(x+\rho z) dS_z = u(x).$$

Hence

$$\fint_{\partial B(x,r)} u(y) dS_y = u(x). \tag{2.21}$$

So, by the polar coordinate formula,

$$\int_{B(x,r)} u(y)\, dy = \int_0^r \left(\int_{\partial B(x,\rho)} u(y) dS_y\right) d\rho$$

$$= u(x)\int_0^r |\partial B(x,\rho)|\, d\rho = u(x)|B(x,r)|.$$

This proves the **ball mean-value property**

$$u(x) = \fint_{B(x,r)} u(y)\, dy. \tag{2.22}$$

□

Remark 2.6. The ball mean-value property (2.22) holds for all balls $B(x,r) \subset\subset \Omega$ with $0 < r < r_x$, where $0 < r_x \le \operatorname{dist}(x,\partial\Omega)$ is a number depending x, if and only if the spherical mean-value property (2.21) holds for all spheres $\partial B(x,r)$ with $0 < r < r_x$. This follows from the polar coordinate formula as above:

$$\int_{B(x,r)} u(y)\, dy = \int_0^r \left(\int_{\partial B(x,\rho)} u(y) dS_y\right) d\rho.$$

For example, if (2.22) holds for all balls $B(x,r) \subset\subset \Omega$ with $0 < r < r_x$ then

$$\int_0^r \left(\int_{\partial B(x,\rho)} u(y) dS_y\right) d\rho = u(x)\int_0^r |\partial B(x,\rho)|\, d\rho$$

for all $0 < r < r_x$. Differentiating with respect to r yields (2.21) for all spheres $\partial B(x,r)$ with $0 < r < r_x$.

Theorem 2.9 (Converse to mean-value property)**.** *Let $u \in C^2(\Omega)$ satisfy*

$$u(x) = \fint_{\partial B(x,r)} u(y) dS$$

for all $\partial B(x,r)$ with $0 < r < r_x$, where $0 < r_x \le \operatorname{dist}(x,\partial\Omega)$ is a number depending x. Then u is harmonic in Ω.

Proof. If $\Delta u(x_0) \neq 0$, say, $\Delta u(x_0) > 0$, then there exists a ball $B(x_0, r)$ with $0 < r < r_{x_0}$ such that $\Delta u(y) > 0$ on $B(x_0, r)$. Consider function

$$h(\rho) = \fint_{\partial B(x_0,\rho)} u(y)dS \quad (0 < \rho < r_{x_0}).$$

The assumption says that $h(\rho)$ is constant on $\rho \in (0, r_{x_0})$; however, by the computation as above

$$h'(r) = \frac{r}{n} \fint_{B(x_0,r)} \Delta u(y)\, dy > 0,$$

a contradiction. □

The following result shows that if u is only continuous and satisfies mean-value property then u is in fact C^∞ and thus harmonic.

Theorem 2.10 (Mean-value property and C^∞-regularity)**.** *Let $u \in C(\Omega)$ satisfy*

$$u(x) = \fint_{\partial B(x,r)} u(y)dS$$

for all spheres $\partial B(x, r)$ with $0 < r < \operatorname{dist}(x, \partial\Omega)$. Then $u \in C^\infty(\Omega)$ is harmonic in Ω. Therefore, every harmonic function on Ω is in $C^\infty(\Omega)$.

Proof. Let $u_\epsilon = u * \eta_\epsilon$ be the smooth mollification of u, where $\eta_\epsilon(x) = \epsilon^{-n}\eta(|x|/\epsilon)$ is the standard radial mollifier. For each sufficiently small $\epsilon > 0$, let

$$\Omega_\epsilon = \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) > \epsilon\}.$$

Then $u_\epsilon \in C^\infty(\Omega_\epsilon)$. (For more on smooth mollification, see Appendix C.5.)
Let $x \in \Omega_\epsilon$; then $B(x, \epsilon) \subset\subset \Omega$. We show $u_\epsilon(x) = u(x)$; hence $u = u_\epsilon$ on Ω_ϵ and so $u \in C^\infty(\Omega_\epsilon)$. This proves that u is in $C^\infty(\Omega)$ and, by the previous theorem, u is harmonic in Ω. Indeed,

$$u_\epsilon(x) = \int_\Omega \eta_\epsilon(x-y)u(y)\, dy = \epsilon^{-n} \int_{B(x,\epsilon)} \eta(|x-y|/\epsilon)u(y)\, dy$$

$$= \epsilon^{-n} \int_0^\epsilon \eta(r/\epsilon) \left(\int_{\partial B(x,r)} u(y)dS_y \right) dr = \epsilon^{-n} u(x) \int_0^\epsilon \eta(r/\epsilon) n\alpha_n r^{n-1}\, dr$$

$$= u(x) \int_{B(0,\epsilon)} \eta_\epsilon(y)\, dy = u(x).$$

□

Remark 2.7. Theorem 2.10 still holds if the mean-value property is satisfied for all sufficiently small spheres $\partial B(x, r)$. See Lemmas 2.20 and 2.21 below.

2.4. Maximum principles

Theorem 2.11 (Maximum principle)**.** *Let Ω be a bounded open set in $\mathbb{R}^n$. Assume $u \in C^2(\Omega) \cap C(\bar{\Omega})$ is a harmonic function in Ω.*

(1) *Then the* **maximum principle** *holds:*

$$\max_{\bar{\Omega}} u = \max_{\partial\Omega} u.$$

(2) (**Strong maximum principle**) *If, in addition, Ω is connected and there exists an interior point $x_0 \in \Omega$ such that $u(x_0) = \max_{\bar{\Omega}} u$, then $u(x) \equiv u(x_0)$ for all $x \in \bar{\Omega}$. In other words, if u is a non-constant harmonic function in a bounded domain Ω, then*

$$u(x) < \max_{\partial\Omega} u \quad (x \in \Omega).$$

Proof. Explain why (2) implies (1). So we only prove (2). To this end, set

$$S = \{x \in \Omega \mid u(x) = u(x_0)\}.$$

This set is nonempty since $x_0 \in S$. It is relatively closed in Ω since u is continuous. We show S is an open set. Note that the only nonempty subset that is both open and relatively closed in a connected set is the set itself. Hence $S = \Omega$ because Ω is connected. This proves the conclusion (2). To show S is open, take any $x \in S$; that is, $u(x) = u(x_0)$. Assume $B(x,r) \subset\subset \Omega$. Since $u(x) = u(x_0) = \max_{\bar{\Omega}} u$, by the mean-value property,

$$u(x) = ⨍_{B(x,r)} u(y)\,dy \le ⨍_{B(x,r)} u(x)\,dy = u(x).$$

So equality must hold, which implies $u(y) = u(x)$ for all $y \in B(x,r)$; so, $B(x,r) \subset S$ and hence $x \in S$ is an interior point of S, proving that S is open. □

If u is harmonic then $-u$ is also harmonic; hence, applying the maximum principles to $-u$, one also has the **minimum principles**: all previous results are valid if maximum is changed to minimum. In particular, we have the following **positivity** result for all bounded *open connected sets* (domains) Ω:

If $u \in C^2(\Omega) \cap C(\bar{\Omega})$ is harmonic and $u|_{\partial\Omega} \ge 0$ but $\not\equiv 0$, then $u(x) > 0$ for all $x \in \Omega$.

Proof. By the minimum principle, $u \ge 0$ on $\bar{\Omega}$. If $u(x_0) = 0$ at some $x_0 \in \Omega$, then by the strong minimum principle $u \equiv 0$ on $\bar{\Omega}$, which implies $u \equiv 0$ on $\partial\Omega$, a contradiction. □

Theorem 2.12 (Uniqueness for Dirichlet problem)**.** *Given f and g, the Dirichlet problem for Poisson's equation*

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega \end{cases}$$

can have at most one solution $u \in C^2(\Omega) \cap C(\bar{\Omega})$.

Proof. Let u_1, u_2 be any two solutions to the problem in $C^2(\Omega) \cap C(\bar{\Omega})$. Let $u = u_1 - u_2$. Then $u \in C^2(\Omega) \cap C(\bar{\Omega})$ is harmonic in Ω and $u = 0$ on $\partial\Omega$. Hence $\max_{\partial\Omega} = \min_{\partial\Omega} = 0$; so, by the maximum and minimum principles,

$$\max_{\bar{\Omega}} u = \max_{\partial\Omega} u = 0, \quad \min_{\bar{\Omega}} u = \min_{\partial\Omega} u = 0,$$

which implies $u \equiv 0$ on $\bar{\Omega}$. Hence $u_1 \equiv u_2$ on $\bar{\Omega}$. □

2.5. Estimates of higher-order derivatives and Liouville's theorem

Theorem 2.13 (Local estimates on derivatives)**.** *Assume u is harmonic in Ω. Then*

$$|D^\alpha u(x)| \le \frac{C_k}{r^{n+k}} \|u\|_{L^1(B(x,r))} \tag{2.23}$$

for each ball $B(x,r) \subset\subset \Omega$ and each multi-index α of order $|\alpha| = k$.

Proof. We use induction on k.

1. If $k=1$, say $\alpha=(1,0,\cdots,0)$ then $D^\alpha u = u_{x_1}$. Since $u\in C^\infty$ is harmonic, u_{x_1} is also harmonic; hence, if $B(x,r)\subset\subset\Omega$,

$$u_{x_1}(x) = ⨍_{B(x,r/2)} u_{x_1}(y)dy = \frac{2^n}{\alpha_n r^n}\int_{B(x,r/2)} u_{x_1}(y)dy$$
$$= \frac{2^n}{\alpha_n r^n}\int_{\partial B(x,r/2)} \nu_1 u(y)\,dS.$$

So

$$|u_{x_1}(x)| \le \frac{2^n}{\alpha_n r^n}\int_{\partial B(x,r/2)} |u(y)|\,dS \le \frac{2n}{r}\max_{y\in\partial B(x,r/2)}|u(y)|.$$

However, for each $y\in\partial B(x,r/2)$, one has $B(y,r/2)\subset\subset B(x,r)\subset\subset\Omega$ and hence

$$|u(y)| = \left| ⨍_{B(y,r/2)} u(z)dz\right| \le \frac{2^n}{\alpha_n r^n}\|u\|_{L^1(B(x,r))}.$$

Combining the inequalities above, we have

$$|u_{x_1}(x)| \le \frac{2^{n+1}n}{\alpha_n r^{n+1}}\|u\|_{L^1(B(x,r))}.$$

This proves (2.23) with $k=1$, with $C_1=\frac{2^{n+1}n}{\alpha_n}$.

2. Assume now $k\ge 2$. Let $|\alpha|=k$. Then, for some i, $\alpha=\beta+(0,\cdots,1,0,\cdots,0)$, where $|\beta|=k-1$. So $D^\alpha u=(D^\beta u)_{x_i}$ and, as above,

$$|D^\alpha u(x)| \le \frac{nk}{r}\|D^\beta u\|_{L^\infty(B(x,r/k))}.$$

If $y\in B(x,r/k)$ then $B(y,\frac{k-1}{k}r)\subset\subset B(x,r)\subset\subset\Omega$; hence, by induction assumption with $D^\beta u$ at y,

$$|D^\beta u(y)| \le \frac{C_{k-1}}{(\frac{k-1}{k}r)^{n+k-1}}\|u\|_{L^1(B(y,\frac{k-1}{k}r))} \le \frac{C_{k-1}(\frac{k}{k-1})^{n+k-1}}{r^{n+k-1}}\|u\|_{L^1(B(x,r))}.$$

Combining the previous two inequalities, we have

$$|D^\alpha(x)| \le \frac{C_k}{r^{n+k}}\|u\|_{L^1(B(x,r))},$$

where $C_k\ge C_{k-1}nk(\frac{k}{k-1})^{n+k-1}$. For example, we can choose

$$C_k = \frac{(2^{n+1}nk)^k}{\alpha_n},\quad k=1,2,\cdots.$$

□

Theorem 2.14 (Liouville's Theorem)**.** *Suppose u is a bounded harmonic function on whole $\mathbb{R}^n$. Then u is constant.*

Proof. Let $|u(y)|\le M$ for all $y\in\mathbb{R}^n$. By (2.23) with $k=1$, for each $i=1,2,\cdots,n$,

$$|u_{x_i}(x)| \le \frac{C_1}{r^{n+1}}\|u\|_{L^1(B(x,r))} \le \frac{C_1}{r^{n+1}}M\alpha_n r^n = \frac{MC_1}{r}.$$

This inequality holds for all r since $B(x,r)\subset\subset\mathbb{R}^n$. Taking $r\to\infty$ we have $u_{x_i}(x)=0$ for each $i=1,2,\cdots,n$. Hence u is constant. □

Theorem 2.15 (Representation formula)**.** *Let $n \geq 3$ and $f \in C_c^\infty(\mathbb{R}^n)$. Then any bounded solution of*

$$-\Delta u = f \qquad \text{on } \mathbb{R}^n$$

has the form

$$u(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y)\, dy + C$$

for some constant C.

Proof. Since $\Phi(y) \to 0$ as $|y| \to \infty$ if $n \geq 3$, we can prove that the (**Newton's potential**) function

$$\tilde{u}(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y)\, dy$$

is bounded on $\mathbb{R}^n$. This function solves Poisson's equation $-\Delta \tilde{u} = f$. Hence $u - \tilde{u}$ is a bounded harmonic function on $\mathbb{R}^n$. By Liouville's Theorem, $u - \tilde{u} = C$, a constant. □

Theorem 2.16 (Compactness of sequence of harmonic functions)**.** *Suppose $\{u^j\}$ be a sequence of harmonic functions in Ω and*

$$|u^j(x)| \leq M \quad (x \in \Omega,\ j = 1, 2, \cdots).$$

Let $V \subset\subset \Omega$. Then there exists a subsequence $\{u^{j_k}\}$ and a harmonic function $\bar{u}$ in V such that

$$\lim_{k\to\infty} \|u^{j_k} - \bar{u}\|_{L^\infty(V)} = 0.$$

Proof. Let $0 < r < \operatorname{dist}(V, \partial\Omega)$. Then $B(x,r) \subset\subset \Omega$ for all $x \in \bar{V}$. Applying the local estimate above, we have

$$|Du^j(x)| \leq \frac{CM}{r} \quad (x \in \bar{V},\ j = 1, 2, \cdots).$$

By Arzela-Ascoli's theorem, there exists a subsequence of $\{u^j\}$ which converges uniformly in V to a function $\bar{u}$ in $\bar{V}$. This uniform limit function $\bar{u}$ certainly satisfies the mean-value property in V since each u^j does and hence is harmonic in V. □

Theorem 2.17 (Harnack's inequality)**.** *For each subdomain $V \subset\subset \Omega$, there exists a constant $C = C(V, \Omega)$ such that inequality*

$$\sup_V u \leq C \inf_V u$$

holds for all nonnegative *harmonic functions u on Ω.*

Proof. Let $r = \frac{1}{4}\operatorname{dist}(V, \partial\Omega)$. Let $x, y \in V$, $|x-y| \leq r$. Then $B(y,r) \subset B(x,2r) \subset\subset \Omega$ and hence

$$u(x) = \fint_{B(x,2r)} u(z)dz \geq \frac{1}{\alpha_n 2^n r^n} \int_{B(y,r)} u(z)dz = \frac{1}{2^n} \fint_{B(y,r)} u(z)dz = \frac{1}{2^n} u(y).$$

Thus $2^n u(y) \geq u(x) \geq \frac{1}{2^n} u(y)$ if $x, y \in V$ and $|x-y| \leq r$.

Since V is connected and $\bar{V}$ is compact in Ω, we can cover $\bar{V}$ by a chain of finitely many balls $\{B_i\}_{i=1}^N$, each of which has radius $r/2$ and $B_i \cap B_{i-1} \neq \emptyset$ for $i = 1, 2, \cdots, N$. Then

$$u(x) \geq \frac{1}{2^{n(N+1)}} u(y) \quad \forall\ x,\ y \in V.$$

□

2.6. Perron's method for existence using subharmonic functions

Definition 2.8. We call $u\colon \Omega \to \mathbb{R}$ **subharmonic in** Ω if $u \in C(\Omega)$ and if for every $\xi \in \Omega$ the inequality

$$u(\xi) \leq \fint_{\partial B(\xi,\rho)} u(x)dS := M_u(\xi,\rho)$$

holds for all sufficiently small $\rho > 0$. We denote by $\sigma(\Omega)$ the set of all subharmonic functions in Ω.

Lemma 2.18. *For $u \in C(\bar{\Omega}) \cap \sigma(\Omega)$,*

$$\max_{\bar{\Omega}} = \max_{\partial\Omega} u.$$

For $u \in C(\Omega)$ and $B(\xi,\rho) \subset\subset \Omega$, we define $u_{\xi,\rho}(x)$ by

$$u_{\xi,\rho}(x) = \begin{cases} u(x) & \text{if } x \in \Omega \setminus B(\xi,\rho), \\ \frac{\rho^2-|x-\xi|^2}{n\alpha_n\rho} \int_{\partial B(\xi,\rho)} \frac{u(y)}{|y-x|^n}\, dS_y & \text{if } x \in B(\xi,\rho). \end{cases}$$

Note that in $B(\xi,\rho)$ the function $u_{\xi,\rho}$ is simply Poisson's integral of $u|_{\partial B(\xi,\rho)}$ and hence is harmonic and takes the same boundary on $\partial B(\xi,\rho)$. Therefore $u_{\xi,\rho}$ is in $C(\Omega)$.

Lemma 2.19. *For $u \in \sigma(\Omega)$ and $B(\xi,\rho) \subset\subset \Omega$, we have $u_{\xi,\rho} \in \sigma(\Omega)$ and*

$$u(x) \leq u_{\xi,\rho}(x) \quad \forall\, x \in \Omega. \tag{2.24}$$

Proof. By definition, (2.24) holds if $x \notin B(\xi,\rho)$. Note that $u - u_{\xi,\rho}$ is subharmonic in $B(\xi,\rho)$ and equals zero on $\partial B(\xi,\rho)$; hence, by the previous lemma, $u \leq u_{\xi,\rho}$ on $B(\xi,\rho)$. Hence (2.24) is proved. To show that $u_{\xi,\rho}$ is subharmonic in Ω, we have to show that for any $x \in \Omega$

$$u_{\xi,\rho}(x) \leq \fint_{\partial B(x,r)} u_{\xi,\rho}(y)\, dS_y = M_{u_{\xi,\rho}}(x,r) \tag{2.25}$$

for all sufficiently small $r > 0$. If $x \notin \partial B(\xi,\rho)$, then there exists a ball $B(x,r') \subset\subset \Omega$ such that either $B(x,r') \subset \Omega \setminus \bar{B}(\xi,\rho)$ or $B(x,r') \subset B(\xi,\rho)$ and hence either $u_{\xi,\rho}(y) = u(y)$ for all $y \in B(x,r')$ or $u_{\xi,\rho}(y)$ is harmonic in $B(x,r')$. In first case, (2.25) holds if $0 < r < r'$ is sufficiently small since u is subharmonic, while in second case, (2.25) holds for all $0 < r < r'$ since $u_{\xi,\rho}$ is harmonic. We need to show (2.25) if $x \in \partial B(\xi,\rho)$. In this case, by (2.24),

$$u_{\xi,\rho}(x) = u(x) \leq M_u(x,r) \leq M_{u_{\xi,\rho}}(x,r)$$

for all sufficiently small $r > 0$. □

Lemma 2.20. *For $u \in \sigma(\Omega)$,*

$$u(\xi) \leq M_u(\xi,\rho)$$

whenever $B(\xi,\rho) \subset\subset \Omega$.

Proof. Let $B(\xi,\rho) \subset\subset \Omega$. Then

$$u(\xi) \leq u_{\xi,\rho}(\xi) = M_{u_{\xi,\rho}}(\xi,\rho) = M_u(\xi,\rho).$$

□

Lemma 2.21. *u is harmonic in Ω if and only if both u and $-u$ belong to $\sigma(\Omega)$.*

Proof. If $u, -u \in \sigma(\Omega)$, then for all $B(\xi,\rho) \subset\subset \Omega$,

$$u \le u_{\xi,\rho}, \quad -u \le -u_{\xi,\rho} \quad \text{in } \Omega.$$

Hence $u \equiv u_{\xi,\rho}$; in particular, u is harmonic in $B(\xi,\rho)$. □

Given $g \in C(\partial\Omega)$, define

$$\sigma_g(\Omega) = \{u \in C(\bar{\Omega}) \cap \sigma(\Omega) \mid u \le g \text{ on } \partial\Omega\},$$

and

$$w_g(x) = \sup_{u\in\sigma_g(\Omega)} u(x) \quad (x \in \Omega).$$

Set

$$m = \min_{\partial\Omega} g, \quad M = \max_{\partial\Omega} g.$$

Then m, M are finite, and $m \in \sigma_g(\Omega)$. So $\sigma_g(\Omega)$ is non-empty. Also, by the maximum principle,

$$u(x) \le \max_{\partial\Omega} u = M \quad \text{for } u \in \sigma_g(\Omega),\ x \in \bar{\Omega}.$$

Hence the function w_g is well-defined in Ω.

The following has been left as a homework problem.

Lemma 2.22. *Let $v_1, v_2, \cdots, v_k \in \sigma_g(\Omega)$ and $v = \max\{v_1, v_2, \cdots, v_k\}$. Then $v \in \sigma_g(\Omega)$.*

Lemma 2.23. *w_g is harmonic in Ω.*

Proof. Let $B(\xi,\rho) \subset\subset \Omega$. Let $x^1, x^2, \cdots,$ be a sequence of points in $B(\xi,\rho')$, where $0 < \rho' < \rho$. We can find $u_k^j \in \sigma_g(\Omega)$ such that

$$w_g(x^k) = \lim_{j\to\infty} u_k^j(x^k) \quad (k = 1, 2, \cdots).$$

Define

$$u^j(x) = \max\{m, u_1^j(x), u_2^j(x), \cdots, u_j^j(x)\} \quad (x \in \bar{\Omega}).$$

Then $u^j \in \sigma_g(\Omega)$, $m \le u^j(x) \le M$ $(x \in \bar{\Omega})$, and

$$\lim_{j\to\infty} u^j(x^k) = w_g(x^k) \quad (k = 1, 2, \cdots).$$

Let $v^j = u^j_{\xi,\rho}$; then v^j is harmonic in $B(\xi,\rho)$,

$$m \le v^j(x) \le M \quad (x \in \bar{\Omega}),$$

and

$$\lim_{j\to\infty} v^j(x^k) = w_g(x^k) \quad (k = 1, 2, \cdots).$$

Since $\{v^j\}$ is a bounded sequence of harmonic functions in $B(\xi,\rho)$, by the compactness theorem (Theorem 2.16), there exists a subsequence uniformly convergent to a harmonic function W on $B(\xi,\rho')$. Hence

$$W(x^k) = w_g(x^k) \quad (k = 1, 2, \cdots). \tag{2.26}$$

Of course W depends on the choice of $\{x^k\}$ and the subsequence of $\{v^j\}$. However the function w_g is independent of all these choices.

Let $x^0 \in B(\xi,\rho')$. We first choose $\{x^k\}$ in $B(\xi,\rho')$ such that $x^1 = x^0$ and $x^k \to x^0$. With this choice of $\{x^k\}$, by (2.26) and the continuity of W,

$$w_g(x^0) = W(x^0) = \lim_{k\to\infty} W(x^k) = \lim_{k\to\infty} w_g(x^k).$$

Since $\{x^k\}$ is arbitrary, this proves the continuity of w_g on $B(\xi, \rho')$. So w_g is continuous in Ω.

We now choose $\{x^k\}$ to be a dense sequence in $B(\xi, \rho')$. Then (2.26) implies that $w_g \equiv W$ in $B(\xi, \rho')$. Since W is harmonic in $B(\xi, \rho')$, so is w_g in $B(\xi, \rho')$. This proves that w_g is harmonic in Ω. $\square$

Therefore, we have proved that the function w_g is harmonic in Ω. To study the behavior of $w_g(x)$ as x approaches a boundary point $\eta \in \partial\Omega$, we need some property of the boundary $\partial\Omega$ at point η.

A point $\eta \in \partial\Omega$ is called **regular** if there exists a **barrier function** $Q_\eta(x) \in C(\bar{\Omega}) \cap \sigma(\Omega)$ such that

$$Q_\eta(\eta) = 0, \quad Q_\eta(x) < 0 \quad (x \in \partial\Omega \setminus \{\eta\}).$$

Lemma 2.24. *If $\eta \in \partial\Omega$ is regular, then*

$$\lim_{x \to \eta,\, x \in \Omega} w_g(x) = g(\eta).$$

Proof. 1. We first prove

$$\liminf_{x \to \eta,\, x \in \Omega} w_g(x) \geq g(\eta). \tag{2.27}$$

Let $\epsilon > 0, K > 0$ be constants and

$$u(x) = g(\eta) - \epsilon + KQ_\eta(x).$$

Then $u \in C(\bar{\Omega}) \cap \sigma(\Omega)$, $u(\eta) = g(\eta) - \epsilon$, and

$$u(x) \leq g(\eta) - \epsilon \quad (x \in \partial\Omega).$$

Since g is continuous, there exists a $\delta > 0$ such that $g(x) > g(\eta) - \epsilon$ for $x \in \partial\Omega \cap B(\eta, \delta)$. Hence

$$u(x) \leq g(x) \quad (x \in \partial\Omega \cap B(\eta, \delta)).$$

Since $Q_\eta(x) < 0$ on compact set $\partial\Omega \setminus B(\eta, \delta)$, it follows that $Q_\eta(x) \leq -\gamma$ on $\partial\Omega \setminus B(\eta, \delta)$, where $\gamma > 0$ is a constant. Let $K = \frac{M-m}{\gamma} \geq 0$. Then

$$u(x) = g(\eta) - \epsilon + KQ_\eta(x) \leq M - K\gamma = m \leq g(x) \quad (x \in \partial\Omega \setminus B(\eta, \delta)).$$

Thus $u \leq g$ on $\partial\Omega$ and, by definition, $u \in \sigma_g(\Omega)$. So

$$u(x) \leq w_g(x) \quad (x \in \Omega).$$

Then

$$g(\eta) - \epsilon = \lim_{x \to \eta, x \in \Omega} u(x) \leq \liminf_{x \to \eta, x \in \Omega} w_g(x).$$

This proves (2.27).

2. We now prove

$$\limsup_{x \to \eta,\, x \in \Omega} w_g(x) \leq g(\eta). \tag{2.28}$$

We consider function $-w_{-g}(x)$ which is defined by

$$-w_{-g}(x) = -\sup_{v \in \sigma_{-g}(\Omega)} u(x) = \inf U(x),$$

where $U = -v$, with $v \in \sigma_{-g}(\Omega)$, satisfies

$$-U \in C(\bar{\Omega}) \cap \sigma(\Omega), \quad -U \leq -g \quad \text{on } \partial\Omega.$$

For each $u \in \sigma_g(\Omega)$ and each $U = -v$ as above, it follows that $u - U \in \sigma(\Omega) \cap C(\bar{\Omega})$ and

$$u - U \le g + (-g) = 0 \quad \text{on } \partial\Omega.$$

Hence, by the maximum principle, $u - U \le 0$ in $\bar{\Omega}$. This proves that $u(x) \le U(x)$ $(x \in \Omega)$ for any u and U above and hence

$$w_g(x) \le -w_{-g}(x) \quad \forall\, x \in \Omega.$$

Applying (2.27) above to w_{-g}, we deduce (2.28) as follows:

$$\limsup_{x\to\eta,\, x\in\Omega} w_g(x) \le \limsup_{x\to\eta,\, x\in\Omega} (-w_{-g}(x)) = -\liminf_{x\to\eta,\, x\in\Omega} w_{-g}(x) \le g(\eta).$$

□

Theorem 2.25. *Let $\Omega \subset \mathbb{R}^n$ be bounded. The Dirichlet problem*

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega \end{cases}$$

has a solution $u \in C^2(\Omega) \cap C(\bar{\Omega})$ for all continuous boundary data $g \in C(\partial\Omega)$ if and only if every point $\eta \in \partial\Omega$ is regular.

Proof. If every point $\eta \in \partial\Omega$ is regular, then for each $g \in C(\partial\Omega)$, the function $w_g(x)$ defined above is harmonic in Ω and has boundary limit g; so, extending w_g to $\partial\Omega$ by g gives a solution u to the problem in $C^2(\Omega) \cap C(\bar{\Omega})$.

On the other hand, given $\eta \in \partial\Omega$, if the Dirichlet problem is solvable for boundary data $g(x) = -|x-\eta|$ in $C^2(\Omega) \cap C(\bar{\Omega})$, then the solution u is a barrier function at η, $u(x) = Q_\eta(x)$, (Explain why); hence η is regular. □

Remark 2.9. A domain Ω is said to have the **exterior ball property at boundary point** $\eta \in \partial\Omega$ if there is a closed ball $B = \bar{B}(x_0, \rho)$ in the exterior domain $\mathbb{R}^n \setminus \Omega$ such that $B \cap \partial\Omega = \{\eta\}$. In this case, η is regular point of $\partial\Omega$ since the barrier function at η can be chosen as

$$Q_\eta(x) = \Phi(x - x_0) - \phi(\rho),$$

where $\Phi(x) = \phi(|x|)$ is the fundamental solution above.

We say that Ω has the **exterior ball property** if it has this property at every point $\eta \in \partial\Omega$. For such domains, the Dirichlet problem is always uniquely solvable in $C^2(\Omega) \cap C(\bar{\Omega})$.

In particular, if Ω is strictly convex, then Ω has the exterior ball property.

2.7. Maximum principles for second-order linear elliptic equations

(*This material is from Section 6.4 of the textbook.*)

The maximum principle for harmonic functions depends heavily on the mean-value property of harmonic functions and so does the Perron's method for the solvability of Dirichlet problem of Laplace's equation. Therefore, the proof of maximum principles and the technique of Perron's method are strongly limited to Laplace's equation and can not be used to other more general important PDEs in geometry, physics and applications.

We study the maximum principles for second-order linear *elliptic* PDEs in this section. Existence of the so-called *weak solutions* for some of these equations is based on **energy method** in Sobolev spaces and other modern methods of PDEs and will be briefly introduced later; if you are interested in more of the modern theory of PDEs, you should register in the sequence courses MTH 940-941.

2.7.1. Second-order linear elliptic PDEs. We consider the second-order linear differential operator

$$Lu(x) = -\sum_{i,j=1}^{n} a^{ij}(x)D_{ij}u(x) + \sum_{i=1}^{n} b^i(x)D_iu(x) + c(x)u(x).$$

Here $D_iu = u_{x_i}$ and $D_{ij}u = u_{x_ix_j}$. Without the loss of generality, we assume that $a^{ij}(x) = a^{ji}(x)$. Throughout this section we assume that all functions a^{ij}, b^i and c are well-defined everywhere in Ω.

Definition 2.10. The operator Lu is called **elliptic** in Ω if there exists $\lambda(x) > 0$ $(x \in \Omega)$ such that

$$\sum_{i,j=1}^{n} a^{ij}(x)\xi_i\xi_j \geq \lambda(x)\sum_{k=1}^{n}\xi_k^2, \quad \forall\, x \in \Omega,\ \xi \in \mathbb{R}^n.$$

If $\lambda(x) \geq \lambda_0 > 0$ for all $x \in \Omega$, we say Lu is **uniformly elliptic** in Ω.

So, if Lu is elliptic, then, for each $x \in \Omega$, the matrix $(a^{ij}(x))$ is *positive definite*, with smallest eigenvalue not less than $\lambda(x) > 0$. We use the following notations:

$$L_0u(x) = -\sum_{i,j=1}^{n} a^{ij}(x)D_{ij}u(x) + \sum_{i=1}^{n} b^i(x)D_iu(x),$$

$$\tilde{L}u(x) = -\sum_{i,j=1}^{n} a^{ij}(x)D_{ij}u(x) + \sum_{i=1}^{n} b^i(x)D_iu(x) + c^+(x)u(x),$$

where $c^+(x) = \max\{0, c(x)\}$. We also denote $c^-(x) = \min\{0, c(x)\}$. (**Warning:** I am using a different notation for $c^-(x)$ from the Textbook, where $c^-(x) = -\min\{0, c(x)\}$.)

First we recall a fact in linear algebra.

Lemma 2.26. *If $A = (a^{ij})$ is a positive definite matrix, then there is an invertible matrix $B = (B_{ij})$ such that $A = B^TB$, i.e.,*

$$a^{ij} = \sum_{k=1}^{n} b_{ki}b_{kj} \quad (i, j = 1, 2, \cdots, n).$$

2.7.2. Weak maximum principle and the uniqueness.

Lemma 2.27. *Let Lu be elliptic in Ω and $u \in C^2(\Omega)$ satisfy $Lu < 0$ in Ω. If $c(x) \geq 0$, then u can not attain a nonnegative maximum in Ω. If $c \equiv 0$, then u can not attain a maximum in Ω.*

Proof. We argue by contradiction. Suppose, for some $x_0 \in \Omega$, $u(x_0)$ is a maximum of u in Ω. Then $D_ju(x_0) = 0$, and (by the **second-derivative test**)

$$\frac{d^2u(x_0 + t\xi)}{dt^2}\Big|_{t=0} = \sum_{i,j=1}^{n} D_{ij}u(x_0)\xi_i\xi_j \leq 0,$$

for any $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$. (This is to say the Hessian matrix $(D_{ij}u(x_0)) \leq 0$.) We write

$$a^{ij}(x_0) = \sum_{k=1}^{n} b_{ki}b_{kj} \quad (i, j = 1, 2, \cdots, n),$$

where $B = (b_{ik})$ is a matrix. Hence

$$\sum_{i,j=1}^{n} a^{ij}(x_0)D_{ij}u(x_0) = \sum_{k=1}^{n}\sum_{i,j=1}^{n} D_{ij}u(x_0)b_{ki}b_{kj} \le 0,$$

which implies that $Lu(x_0) \ge c(x_0)u(x_0) \ge 0$ either when $c \ge 0$ and $u(x_0) \ge 0$ or when $c \equiv 0$. This gives a contradiction to $Lu(x_0) < 0$ in both cases. □

Theorem 2.28 (Weak Maximum Principle)**.** *Let Lu be elliptic in Ω and*

$$|b^i(x)|/\lambda(x) \le M \quad (x \in \Omega,\ i = 1, 2, \cdots, n) \tag{2.29}$$

for some constant $M > 0$. If $c(x) = 0$ and $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfies $Lu \le 0$ (in this case we say u is a **sub-solution of** *L) in a bounded domain Ω, then*

$$\max_{\bar{\Omega}} u = \max_{\partial\Omega} u.$$

Proof. 1. Let $v(x) = e^{\alpha x_1}$. Then

$$Lv(x) = (-a^{11}(x)\alpha^2 + b^1(x)\alpha)e^{\alpha x_1} \le \alpha a^{11}(x)\left[-\alpha + \frac{b^1(x)}{a^{11}(x)}\right]e^{\alpha x_1} < 0$$

if $\alpha > M + 1$, as $\frac{|b^1(x)|}{a^{11}(x)} \le \frac{|b^1(x)|}{\lambda(x)} \le M$.

2. Consider $w(x) = u(x) + \varepsilon v(x)$ for any $\varepsilon > 0$. Then $Lw = Lu + \varepsilon Lv < 0$ in Ω. It follows from Lemma 3.6 that for any $x \in \bar{\Omega}$,

$$u(x) + \varepsilon v(x) \le \max_{\partial\Omega}(u + \varepsilon v) \le \max_{\partial\Omega} u + \varepsilon \max_{\partial\Omega} v.$$

Setting $\varepsilon \to 0^+$ proves the theorem. □

Remark 2.11. (a) The weak maximum principle still holds if (a^{ij}) is nonnegative definite, i.e., $\lambda(x) \ge 0$, but satisfies $\frac{|b^k|}{a^{kk}(x)} \le M$ for some k. (Then use $v = e^{\alpha x_k}$.)

(b) If Ω is unbounded, but is bounded in a slab $|x_1| \le N$, then the proof is still valid if the maximum is changed to the supremum.

From the proof of Theorem 3.8 we can easily see the following

Theorem 2.29. *Let Lu be elliptic in Ω and satisfy* (2.29)*. Let $c(x) \ge 0$ and $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$. Then*

$$\max_{\bar{\Omega}} u \le \max_{\partial\Omega} u^+ \quad \text{if } Lu \le 0,$$
$$\max_{\bar{\Omega}} |u| = \max_{\partial\Omega} |u| \quad \text{if } Lu = 0,$$

where $u^+(x) = \max\{0, u(x)\}$.

Proof. 1. Let $Lu \le 0$. Let $\Omega^+ = \{x \in \Omega \mid u(x) > 0\}$. If Ω^+ is empty, the result is trivial. Assume Ω^+ is not empty, then $L_0u(x) = Lu(x) - c(x)u(x) \le 0$ in Ω^+. Note that $\max_{\partial(\Omega^+)} u = \max_{\partial\Omega} u^+$. It follows from Theorem 3.8 that

$$\max_{\bar{\Omega}} u = \max_{\overline{\Omega^+}} u = \max_{\partial(\Omega^+)} u = \max_{\partial\Omega} u^+.$$

2. Assume $Lu(x) = 0$. We apply the inequality in Step 1 to u and $v = -u$, noticing that $v^+ = -u^- = -\min\{0, u\}$, to deduce

$$\max_{\bar{\Omega}} u \le \max_{\partial\Omega} u^+, \quad -\min_{\bar{\Omega}} u = \max_{\bar{\Omega}} v \le \max_{\partial\Omega} v^+ = -\min_{\partial\Omega} u^-. \tag{2.30}$$

This yields $\max_{\bar\Omega}|u| = \max_{\partial\Omega}|u|$. □

Theorem 2.30. *Let Lu be elliptic in Ω and satisfy (2.29). If $c(x)\ge 0$ and Ω is bounded, then solution to the Dirichlet problem*

$$Lu = f \quad \text{in } \Omega, \quad u|_{\partial\Omega} = g$$

is unique in $C^2(\Omega)\cap C(\bar\Omega)$.

Remark 2.12. (a) The maximum principle and the uniqueness fail if $c(x)<0$. For example, if $n=1$, the function $u(x)=\sin x$ satisfies

$$-u'' - u = 0 \quad \text{in } \Omega=(0,\pi), \quad u(0)=u(\pi)=0.$$

But $u\not\equiv 0$.

(b) Nontrivial solutions can be constructed for equation $-\Delta u - cu = 0$ with zero Dirichlet boundary data in the cube $\Omega=(0,\pi)^n$ in dimensions $n\ge 2$, where $c>0$ is a constant.

2.7.3. Strong maximum principle.

The following lemma is needed to prove a version of the strong maximum principle for elliptic equations.

Lemma 2.31 (Hopf's Lemma)**.** *Let Lu be uniformly elliptic with bounded coefficients in a ball B and $u\in C^2(\bar B)$ satisfy $Lu\le 0$ in B. Assume that $x^0\in\partial B$ such that $u(x)<u(x^0)$ for every $x\in B$.*

(a) If $c(x)=0$ in $\bar B$, then $\frac{\partial u}{\partial\nu}(x^0)>0$, where ν is outer unit normal to B at x^0.

(b) If $c(x)\ge 0$, then the same conclusion holds provided $u(x^0)\ge 0$.

(c) If $u(x^0)=0$, the same conclusion holds no matter what sign of $c(x)$ is.

Proof. 1. Without the loss of generality, we assume that $B=B(0,R)$. Consider

$$v(x) = e^{-\alpha|x|^2} - e^{-\alpha R^2}.$$

Recall $\tilde L u = Lu - c(x)u + c^+(x)u = Lu - c^-(x)u$ is an elliptic operator with nonnegative coefficient for the zero-th order term; so, the weak maximum principle applies to $\tilde L$. (**Warning:** Again, $c^- = \min\{0,c\}$ here, while the book uses $c^- = -\min\{0,c\}$.) It is easy to check that

$$\begin{aligned}
\tilde L v(x) &= \Big[-4\sum_{i,j} a^{ij}(x)\alpha^2 x_i x_j + 2\alpha\sum_i (a^{ii}(x) - b^i(x)x_i)\Big]e^{-\alpha|x|^2} + c^+(x)v(x)\\
&\le \Big[-4\lambda_0\alpha^2|x|^2 + 2\alpha\,\mathrm{tr}(A(x)) + 2\alpha|b(x)||x| + c^+(x)\Big]e^{-\alpha|x|^2} \qquad (2.31)\\
&< 0 \quad (\text{here } \; A(x)=(a^{ij}(x)),\; b(x)=(b^1(x),\cdots,b^n(x)))
\end{aligned}$$

on $\frac{R}{2}\le|x|\le R$, provided $\alpha>0$ is fixed and sufficiently large.

2. For any $\varepsilon>0$, consider $w_\varepsilon(x)=u(x)-u(x^0)+\varepsilon v(x)$. Then

$$\begin{aligned}
\tilde L w_\varepsilon(x) &= \varepsilon\tilde L v(x) + Lu(x) - c^-(x)u(x) - c^+(x)u(x^0)\\
&< -c^-(x)u(x) - c^+(x)u(x^0) \le 0
\end{aligned}$$

on $\frac{R}{2}\le|x|\le R$ in all cases of (a), (b) and (c).

3. By assumption, $u(x)<u(x^0)$ for all $|x|=\frac{R}{2}$; hence, there is a $\varepsilon>0$ such that $w_\varepsilon(x)<0$ for all $|x|=\frac{R}{2}$. In addition, since $v|_{\partial B}=0$, we have $w_\varepsilon(x)=u(x)-u(x^0)\le 0$ on

$|x| = R$. The weak maximum principle implies that $w_\varepsilon(x) \le 0$ for all $\frac{R}{2} \le |x| \le R$. Hence x^0 is a maximum point of w_ε on $\frac{R}{2} \le |x| \le R$ and thus

$$0 \le \frac{\partial w_\varepsilon}{\partial \nu}(x^0) = \frac{\partial u}{\partial \nu}(x^0) + \varepsilon \frac{\partial v}{\partial \nu}(x^0) = \frac{\partial u}{\partial \nu}(x^0) - 2\varepsilon R\alpha e^{-\alpha R^2}.$$

That is

$$\frac{\partial u}{\partial \nu}(x^0) \ge 2\varepsilon R\alpha e^{-\alpha R^2} > 0, \quad \text{as required.}$$

□

Theorem 2.32 (Strong Maximum Principle). *Let L be uniformly elliptic in a bounded domain Ω with bounded coefficients and $u \in C^2(\Omega)$ such that $Lu \le 0$ in Ω.*

(a) If $c(x) \ge 0$, then u can not attain a nonnegative maximum in Ω unless u is a constant.

(b) If $c = 0$, then u can not attain a maximum in Ω unless u is a constant.

Proof. We proceed by contradiction. Suppose that u is not a constant but there exists a $\bar{x} \in \Omega$ such that $u(\bar{x}) \ge u(x)$ for all $x \in \Omega$. Let $M = u(\bar{x})$. We assume $M \ge 0$ if $c(x) \ge 0$. Then the sets

$$\Omega^- = \{x \in \Omega \mid u(x) < M\}, \quad \Omega_0 = \{x \in \Omega \mid u(x) = M\}$$

are both nonempty (since $u \not\equiv M$), and thus $\emptyset \ne \partial\Omega_0 \cap \Omega \subset C = \partial\Omega^- \cap \Omega$. Let $y \in \Omega^-$ be such that $\operatorname{dist}(y, C) < \operatorname{dist}(y, \partial\Omega)$. Let $B = B(y, R)$ be the largest ball contained in Ω^- and centered at y. Then there exists a point $x^0 \in C$, with $x^0 \in \partial B$. Hence $u(x) < u(x^0)$ for all $x \in B$. Then Hopf's Lemma would imply that $\frac{\partial u}{\partial \nu}(x^0) > 0$, where ν is the outnormal of B at x^0; this contradicts to the fact that $D_j u(x^0) = 0$ since u has maximum at $x^0 \in \Omega$. □

Theorem 2.33. *Let $u \in C^2(\Omega) \cap C(\bar{\Omega})$. If $Lu = 0$, $c(x) \ge 0$ and u is not a constant, then*

$$|u(x)| < \max_{y \in \partial\Omega} |u(y)| \quad \forall\, x \in \Omega.$$

If $Lu = 0$, $c = 0$ and u is not a constant, then

$$\min_{y \in \partial\Omega} u(y) < u(x) < \max_{y \in \partial\Omega} u(y) \quad \forall\, x \in \Omega.$$

Finally, we state without proof the following **Harnack's inequality** for nonnegative solutions of elliptic equations. (The proof for some special cases can be found in Textbook, which actually follows from the proof of the similar result for parabolic equations later.)

Theorem 2.34 (Harnack's inequality). *Let $V \subset\subset \Omega$ be connected and Lu be uniformly elliptic in Ω with bounded coeffients. Then, there exists $C = C(V, \Omega, L) > 0$ such that*

$$\sup_V u \le C \inf_V u$$

for each nonnegative C^2 function u of $Lu = 0$ in Ω.

2.8. Weak solutions and existence in Sobolev spaces

(*This material is from parts of Chapter 5 and Sections 6.1 and 6.2 of the Texttbook.*)

We study the second-order linear elliptic operator in **divergence form**:

$$Lu(x) = -\sum_{i,j=1}^{n} D_j(a^{ij}(x) D_i u(x)) + \sum_{i=1}^{n} b^i(x) D_i u(x) + c(x)u(x).$$

Here we assume that $a^{ij}(x) = a^{ji}(x)$ and a^{ij}, b^i and c are all bounded functions in Ω.

Definition 2.13. The operator Lu is called **elliptic** if there exists $\lambda(x) > 0$ such that

$$\sum_{i,j=1}^{n} a^{ij}(x)\xi_i\xi_j \geq \lambda(x)\sum_{k=1}^{n}\xi_k^2, \quad \forall\, x \in \Omega,\ \xi \in \mathbb{R}^n.$$

If $\lambda(x) \geq \lambda_0 > 0$ for all $x \in \Omega$, we say Lu is **uniformly elliptic** in Ω.

Let Ω be a bounded domain in $\mathbb{R}^n$. We study the following Dirichlet problem:

$$\begin{cases} Lu = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.32}$$

If a^{ij} are smooth and u is a C^2 solution to the Dirichlet problem, then for each function $v \in C_c^\infty(\Omega)$, $vLu = 0$ on Ω. So $\int_\Omega vLu\,dx = 0$. Using integration by parts, we have

$$\int_\Omega \Big(\sum_{i,j=1}^{n} a^{ij}(x)u_{x_i}(x)v_{x_j}(x) + \sum_{i=1}^{n} b^i(x)v(x)u_{x_i}(x) + c(x)u(x)v(x)\Big)dx = \int_\Omega fv\,dx.$$

Define

$$B[u,v] := \int_\Omega \Big(\sum_{i,j=1}^{n} a^{ij}(x)u_{x_i}(x)v_{x_j}(x) + \sum_{i=1}^{n} b^i(x)v(x)u_{x_i}(x) + c(x)u(x)v(x)\Big)dx. \tag{2.33}$$

In $B[u,v]$ we do not need u to be C^2 but we only need Du exists and is in $L^2(\Omega)$ in order to define $B[u,v]$. In fact, the definition of Du can be even extended as long as it is in $L^2(\Omega)$.

2.8.1. Sobolev spaces $H^1(\Omega)$ and $H_0^1(\Omega)$. A function $u \in L^2(\Omega)$ is called **weakly differentiable** in Ω if there exist functions w_i $(i = 1, 2, \cdots, n)$ in $L^1_{loc}(\Omega)$ such that

$$\int_\Omega u\phi_{x_i}\,dx = -\int_\Omega \phi w_i\,dx \quad (i = 1, 2, \cdots, n)$$

for all test functions $\phi \in C_c^\infty(\Omega)$. In this case we call w_i the **weak derivative** of u in Ω with respect to x_i and denote $w_i = u_{x_i} = D_i u$. Moreover, denote $Du = \nabla u = (w_1, w_2, \cdots, w_n)$ to be the **weak gradient vector** of u in Ω.

Define

$$H^1(\Omega) = \{u \in L^2(\Omega) \mid u \text{ is weakly differentiable with } D_i u \in L^2(\Omega) \text{ for } i = 1, \cdots, n\}.$$

Then $H^1(\Omega)$ is a linear subspace of $L^2(\Omega)$. In $H^1(\Omega)$ we define

$$(u,v) = \int_\Omega (uv + Du \cdot Dv)\,dx, \quad \|u\| = (u,u)^{1/2} \quad (u,\, v \in H^1(\Omega)).$$

Theorem 2.35. *(u,v) defines an inner product on $H^1(\Omega)$ that makes $H^1(\Omega)$ a Hilbert space.*

We define $H_0^1(\Omega)$ to be the closure of $C_c^\infty(\Omega)$ in the Hilbert space $H^1(\Omega)$.

EXAMPLE 2.36. Let $\Omega = B(0,1)$ be the unit ball in $\mathbb{R}^n$.

(a) Let $\alpha > 0$. Show that $u = |x|^{-\alpha} \in H^1(\Omega)$ if and only if $\alpha < \frac{n-2}{2}$.

(b) Show that $u = 1 - |x| \in H_0^1(\Omega)$.

Theorem 2.37. *Let Ω be bounded. Then $H_0^1(\Omega)$ is a Hilbert space under the inner product*

$$(u,v)_{H_0^1} = \int_\Omega Du\cdot Dv\,dx \quad (u,\, v\in H_0^1(\Omega)).$$

Proof. 1. We first prove the **Poincaré's inequality**

$$\|u\|_{L^2(\Omega)} \le C\|Du\|_{L^2(\Omega)} \quad (u\in H_0^1(\Omega)). \tag{2.34}$$

To prove this, assume $\Omega \subset\subset Q \equiv (-a,a)^n$. Fix any $u\in C_c^\infty(\Omega)$. Extend u to $\bar{Q}$ by zero outside Ω. For each $y\in Q$, write $y=(y_1,y')$ with $y_1\in(-a,a)$ and $y'\in Q'\equiv(-a,a)^{n-1}$. Then

$$u(y_1,y') = \int_{-a}^{y_1} u_{x_1}(x_1,y')\,dx_1 \quad (y_1\in(-a,a),\, y'\in Q').$$

By Hölder's inequality,

$$|u(y_1,y')|^2 \le (y_1+a)\int_{-a}^{y_1}|u_{x_1}(x_1,y')|^2\,dx_1 \le 2a\int_{-a}^{a}|u_{x_1}(x_1,y')|^2\,dx_1,$$

and consequently,

$$\int_{-a}^{a}|u(y_1,y')|^2\,dy_1 \le 4a^2\int_{-a}^{a}|u_{x_1}(x_1,y')|^2\,dx_1 \quad (y'\in Q').$$

Integrating over $y'\in Q'=(-a,a)^{n-1}$, we deduce

$$\int_Q |u(y)|^2\,dy \le 4a^2\int_Q |u_{x_1}(y)|^2\,dy.$$

Therefore,

$$\|u\|_{L^2(\Omega)} \le 2a\|Du\|_{L^2(\Omega)} \quad (u\in C_c^\infty(\Omega)).$$

Since $H_0^1(\Omega)$ is the closure of $C_c^\infty(\Omega)$ in $H^1(\Omega)$, we have deduced (2.34) from the above inequality.

2. Suppose $\{u_j\}$ is a Cauchy sequence in $H_0^1(\Omega)$ under the H_0^1-norm: $\|u\|_{H_0^1} = (u,u)_{H_0^1}^{1/2}$. Then, by (2.34), $\{u_j\}$ is also a Cauchy sequence in $H^1(\Omega)$. Hence there exists a subsequence $\{u_{j_k}\}$ converging to some $\bar{u}\in H^1(\Omega)$ under the H^1-norm. Certainly $\bar{u}\in H_0^1(\Omega)$ and $\{u_{j_k}\}$ also converges to $\bar{u}$ in the H_0^1-norm. This proves that $H_0^1(\Omega)$ is complete under the H_0^1-norm, and hence $H_0^1(\Omega)$ is a Hilbert space. □

Remark 2.14. (a) For a bounded domain Ω with Lipschitz boundary, the **Sobolev-Rellich-Kondrachov compactness theorem** asserts that $H^1(\Omega)$ is **compactly embedded** in $L^2(\Omega)$; that is, every bounded sequence in $H^1(\Omega)$ has a subsequence convergent strongly in $L^2(\Omega)$.

(b) By this compactness theorem, we have another version of the **Poincaré's inequality**: For bounded domains Ω with Lipschitz boundary,

$$\|u - ⨍_\Omega u\|_{L^2(\Omega)} \le C\|Du\|_{L^2(\Omega)} \quad (u\in H^1(\Omega)). \tag{2.35}$$

2.8.2. Weak solution to Dirichlet problem.

Definition 2.15. A **weak solution to Dirichlet problem** (2.32) is a function $u \in H_0^1(\Omega)$ such that

$$B[u,v] = \int_\Omega f v \, dx \quad \forall\, v \in H_0^1(\Omega),$$

where $B[u,v]$ is defined by (2.33) with u_{x_i}, v_{x_i} taken as weak derivatives.

The existence of weak solution falls into the general framework of **Lax-Milgram Theorem** in functional analysis.

Let H denote a real Hilbert space with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$. A function $B[\cdot,\cdot] : H \times H \to \mathbb{R}$ is called a **bilinear form** if

$$B[au+bv, w] = aB[u,w] + bB[v,w]$$

$$B[w, au+bv] = aB[w,u] + bB[w,v]$$

for all $u, v, w \in H$ and $a, b \in \mathbb{R}$.

Theorem 2.38. (Lax-Milgram Theorem) *Let H be a Hilbert space and $B\colon H \times H \to \mathbb{R}$ be a bilinear form. Assume*

(i) *B is* **bounded***; i.e., $|B[u,v]| \le \alpha \|u\|\|v\| \ \ \forall\, u, v \in H$, for some $\alpha > 0$; and*

(ii) *B is* **strongly positive** *(also called* **coercive***); i.e., $B[u,u] \ge \beta \|u\|^2 \ \ \forall\, u \in H$, for some $\beta > 0$.*

Then, for each $l \in H^*$ (that is, $l\colon H \to \mathbb{R}$ is a **bounded linear functional** *on H), there exists a unique $u \in H$ such that*

$$B[u,v] = l(v) \quad (v \in H). \tag{2.36}$$

Moreover, the solution u satisfies $\|u\| \le \frac{1}{\beta}\, \|l\|_{H^*}$.

Proof. For each fixed $u \in H$, the functional $v \mapsto B[u,v]$ is in H^*, and hence by the **Riesz representation theorem**, there exists a unique element $w = Au \in H$ such that

$$B[u,v] = (w,v) = (Au, v) \quad \forall\, v \in H.$$

This defines a map $A : H \to H$ and it is easy to see that A is linear. From (i), $\|Au\|^2 = B[u, Au] \le \alpha\|u\|\|Au\|$, and hence $\|Au\| \le \alpha\|u\|$ for all $u \in H$; that is, A is bounded. Furthermore, by (ii), $\beta\|u\|^2 \le B[u,u] = (Au,u) \le \|Au\|\|u\|$ and hence $\|Au\| \ge \beta\|u\|$ for all $u \in H$. By the Riesz representation theorem again, we have a unique $w_0 \in H$ such that $l(v) = (w_0, v)$ for all $v \in H$ and $\|l\|_{H^*} = \|w_0\|$. We will show that the equation $Au = w_0$ has a unique solution $u \in H$. The uniqueness of u follows easily from the property $\|Au - Av\| \ge \beta\|u - v\|$ for all $u, v \in H$; the existence of u will be proved by using the **contraction mapping theorem**. (See the textbook for a different proof.) Note that solution u to $Au = w_0$ is equivalent to a fixed-point of map $T\colon H \to H$ defined by $T(v) = v - tAv + tw_0$ $(v \in H)$ for any fixed $t > 0$. We show that, for $t > 0$ small enough, map T is a strict contraction. Note that for all $v, w \in H$ we have $\|T(v) - T(w)\| = \|(I - tA)(v - w)\|$. We compute that for all $u \in H$

$$\begin{aligned} \|(I - tA)u\|^2 &= \|u\|^2 + t^2\|Au\|^2 - 2t(Au, u) \\ &\le \|u\|^2(1 + t^2\alpha^2 - 2\beta t) \\ &\le \gamma\|u\|^2, \end{aligned}$$

for some $0 < \gamma < 1$ if we choose t such that $0 < t < \frac{2\beta}{\alpha^2}$. Therefore, map $T\colon H \to H$ is a contraction (with constant $\sqrt{\gamma} < 1$) on H and thus has a fixed point u by contraction mapping theorem. This u solves $Au = w_0$. Moreover, from $\|f\|_{H^*} = \|w_0\| = \|Au\| \ge \beta\|u\|$, we have $\|u\| \le \frac{1}{\beta}\|l\|_{H^*}$. The proof is complete. □

2.8.3. Existence of weak solutions. We state the following special existence theorem.

Theorem 2.39 (Existence of weak solutions)**.** *Let Lu be uniformly elliptic in Ω with bounded coefficients. Then there exists a constant M depending on the ellipticity constant and the L^∞-bound of coefficients b^i such that, if $c(x) \ge M$, then, for each $f \in L^2(\Omega)$, the Dirichlet problem* (2.32) *has a unique weak solution $u \in H_0^1(\Omega)$.*

Proof. 1. Let

$$l(v) = \int_\Omega f(x)v(x)\,dx \quad (v \in H_0^1(\Omega)).$$

Then l is a bounded linear functional on $H_0^1(\Omega)$. (This uses Poincaré's inequality (2.34).)

2. Using the boundedness of the coefficients, by Cauchy's inequality and Poincaré's inequality (2.34), we deduce that

$$|B[u,v]| \le \alpha\|Du\|_{L^2(\Omega)}\|Dv\|_{L^2(\Omega)} = \alpha\|u\|_{H_0^1}\|v\|_{H_0^1} \quad (u,\, v \in H_0^1(\Omega)).$$

(This also uses Poincaré's inequality (2.34).)

3. By the uniform ellipticity with constant $\lambda_0 > 0$,

$$\int_\Omega \sum_{i,j=1}^n a^{ij}(x)u_{x_i}u_{x_j}\,dx \ge \lambda_0\|Du\|^2_{L^2(\Omega)} = \lambda_0\|u\|^2_{H_0^1}.$$

By the boundedness of b^i and Cauchy's inequality with ε, we have

$$\left|\int_\Omega \sum_{i=1}^n b^i(x)u_{x_i}u\,dx\right| \le \frac{\lambda_0}{2}\|Du\|^2_{L^2(\Omega)} + M\|u\|^2_{L^2(\Omega)} = \frac{\lambda_0}{2}\|u\|^2_{H_0^1} + M\|u\|^2_{L^2(\Omega)},$$

where M depends only on λ_0 and the L^∞-bound of the coefficients b^i $(i = 1, 2, \cdots, n)$. Note that $M = 0$ if all b^i's are zero (i.e., Lu has no first-order terms). Therefore,

$$B[u,u] \ge \frac{\lambda_0}{2}\|u\|^2_{H_0^1} + \int_\Omega (c(x) - M)u^2\,dx \quad (u \in H_0^1(\Omega)).$$

This is called the **energy estimate**.

4. By the energy estimate above, we deduce that if $c(x) \ge M$ then

$$B[u,u] \ge \beta\|u\|^2_{H_0^1(\Omega)},\ \beta = \frac{\lambda_0}{2} > 0 \quad (u \in H_0^1(\Omega)).$$

Consequently, the bilinear form $B[u,v]$ on Hilbert space $H = H_0^1(\Omega)$ satisfies all conditions of the Lax-Milgram Theorem; therefore, we have the unique existence of weak solution. □

Remark 2.16. Can we show the weak solution is smooth enough to be a classical solution? This is the **regularity** problem, which will not be studied in this course.

Also, what about if $c(x)$ is not sufficiently large? This is related to the **eigenvalue problem** of Lu and will not be studied in this course, either.

Suggested exercises

Materials covered are from Chapters 2 and 6 of the textbook with some new materials added. So complete the arguments that are left in lectures. Also try working on the following problems related to the covered materials.

Chapter 2: Problems 2–11.

Chapter 6: Problems 1, 4, 5, 6, 8, 9, 10, 12.

Homework # 3.

(1) (15 points) Let Ω be a bounded domain in $\mathbf{R}^n$ with smooth boundary $\partial\Omega$. Let $G(x,y)$ be Green's function for Ω and $K(x,y) = -\frac{\partial G}{\partial \nu_y}(x,y)$ $(x \in \Omega,\ y \in \partial\Omega)$ be Poisson's kernel for Ω. Then, for all $u \in C^2(\bar{\Omega})$, we have

$$u(x) = \int_{\partial\Omega} K(x,y)u(y)dS_y - \int_{\Omega} G(x,y)\Delta u(y)dy \quad (x \in \Omega).$$

Prove the following statements:

$$G(x,y) > 0\ (x,y \in \Omega,\ x \neq y); \quad K(x,y) \geq 0\ (x \in \Omega,\ y \in \partial\Omega); \quad \int_{\partial\Omega} K(x,y)dS_y = 1\ (x \in \Omega).$$

(2) (15 points) Let Ω be the ellipsoid $\frac{x_1^2}{2} + \frac{x_2^2}{3} + \frac{x_3^2}{4} < 1$ in $\mathbf{R}^3$. Use (without proof) the fact that Ω has Green's function $G(x,y)$ to prove that for all $u \in C^2(\bar{\Omega})$

$$\max_{\bar{\Omega}} |u| \leq \max_{\partial\Omega} |u| + \frac{6}{13} \max_{\bar{\Omega}} |\Delta u|.$$

(3) (15 points) Let $n \geq 3$ and $\Omega \subset \mathbf{R}^n$ be a bounded domain and $0 \in \Omega$. Suppose that u is continuous on $\bar{\Omega}\backslash\{0\}$, harmonic in $\Omega\backslash\{0\}$, $u = 0$ on $\partial\Omega$, and $\lim_{x\to 0} |x|^{n-2} u(x) = 0$. Prove that $u(x) \equiv 0$ on $\Omega \setminus \{0\}$.

(4) (15 points) Let u be the solution of

$$\begin{cases} \Delta u = 0 & \text{in } \mathbf{R}^n_+ \\ u = g & \text{on } \partial\mathbf{R}^n_+ \end{cases}$$

given by Poisson's formula for the half-space, where g is continuous, bounded on $\partial\mathbf{R}^n_+$, and $g(x) = |x|$ for $x \in \partial\mathbf{R}^n_+, |x| \leq 1$. Show that Du is *unbounded* near $x = 0$.

(5) (20 points) Recall that $u\colon \Omega \to \mathbf{R}$ is *subharmonic in* Ω if $u \in C(\Omega)$ and if for every $\xi \in \Omega$ the inequality

$$u(\xi) \leq \fint_{\partial B(\xi,\rho)} u(x)dS := M_u(\xi,\rho)$$

holds for all *sufficiently small* $\rho > 0$. We denote by $\sigma(\Omega)$ the set of all subharmonic functions in Ω.

(a) For $u \in C(\bar{\Omega}) \cap \sigma(\Omega)$, show that $\max_{\bar{\Omega}} = \max_{\partial\Omega} u$.

(b) If $u_1, u_2, \cdots, u_k \in \sigma(\Omega)$, show that $u = \max\{u_1, \cdots, u_k\} \in \sigma(\Omega)$.

(c) Let $u \in C^2(\Omega)$. Show that $u \in \sigma(\Omega)$ if and only if $-\Delta u \leq 0$ in Ω.

Homework # 4.

(1) (10 points) Let $Lu = -\sum_{i,j=1}^{n} a^{ij}(x)u_{x_ix_j} + \sum_{i=1}^{n} b^i(x)u_{x_i}$ be a uniformly elliptic operator in a bounded domain Ω. If σ is a C^1 function on $\mathbf{R}$ with $\sigma'(s) \geq 0$ for all $s \in \mathbf{R}$, show that, given any f and g, the Dirichlet problem

$$\begin{cases} Lu + \sigma(u) = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

can have *at most one* solution $u \in C^2(\Omega) \cap C(\bar{\Omega})$.

(2) (10 points) Let $a^{ij} \in C^1(\bar{\Omega})$ and $Lu = -\sum_{i,j=1}^{n} a^{ij}(x)u_{x_ix_j}$ be a uniformly elliptic operator in a bounded domain Ω. Let $u \in C^3(\bar{\Omega})$ be a solution of $Lu = 0$ in Ω. Set $v = |Du|^2 + \lambda u^2$. Show that $Lv \leq 0$ in Ω if λ is large enough. Deduce

$$\|Du\|_{L^\infty(\Omega)} \leq C\left(\|Du\|_{L^\infty(\partial\Omega)} + \|u\|_{L^\infty(\partial\Omega)}\right)$$

for some constant C.

(3) (10 points) Given $f \in L^2(\Omega)$. Formulate and prove the existence of **weak solutions** to the *Neumann boundary problem* in $H^1(\Omega)$:

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

(What is the necessary condition of f for existence of smooth solutions u? Show the condition is *necessary and sufficient* for the existence of weak solution.)

Heat Equation

The **heat equation**, also known as the *diffusion equation*, describes in typical physical applications the evolution in time of the density u of some quality such as heat, chemical concentration, etc. Let V be any smooth subdomain, in which there is no source or sink, the rate of change of the total quantity within V equals the negative of the net flux $\mathbf{F}$ through ∂V:

$$\frac{d}{dt}\int_V u dx = -\int_{\partial V} \mathbf{F} \cdot \nu dS.$$

The divergence theorem tells us

$$\frac{d}{dt}\int_V u dx = -\int_V \operatorname{div} \mathbf{F} dx.$$

Since V is arbitrary, we should have

$$u_t = -\operatorname{div} \mathbf{F}.$$

For many applications $\mathbf{F}$ is proportional to the gradient of u, but points in opposite direction (flux is from regions of higher concentration to lower concentration):

$$\mathbf{F} = -a\nabla u \quad (a > 0).$$

Therefore we obtain the equation

$$u_t = a \operatorname{div} \nabla u = a\Delta u,$$

which is called the heat equation when $a = 1$.

If there is a source in Ω, we obtain the following nonhomogeneous heat equation

$$u_t - \Delta u = f(x,t) \quad x \in \Omega,\ t \in (0,\infty).$$

3.1. Fundamental solution of heat equation

As in Laplace's equation case, we would like to find some special solutions to the heat equation. The textbook gives one way to find such a specific solution, and the problem in the book gives another way. Here we discuss yet another way of finding a special solution to the heat equation.

3.1.1. Fundamental solution and the heat kernel. We start with the following observations:

(1) If $u_j(x,t)$ are solution to the *one-dimensional heat equation* $u_t = u_{xx}$ for $x \in \mathbb{R}$ and $t > 0$, $j = 1, \ldots, n$, then

$$u(x_1, \ldots, x_n, t) = u_1(x_1, t) \cdots u_n(x_n, t)$$

is a solution to the heat equation $u_t = \Delta u$ for $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$ and $t > 0$. This simple fact is left as an exercise.

(2) If $u(x,t)$ is a solution to the one-dimensional heat equation $u_t = u_{xx}$, then so is $w(\lambda)u(\lambda x, \lambda^2 t)$ for any real λ. Especially, there should be a solution of the form $u(x,t) = w(t)v(\frac{x^2}{t})$. A direct computation yields

$$u_t(x,t) = w'(t)v(\tfrac{x^2}{t}) - w(t)v'(\tfrac{x^2}{t})\tfrac{x^2}{t^2},$$
$$u_x(x,t) = w(t)v'(\tfrac{x^2}{t})\tfrac{2x}{t},$$
$$u_{xx}(x,t) = w(t)v''(\tfrac{x^2}{t})\tfrac{4x^2}{t^2} + w(t)v'(\tfrac{x^2}{t})\tfrac{2}{t}.$$

For $u_t = u_{xx}$, we need

$$w'(t)v - \frac{w}{t}[v''\frac{4x^2}{t} + 2v' + v'\frac{x^2}{t}] = 0.$$

Separation of variables yields that

$$\frac{w'(t)t}{w(t)} = \frac{4sv''(s) + 2v'(s) + sv'(s)}{v(s)}$$

with $s = \frac{x^2}{t}$. Therefore, both sides of this equality must be constant, say, λ. So

$$s(4v'' + v') + \frac{1}{2}(4v' - 2\lambda v) = 0;$$

this equation is satisfied if we choose $\lambda = -1/2$ and $4v' + v = 0$ and hence $v(s) = e^{-\frac{s}{4}}$. In this case,

$$\frac{w'(t)t}{w(t)} = -\frac{1}{2},$$

from which we have $w(t) = t^{-\frac{1}{2}}$. Therefore

$$u = u_1(x,t) = \frac{1}{\sqrt{t}}\, e^{-\frac{x^2}{4t}}$$

is a solution of $u_t = u_{xx}$. By observation (1) above, function

$$u(x) = u_1(x_1,t)u_1(x_2,t) \cdots u_1(x_n,t) = \frac{1}{t^{n/2}}\, e^{-\frac{|x|^2}{4t}} \quad (x \in \mathbb{R}^n,\ t > 0)$$

is a solution of the heat equation $u_t = \Delta u$ for $t > 0$ and $x \in \mathbb{R}^n$.

Definition 3.1. The function

$$\Phi(x,t) = \begin{cases} \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}} & t > 0, \\ 0 & t < 0 \end{cases}$$

is called the **fundamental solution of the heat equation** $u_t = \Delta u$.

The constant $\frac{1}{(4\pi)^{n/2}}$ in the fundamental solution $\Phi(x,t)$ is to make valid the following

Lemma 3.1. *For each $t > 0$,*

$$\int_{\mathbb{R}^n} \Phi(x,t)dx = 1.$$

Proof. This is a straight forward computation based on the fact (**verify!**)

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}.$$

□

Note that, unlike the fundamental solution of Laplace's equation, the fundamental solution $\Phi(x,t)$ of the heat equation is C^∞ in $x \in \mathbb{R}^n$ for each $t > 0$. Furthermore, all *space-time derivatives* $D^\alpha \Phi$ are integrable on $\mathbb{R}^n$ for each $t > 0$. Also notice that as $t \to 0^+$ it follows that $\Phi(x,t) \to 0$ $(x \neq 0)$ and $\Phi(0,t) \to \infty$. In the following, we will see that $\Phi(\cdot, t) \to \delta_0$ in distribution as $t \to 0^+$.

We call the function

$$K(x,y,t) = \Phi(x-y,t) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}} \quad (x,\ y \in \mathbb{R}^n,\ t > 0)$$

the **heat kernel** in $\mathbb{R}^n$.

3.1.2. Initial-value problem. We now study the **initial-value problem** or **Cauchy problem** of the heat equation

$$\begin{cases} u_t = \Delta u, & x \in \mathbb{R}^n,\ t \in (0,\infty), \\ u(x,0) = g(x), & x \in \mathbb{R}^n. \end{cases} \tag{3.1}$$

Define

$$\begin{aligned} u(x,t) &= \int_{\mathbb{R}^n} \Phi(x-y,t) g(y) dy = \int_{\mathbb{R}^n} K(x,y,t) g(y) dy \\ &= \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} g(y) dy \quad (x \in \mathbb{R}^n,\ t > 0). \end{aligned} \tag{3.2}$$

Theorem 3.2. *Assume $g \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, and define u by (3.2). Then*

(i) $u \in C^\infty(\mathbb{R}^n \times (0,\infty)) \cap L^\infty(\mathbb{R}^n \times (0,\infty))$,

(ii) $u_t = \Delta u \quad (x \in \mathbb{R}^n,\ t > 0)$,

(iii) for each $x^0 \in \mathbb{R}^n$,

$$\lim_{x \to x^0,\, t \to 0^+} u(x,t) = g(x^0).$$

Proof. 1. Clearly, $K(x,y,t) > 0$ and $\int_{\mathbb{R}^n} K(x,y,t)\, dy = 1$ for all $x \in \mathbb{R}^n$ and $t > 0$. Hence

$$|u(x,t)| \le \|g\|_{L^\infty} \int_{\mathbb{R}^n} K(x,y,t)\, dy = \|g\|_{L^\infty} \quad (x \in \mathbb{R}^n,\ t > 0).$$

The fact $u \in C^\infty(\mathbb{R}^n \times (0,\infty))$ and solves the heat equation follows from the integrability of $D^\alpha \Phi$ in $\mathbb{R}^n$, differentiation under the integral, and the fact $K_t = \Delta_x K$ in $\mathbb{R}^n \times \mathbb{R}^n \times (0,\infty)$.

2. For any fixed $x^0 \in \mathbb{R}^n$ and $\epsilon > 0$, choose a $\delta > 0$ such that

$$|g(x^0) - g(y)| < \frac{\epsilon}{2} \quad \text{if} \quad |x^0 - y| < \delta.$$

Assume $|x - x^0| < \delta/2$. Then,

$$|u(x,t) - g(x^0)| = \left| \int_{\mathbb{R}^n} \Phi(x-y,t)(g(y) - g(x^0)\, dy \right|$$

$$\le \int_{B(x^0,\delta)} \Phi(x-y,t)|g(y)-g(x^0)|\,dy + \int_{\mathbb{R}^n\setminus B(x^0,\delta)} \Phi(x-y,t)|g(y)-g(x^0)|\,dy$$
$$:= I + J.$$

Now $I < \epsilon/2$. To estimate J, note that if $|y-x^0| \ge \delta$, since $|x-x^0| < \delta/2 \le |y-x^0|/2$, then

$$|y-x^0| \le |y-x| + |x-x^0| \le |y-x| + |y-x^0|/2.$$

Thus $|y-x| \ge \frac{1}{2}|y-x^0|$. Consequently

$$J \le 2\|g\|_{L^\infty} \int_{\mathbb{R}^n\setminus B(x^0,\delta)} \Phi(x-y,t)\,dy$$
$$\le \frac{C}{t^{n/2}} \int_{\mathbb{R}^n\setminus B(x^0,\delta)} e^{-\frac{|y-x|^2}{4t}}\,dy$$
$$\le \frac{C}{t^{n/2}} \int_{\mathbb{R}^n\setminus B(x^0,\delta)} e^{-\frac{|y-x^0|^2}{16t}}\,dy = C\int_{\mathbb{R}^n\setminus B(0,\delta/\sqrt{t})} e^{\frac{-|z|^2}{16}}\,dz,$$

by the change of variable $z = \frac{y-x^0}{\sqrt{t}}$. We select a $0 < t_0 < \delta/2$ such that

$$C\int_{\mathbb{R}^n\setminus B(0,\delta/\sqrt{t_0})} e^{\frac{-|z|^2}{16}}\,dz < \frac{\epsilon}{2}.$$

Then for all $|x-x^0| < t_0,\ 0 < t < t_0$, we have

$$|u(x,t) - g(x^0)| \le I + J < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

□

Remark 3.2. (a) If g is bounded, continuous, $g \ge 0$ and $\not\equiv 0$, then function $u(x,t)$ defined by (3.2) is in fact *positive* for all $x \in \mathbb{R}^n$ and $t > 0$. Moreover, $u(x,t)$ depends on the values of $g(y)$ at all points $y \in \mathbb{R}^n$ no matter how far y and x are away. Even the initial g is compactly supported, its *domain of influence* is still all of $x \in \mathbb{R}^n$. We interpret these observations by saying the heat equation has *infinite propagation speed* for disturbances of initial data.

(b) Even when g is not continuous but only bounded, the function u defined through the heat kernel K above is C^∞ in $\mathbb{R}^n \times (0,\infty)$ (in fact, analytic). Therefore, the heat kernel $K(x,y,t)$ has the *smoothing effect* in the sense that function u becomes infinitely smooth in whole space-time as soon as time $t > 0$ no matter how rough the initial data g is at $t = 0$.

3.1.3. Duhamel's principle and the nonhomogeneous problem. Now we solve the initial-value problem of *nonhomogeneous heat equation*:

$$\begin{cases} u_t - \Delta u = f(x,t) & (x \in \mathbb{R}^n,\ t > 0) \\ u(x,0) = 0 & (x \in \mathbb{R}^n). \end{cases}$$

There is a general method of solving nonhomogeneous problem using the solutions of homogeneous problem with variable initial data; such a method is usually known as the **Duhamel's principle**.

Given $s > 0$, suppose we solve the following homogeneous problem on $\mathbb{R}^n \times (s,\infty)$:

$$\begin{cases} \tilde{u}_t - \Delta \tilde{u} = 0 & \text{in } \mathbb{R}^n \times (s,\infty) \\ \tilde{u} = f & \text{on } \mathbb{R}^n \times \{t = s\}. \end{cases} \tag{3.3}$$

Let $\tilde{u}(x,t) = v(x, t-s)$; then $v(x,t)$ will solve the Cauchy problem

$$\begin{cases} v_t - \Delta v = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ v(x,0) = f(x,s) & \text{on } \mathbb{R}^n \times \{t=0\}. \end{cases}$$

One solution of v to this problem is given above by

$$v(x,t) = \int_{\mathbb{R}^n} K(x,y,t) f(x,s)\, dy \quad (x \in \mathbb{R}^n,\ t > 0).$$

In this way, we obtain a solution to (3.3) as

$$\tilde{u}(x,t) = U(x,t;s) = \int_{\mathbb{R}^n} K(x,y,t-s) f(x,s)\, dy \quad (x \in \mathbb{R}^n,\ t > s).$$

Then, **Duhamel's principle** asserts that the function

$$u(x,t) = \int_0^t U(x,t;s) ds \quad (x \in \mathbb{R}^n,\ t > 0)$$

solves the nonhomogeneous problem above. Rewriting, we have

$$\begin{aligned} u(x,t) &= \int_0^t \int_{\mathbb{R}^n} \Phi(x-y, t-s) f(y,s)\, dyds \\ &= \int_0^t \frac{1}{(4\pi(t-s))^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4(t-s)}} f(y,s)\, dy\, ds \quad (x \in \mathbb{R}^n,\ t > 0). \end{aligned} \tag{3.4}$$

In the following, we use $C_1^2(\Omega \times I)$ to denote the space of functions $u(x,t)$ such that u, Du, D^2u, u_t are continuous in $\Omega \times I$.

Theorem 3.3. *Assume $f \in C_1^2(\mathbb{R}^n \times [0,\infty))$ has compact support in $\mathbb{R}^n \times [0,\infty)$. Define $u(x,t)$ by (3.4). Then $u \in C_1^2(\mathbb{R}^n \times (0,\infty))$ satisfies*

(i) $u_t(x,t) - \Delta u(x,t) = f(x,t) \quad (x \in \mathbb{R}^n,\ t > 0)$,

(ii) for each $x^0 \in \mathbb{R}^n$,

$$\lim_{x \to x^0,\, t \to 0^+} u(x,t) = 0.$$

Proof. By change of variables,

$$u(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(y,s) f(x-y, t-s)\, dyds \quad (x \in \mathbb{R}^n,\ t > 0).$$

1. As f has compact support and $\Phi(y,s)$ is smooth near $s = t > 0$, we have

$$u_t(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(y,s) f(x-y, t-s)\, dyds + \int_{\mathbb{R}^n} \Phi(y,t) f(x-y, 0)\, dy$$

and

$$u_{x_i x_j}(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(y,s) f_{x_i x_j}(x-y, t-s)\, dyds \quad (i,j = 1,2,\cdots,n).$$

This proves $u \in C_1^2(\mathbb{R}^n \times (0,\infty))$.

2.

$$\begin{aligned} u_t(x,t) - \Delta u(x,t) = &\int_0^t \int_{\mathbb{R}^n} \Phi(y,s)[(\partial_t - \Delta_x) f(x-y,t-s)]\, dyds \\ &+ \int_{\mathbb{R}^n} \Phi(y,t) f(x-y,0)\, dy \\ = &\int_\varepsilon^t \int_{\mathbb{R}^n} \Phi(y,s)[(-\partial_s - \Delta_y) f(x-y,t-s)]\, dyds \\ &+ \int_0^\varepsilon \int_{\mathbb{R}^n} \Phi(y,s)[(-\partial_s - \Delta_y) f(x-y,t-s)]\, dyds \\ &+ \int_{\mathbb{R}^n} \Phi(y,t) f(x-y,0)\, dy \\ := &I_\varepsilon + J_\varepsilon + N. \end{aligned}$$

Now $|J_\varepsilon| \le C \int_0^\varepsilon \int_{\mathbb{R}^n} \Phi(y,s) dyds \le \varepsilon C$. Integration by parts, we have

$$\begin{aligned} I_\varepsilon = &\int_\varepsilon^t \int_{\mathbb{R}^n} [(\partial_s - \Delta_y)\Phi(y,s)] f(x-y,t-s)\, dyds \\ &+ \int_{\mathbb{R}^n} \Phi(y,\varepsilon) f(x-y,t-\varepsilon) dy - \int_{\mathbb{R}^n} \Phi(y,t) f(x-y,0)\, dy \\ = &\int_{\mathbb{R}^n} \Phi(y,\varepsilon) f(x-y,t) dy + \int_{\mathbb{R}^n} \Phi(y,\varepsilon)[f(x-y,t-\varepsilon) - f(x-y,t)] dy - N, \end{aligned}$$

since $(\partial_s - \Delta_y)\Phi(y,s) = 0$.

3. Therefore,

$$u_t(x,t) - \Delta u(x,t) = \lim_{\varepsilon\to 0^+} (I_\varepsilon + J_\varepsilon + N)$$

$$= \lim_{\varepsilon\to 0^+} \left[\int_{\mathbb{R}^n} \Phi(y,\varepsilon) f(x-y,t) dy + \int_{\mathbb{R}^n} \Phi(y,\varepsilon)[f(x-y,t-\varepsilon) - f(x-y,t)] dy \right]$$

$$= \lim_{\varepsilon\to 0^+} \int_{\mathbb{R}^n} \Phi(y,\varepsilon) f(x-y,t) dy$$

$$= \lim_{\varepsilon\to 0^+} \int_{\mathbb{R}^n} \Phi(x-y,\varepsilon) f(y,t) dy = f(x,t) \quad (x \in \mathbb{R}^n,\ t > 0),$$

by (iii) of Theorem 3.2. Finally, note $\|u(\cdot,t)\|_{L^\infty} \le t\|f\|_{L^\infty} \to 0$ as $t \to 0^+$. □

Corollary 3.4. *Under the previous hypotheses on f and g, the function*

$$u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t) g(y) dy + \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s) f(y,s) dyds$$

is a solution of

$$\begin{cases} u_t - \Delta u = f(x,t) & (x \in \mathbb{R}^n,\ t > 0) \\ u(x,0) = g(x) & (x \in \mathbb{R}^n). \end{cases}$$

3.1.4. Nonuniqueness for Cauchy problems. The following result shows that Cauchy problems for the heat equation do not have unique solutions if we allow all kinds of solutions. Later, we will show that the problem will have unique solution if the solution satisfies some growth condition.

Theorem 3.5. *There are infinitely many solutions to Problem* (3.1).

Proof. We only need to construct a nonzero solution of the one-dimensional heat equation with 0 initial data. We formally solve the following Cauchy problem

$$\begin{cases} u_t = u_{xx}, & x \in \mathbb{R},\ t \in (-\infty, \infty), \\ u(0,t) = h(t), & t \in \mathbb{R} \\ u_x(0,t) = 0, & t \in \mathbb{R}. \end{cases} \tag{3.5}$$

We have the Taylor expansion in terms of x,

$$u(x,t) = \sum_{j=0}^{\infty} h_j(t) x^j.$$

A formal computation from $u_t = u_{xx}$ gives us

$$h_0(t) = h(t),\ h_1(t) = 0,\ h_j'(t) = (j+2)(j+1)h_{j+2},\ j = 0, 1, 2, \ldots.$$

Therefore

$$h_{2k}(t) = \frac{1}{(2k)!} h^{(k)}(t), \quad h_{2k+1}(t) = 0, \quad k = 0, 1, \ldots,$$

$$u(x,t) = \sum_{k=0}^{\infty} \frac{h^{(k)}(t)}{(2k)!} x^{2k}. \tag{3.6}$$

We choose for some $\alpha > 1$ and define the function $h(t)$ by

$$h(t) = \begin{cases} 0, & t \le 0, \\ e^{-t^{-\alpha}}, & t > 0. \end{cases} \tag{3.7}$$

Then u given above is a honest solution of the heat equation and $u(x,0) = 0$, but this solution, called a **Tychonoff solution**, is not identically zero since $u(0,t) = h(t)$ for all $t > 0$. Actually, $u(x,t)$ is an entire function of x for any real t, but is not analytic in t. ☐

Exercise 3.3 (not required)**.** (a) Let $\alpha > 1$ and h be defined by (3.7) above. Show that there is a constant $\theta > 0$ such that

$$|h^{(k)}(t)| \le \frac{k!}{(\theta t)^k} e^{-\frac{1}{2}t^{-\alpha}} \quad (t > 0,\ \ k = 1, 2, \cdots).$$

(b) For this function h, prove that the function u given by (3.6) is a solution to the heat equation in $\mathbb{R} \times (0, \infty)$ with $u(x,0) = 0$.

3.2. Weak maximum principle and the uniqueness

More practical question is the existence and uniqueness of the **mixed-value problems** in a bounded open set.

3.2.1. Parabolic cylinders and the weak maximum principle. We assume Ω is a bounded open set in $\mathbb{R}^n$, $T > 0$. Consider the **parabolic cylinder**:

$$\Omega_T = \Omega \times (0, T] := \{(x,t) \mid x \in \Omega,\ t \in (0, T]\}.$$

We define the **parabolic boundary** of Ω_T to be

$$\Gamma_T = \partial' \Omega_T := \overline{\Omega_T} \setminus \Omega_T;$$

that is,

$$\Gamma_T = \partial' \Omega_T := (\partial\Omega \times [0, T]) \cup (\Omega \times \{t = 0\}).$$

First we have the **weak maximum principle** for *sub-solutions.*

Theorem 3.6 (Weak maximum principle)**.** *Let $u \in C^2_1(\Omega_T) \cap C(\overline{\Omega_T})$ and satisfy $u_t - \Delta u \le 0$ in Ω_T.* (In this case, we say u is a **subsolution of heat equation**.) *Then*

$$\max_{\overline{\Omega_T}} u = \max_{\partial'\Omega_T} u.$$

Proof. Consider $v = u - \epsilon t$ for any $\epsilon > 0$. Then

$$v_t - \Delta v = u_t - \Delta u - \epsilon \le -\epsilon < 0 \quad ((x,t) \in \Omega_T).$$

Let $v(x^0, t^0) = \max_{\overline{\Omega_T}} v$. Supose $(x^0, t^0) \in \Omega_T$. Then $x^0 \in \Omega$ and $t^0 \in (0, T]$; consequently, at maximum point (x^0, t^0) we have $\Delta v(x^0, t^0) \le 0$ and $v_t(x^0, t^0) \ge 0$; this would limply $v_t - \Delta v \ge 0$ at (x^0, t^0), a contradiction since $v_t - \Delta v < 0$. Hence $(x^0, t^0) \in \partial'\Omega_T$ and so

$$\max_{\overline{\Omega_T}} v = \max_{\partial'\Omega_T} v \le \max_{\partial'\Omega_T} u.$$

So

$$\max_{\overline{\Omega_T}} u \le \max_{\overline{\Omega_T}} (v + \epsilon t) \le \max_{\overline{\Omega_T}} v + \epsilon T = \max_{\partial'\Omega_T} v + \epsilon T \le \max_{\partial'\Omega_T} u + \epsilon T.$$

Setting $\epsilon \to 0^+$, we deduce

$$\max_{\overline{\Omega_T}} u \le \max_{\partial'\Omega_T} u.$$

The opposite inequality is obvious and hence the equality follows. □

Similarly, the **weak minimum principle** holds for the **super-solutions** satisfying $u_t - \Delta u \ge 0$ in Ω_T.

3.2.2. Uniqueness of mixed-value problems. Theorem 3.6 immediately implies the following uniqueness result for the **initial-boundary value problem** (or the **mixed-value problem**) of heat equation in Ω_T. The proof is standard and is left as an exercise.

Theorem 3.7. *The mixed-value problem*

$$\begin{cases} u_t - \Delta u = f(x,t), & (x,t) \in \Omega_T, \\ u(x,0) = g(x), & x \in \Omega, \\ u(x,t) = h(x,t), & x \in \partial\Omega, \ t \in [0,T] \end{cases}$$

can have at most one solution u in $C^2_1(\Omega_T) \cap C(\overline{\Omega_T})$.

3.2.3. Maximum principle for the Cauchy problems. We now extend the maximum principle to the region $\mathbb{R}^n \times (0, T]$.

Theorem 3.8 (Weak maximum principle on $\mathbb{R}^n$)**.** *Let $u \in C^2_1(\mathbb{R}^n \times (0,T]) \cap C(\mathbb{R}^n \times [0,T])$ and satisfy*

$$\begin{cases} u_t - \Delta u \le 0, & \text{for } 0 < t < T, \ x \in \mathbb{R}^n, \\ u(x,t) \le Ae^{a|x|^2} & \text{for } 0 < t < T, \ x \in \mathbb{R}^n, \\ u(x,0) = g(x) & \text{for } x \in \mathbb{R}^n \end{cases}$$

for constants $A, a > 0$. Then

$$\sup_{x \in \mathbb{R}^n,\, t \in [0,T]} u(x,t) = \sup_{z \in \mathbb{R}^n} g(z).$$

Proof. 1. We first assume that $4aT < 1$. Let $\varepsilon > 0$ be such that $4a(T+\epsilon) < 1$. Fix $y \in \mathbb{R}^n, \mu > 0$, and consider

$$v(x,t) = u(x,t) - \frac{\mu}{(T+\epsilon-t)^{n/2}} e^{\frac{|x-y|^2}{4(T+\epsilon-t)}} \quad (x \in \mathbb{R}^n,\ 0 \le t \le T).$$

A direct calculation shows that the function $w(x,t) = \frac{\mu}{(T+\epsilon-t)^{n/2}} e^{\frac{|x-y|^2}{4(T+\epsilon-t)}}$ satisfies $w_t = \Delta w$ (or by noting $w(x,t) = C\Phi(i(x-y), T+\epsilon-t)$, where C is a constant, $i^2 = -1$ and $\Phi(x,t)$ is the fundamental solution of heat equation), and hence

$$v_t - \Delta v \le 0 \quad \text{in } \mathbb{R}^n \times (0,T].$$

Now fix $r > 0$ and let $\Omega = B(y,r)$. Now consider the circular cylinder Ω_T. We have from the maximum principle that

$$v(y,t) \le \max_{\partial'\Omega_T} v \quad (0 \le t \le T). \tag{3.8}$$

2. If $x \in \mathbb{R}^n$, then $v(x,0) < u(x,0) = g(x) \le \sup_{\mathbb{R}^n} g$. If $x \in \partial\Omega,\ 0 \le t \le T$, then

$$\begin{aligned} v(x,t) &= u(x,t) - \frac{\mu}{(T+\epsilon-t)^{n/2}} e^{\frac{r^2}{4(T+\epsilon-t)}} \\ &\le Ae^{a|x|^2} - \frac{\mu}{(T+\epsilon-t)^{n/2}} e^{\frac{r^2}{4(T+\epsilon-t)}} \\ &\le Ae^{a(r+|y|)^2} - \frac{\mu}{(T+\epsilon))^{n/2}} e^{\frac{r^2}{4(T+\epsilon)}} \\ &\le Ae^{a(r+|y|)^2} - \mu(4(a+\gamma))^{n/2} e^{(a+\gamma)r^2}, \end{aligned}$$

where $a + \gamma = \frac{1}{4(T+\epsilon)}$ for a positive number $\gamma > 0$. Comparing the coefficients of r^2 inside both exponentials, we can choose $r > 0$ sufficiently large (depending on μ and y) so that

$$Ae^{a(r+|y|)^2} - \mu(4(a+\gamma))^{n/2} e^{(a+\gamma)r^2} \le \sup_{\mathbb{R}^n} g.$$

Therefore, we have $v(x,t) \le \sup_{\mathbb{R}^n} g$ for all $(x,t) \in \partial'\Omega_T$. Hence, by (3.8),

$$v(y,t) = u(y,t) - \frac{\mu}{(T+\epsilon-t)^{n/2}} \le \max_{\partial'\Omega_T} v \le \sup_{\mathbb{R}^n} g \quad (0 \le t \le T).$$

So,

$$u(y,t) \le \sup_{\mathbb{R}^n} g + \frac{\mu}{(T+\epsilon-t)^{n/2}}.$$

This being valid for all $\mu > 0$ implies that $u(y,t) \le \sup_{\mathbb{R}^n} g$, by letting $\mu \to 0^+$.

3. Finally, if $4aT \ge 1$, we can repeatedly apply the result above on intervals $[0,T_1]$, $[T_1, 2T_1], \cdots$, until we reach to time T, where, say, $T_1 = \frac{1}{8a}$. □

Theorem 3.9. *There exists at most one solution* $u \in C_1^2(\mathbb{R}^n \times (0,T]) \cap C(\mathbb{R}^n \times [0,T])$ *to the Cauchy problem*

$$\begin{aligned} u_t - \Delta u &= f(x,t), \qquad x \in \mathbb{R}^n,\ t \in (0,T), \\ u(x,0) &= g(x), \qquad x \in \mathbb{R}^n \end{aligned}$$

satisfying the growth condition $|u(x,t)| \le Ae^{a|x|^2}$ *for all* $x \in \mathbb{R}^n$, $t \in (0,T)$, *for some constants* $A, a > 0$.

Remark 3.4. The **Tychonoff solution** constructed by power series above cannot satisfy the growth condition $|u(x,t)| \le Ae^{a|x|^2}$ near $t = 0$.

3.2.4. Energy method for uniqueness. We have already proved the uniqueness for the mixed-value problems of heat equation from the weak maximum principle. Now we introduce another method to show the uniqueness concerning more regular solutions and domains.

Assume Ω is a bounded smooth domain in $\mathbb{R}^n$. Consider the mixed-value problem

$$\begin{cases} u_t - \Delta u = f & \text{in } \Omega_T, \\ u = g & \text{on } \partial'\Omega_T. \end{cases}$$

We set the "energy" at t associated with u to be

$$e(t) = \int_\Omega u^2(x,t)dx. \tag{3.9}$$

Here we abused the name of "energy" since the L^2 norm of the temperature has no physical meaning.

Theorem 3.10. *There exists at most one solution in $C_1^2(\overline{\Omega_T})$ of the mixed-value problem.*

Proof. Let u_1, u_2 be the solutions. Then $u = u_2 - u_1 \in C_1^2(\overline{\Omega_T})$ solves

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \Omega_T, \\ u = 0 & \text{on } \partial'\Omega_T. \end{cases}$$

Let $e(t)$ be the energy associated with u. We have by Green's identity,

$$e'(t) = 2\int_\Omega uu_t(x,t)dx = 2\int_\Omega u\Delta u dx = -2\int_\Omega |\nabla u|^2 dx \le 0,$$

which implies that $e(t) \le e(0) = 0$ for all $t \in (0,T)$. Hence $u \equiv 0$ in Ω_T. □

Theorem 3.11. *There exists at most one solution $u \in C_1^2(\overline{\Omega_T})$ of the following Neumann problem*

$$\begin{cases} u_t - \Delta u = f(x,t) & \textit{in } \Omega_T, \\ u(x,0) = g(x) & \textit{for } x \in \Omega, \\ \frac{\partial u}{\partial \nu}(x,t) = h(x,t) & \textit{for } x \in \partial\Omega,\ t \in [0,T]. \end{cases}$$

Proof. The proof is standard and is left as exercise. □

We can also use the energy method to prove the following *backward* uniqueness ressult, which is not easy itself at all.

Theorem 3.12 (Backward uniqueness)**.** *Suppose $u \in C_1^2(\overline{\Omega_T})$ solves*

$$\begin{cases} u_t - \Delta u = 0 & \textit{in } \Omega_T, \\ u(x,t) = 0 & \textit{for } x \in \partial\Omega,\ t \in [0,T]. \end{cases}$$

If $u(x,T) = 0$ for all $x \in \Omega$, then $u \equiv 0$ in Ω_T.

Proof. 1. Let

$$e(t) = \int_\Omega u^2(x,t)\,dx \quad (0 \le t \le T).$$

As before

$$e'(t) = -2\int_\Omega |\nabla u|^2\,dx$$

and hence

$$e''(t) = -4\int_\Omega \nabla u \cdot \nabla u_t\,dx = 4\int_\Omega (\Delta u)^2\,dx.$$

Now

$$\int_\Omega |\nabla u|^2\,dx = -\int_\Omega u\Delta u\,dx \le \left(\int_\Omega u^2\,dx\right)^{1/2}\left(\int_\Omega (\Delta u)^2\,dx\right)^{1/2}.$$

Thus

$$(e'(t))^2 = 4\left(\int_\Omega |\nabla u|^2\,dx\right)^2 \le e(t)e''(t).$$

2. We show that $e(t) = 0$ for all $0 \le t \le T$ and then we are done. We use contradiction. Suppose otherwise that $e(t) \not\equiv 0$. Since $e(T) = 0$, there must exist an interval $[t_1, t_2] \subset [0, T]$, with

$$e(t) > 0 \quad \text{on } t \in [t_1, t_2), \quad e(t_2) = 0.$$

Set $f(t) = \ln e(t)$ for $t \in [t_1, t_2)$. Then

$$f''(t) = \frac{e''(t)e(t) - e'(t)^2}{e(t)^2} \ge 0 \quad (t_1 \le t < t_2).$$

Hence f is convex on $[t_1, t_2)$. Consequently

$$f((1-\tau)t_1 + \tau t) \le (1-\tau)f(t_1) + \tau f(t) \quad (0 < \tau < 1,\ t_1 < t < t_2)$$

and so

$$e((1-\tau)t_1 + \tau t) \le e(t_1)^{1-\tau}e(t)^\tau \quad (0 < \tau < 1,\ t_1 < t < t_2).$$

Letting $t \to t_2^-$, since $e(t_2) = 0$,

$$0 \le e((1-\tau)t_1 + \tau t_2) \le e(t_1)^{1-\tau}e(t_2)^\tau = 0 \quad (0 < \tau < 1).$$

This is a contradiction since $e(t) > 0$ on $[t_1, t_2)$. □

Remark 3.5. The **backward uniqueness theorem** for the heat equation asserts that if two temperature distributions on Ω agree at some time $T > 0$ and have the same boundary values for times $0 \le t \le T$, then these temperature distributions must be identical within Ω at all earlier times. They will remain the same until a time when the boundary data become different.

3.3. Regularity of solutions

In order to establish the regularity of a solution of the heat equation in a bounded domain we use Green's identity and the fundamental solution as we did for Laplace's equation.

Theorem 3.13 (Smoothness)**.** *Suppose $u \in C_1^2(\Omega_T)$ solves the heat equation in Ω_T. Then $u \in C^\infty(\Omega_T)$.*

Proof. 1. Consider the closed circular cylinder

$$C(x,t;r) = \{(y,s) \mid |y - x| \le r,\ t - r^2 \le s \le t\}.$$

Fix $(x_0, t_0) \in \Omega_T$ and choose $r > 0$ so small that $C := C(x_0, t_0; r)$ is contained in Ω_T. Consider two small circular cylinders

$$C' := C(x_0, t_0; \tfrac{3}{4}r), \quad C'' := C(x_0, t_0; \tfrac{1}{2}r).$$

Let $\zeta(x,t) \in C_c^\infty(\mathbb{R}^n \times \mathbb{R})$ such that $\zeta \equiv 0$ outside C, $\zeta \equiv 1$ on C' and $0 \le \zeta \le 1$.

2. We temporarily assume that $u \in C^\infty(C)$. This seems contradicting to what we needed to prove, but the following argument aims to establishing an identity that is valid for C^2_1-solutions once valid for C^∞-solutions of the heat equation. Let

$$v(x,t) = \zeta(x,t)u(x,t) \quad (x \in \mathbb{R}^n,\ 0 \le t \le t_0).$$

Then $v \in C^\infty(\mathbb{R}^n \times (0,t_0])$ and $v = 0$ on $\mathbb{R}^n \times \{t = 0\}$. Furthermore,

$$v_t - \Delta v = \zeta_t u - 2D\zeta \cdot Du - u\Delta\zeta := \tilde{f}$$

in $\mathbb{R}^n \times (0,t_0)$. Note that $\tilde{f}$ is C^∞ and has compact support in $\mathbb{R}^n \times [0,\infty)$; moreover v is bounded on $\mathbb{R}^n \times [0,T]$. Hence, by uniqueness and Theorem 3.3, we have

$$v(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s)\tilde{f}(y,s)\,dyds \quad (x \in \mathbb{R}^n,\ t \in (0,t_0)).$$

Let $(x,t) \in C''$. Then

$$\begin{aligned}
u(x,t) &= \iint_C \Phi(x-y,t-s)\tilde{f}(y,s)\,dyds \\
&= \iint_C \Phi(x-y,t-s)[(\zeta_s - \Delta\zeta)u(y,s) - 2D\zeta \cdot Du]\,dyds \\
&= \iint_C [\Phi(x-y,t-s)(\zeta_s + \Delta\zeta) + 2D_y\Phi(x-y,t-s)\cdot D\zeta]u(y,s)\,dyds \\
&= \iint_{C\setminus C'} [\Phi(x-y,t-s)(\zeta_s + \Delta\zeta) + 2D_y\Phi(x-y,t-s)\cdot D\zeta]u(y,s)\,dyds.
\end{aligned}$$

Let $\Gamma(x,y,t,s) = \Phi(x-y,t-s)(\zeta_s + \Delta\zeta)(y,s) + 2D_y\Phi(x-y,t-s)\cdot D\zeta(y,s)$; then we have

$$u(x,y) = \iint_{C\setminus C'} \Gamma(x,y,t,s)u(y,s)\,dyds \quad ((x,t) \in C''). \tag{3.10}$$

3. We have derived the formula (3.10) for C^∞-solution u of the heat equation in Ω_T. If u is a C^2_1-solution, let $u_\epsilon = \eta_\epsilon * u$ be the mollification of u. Then u_ϵ is a C^∞-solution to the heat equation in $(\Omega_T)_\epsilon$ and hence the formula (3.10) holds for u_ϵ. Then we let $\epsilon \to 0$ and deduce that (3.10) also holds for u in C''. Since $\Gamma(x,y,t,s)$ is C^∞ in $(x,t) \in C''$ and $(y,s) \in C \setminus C'$, the formula (3.10) proves u is $C^\infty(C'')$. □

Let us now record some estimates on the derivatives of solutions to the heat equation.

Theorem 3.14 (Estimates on derivatives)**.** *There exists a constant $C_{k,l}$ for $k,l = 0,1,2,\cdots$ such that*

$$\max_{C(x,t;r/2)} |D^\alpha_x D^l_t u| \le \frac{C_{k,l}}{r^{k+2l+n+2}} \|u\|_{L^1(C(x,t;r))} \quad (|\alpha| = k)$$

for all cylinders $C(x,t;r/2) \subset C(x,t;r) \subset \Omega_T$ and all solutions $u \in C^2_1(\Omega_T)$ of the heat equation in Ω_T.

Proof. 1. Fix some point $(x,t) \in \Omega_T$. Upon shifting the coordinates, we may assume the point is $(0,0)$. Suppose first that the cylinder $C(1) = C(0,0;1) \subset \Omega_T$ and let $C(3/4) = C(0,0;3/4)$ and $C(1/2) = C(0,0;1/2)$. Then, by (3.10) in the previous proof,

$$u(x,t) = \iint_{C(1)\setminus C(3/4)} \Gamma(x,y,t,s)u(y,s)\,dyds \quad ((x,t) \in C(1/2)).$$

Consequently, for each pair of $k, l = 0, 1, 2, \cdots$ and all $|\alpha| = k$,

$$(3.11) \qquad |D_x^\alpha D_t^l u(x,t)| \le \iint_{C(1)\setminus C(3/4)} |D_x^\alpha D_t^l \Gamma(x,y,t,s)| u(y,s)\, dyds \le C_{k,l} \|u\|_{L^1(C(1))}$$

for some constant $C_{k,l}$.

2. Now suppose $C(r) = C(0,0;r) \subset \Omega_T$ and let $C(r/2) = C(0,0;r/2)$. Let u be a solution to the heat equation in Ω_T. We rescale by defining

$$v(x,t) = u(rx, r^2 t) \quad ((x,t) \in C(1)).$$

Then $v_t - \Delta v = 0$ in the cylinder $C(1)$. Note that for all $|\alpha| = k$,

$$D_x^\alpha D_t^l v(x,t) = r^{2l+k} D_y^\alpha D_s^l u(rx, r^2 t)$$

and $\|v\|_{L^1(C(1))} = \frac{1}{r^{n+2}} \|u\|_{L^1(C(r))}$. Then, with (3.11) applied to v, we have

$$|D_y^\alpha D_s^l u(y,s)| \le \frac{C_{k,l}}{r^{2l+k+n+2}} \|u\|_{L^1(C(1))} \quad ((y,s) \in C(r/2)).$$

This completes the proof. □

3.4. Nonnegative solutions

From the example of Tychonoff solution above of heat equation, we know that the initial data can not determine the solution uniquely. Some additional conditions are needed for uniqueness; for example, in terms of the growth condition, as we discussed in the previous section. This section we discuss yet another important uniqueness result due to D. V. Widder for *nonnegative solutions.*

Theorem 3.15 (Widder's Theorem). *Let u be defined and continuous for $x \in \mathbb{R}$, $0 \le t < T$, and let u_t, u_x and u_{xx} exist and be continuous for $x \in \mathbb{R}$, $0 < t < T$. Assume that*

$$u_t = u_{xx}, \quad u(x,0) = g(x), \quad u(x,t) \ge 0.$$

Then u is determined uniquely for $x \in \mathbb{R}$, $0 < t < T$, is real analytic and is represented by

$$u(x,t) = \int_{\mathbb{R}} K(x,y,t) g(y) dy.$$

Proof. 1. For $a > 1$ define the cut-off function $\zeta^a(x)$ by $\zeta^a(x) = 1$ for $|x| \le a-1$; $\zeta^a(x) = 0$ for $|x| \ge a$; $\zeta^a(x) = a - |x|$ for $a - 1 < |x| < a$. Consider the expression

$$v^a(x,t) = \int_{\mathbb{R}} K(x,y,t) \zeta^a(y) g(y) dy.$$

Note that $\zeta^a(y) g(y) \in C_c(\mathbb{R}^n)$, we know that

$$v_t^a - v_{xx}^a = 0 \quad \text{for } x \in \mathbb{R},\ 0 < t < T,$$
$$v^a(x,0) = \zeta^a(x) g(x).$$

Let M_a be the maximum of $g(x)$ for $|x| \le a$. Using $K(x,y,t) \le \frac{1}{\sqrt{2\pi e}|x-y|}$, we have for $|x| > a$,

$$0 \le v^a(x) \le M_a \int_{-a}^{a} K(x,y,t) dy \le \frac{2M_a a}{\sqrt{2\pi e}} \frac{1}{|x| - a}.$$

Let $\epsilon > 0$ and let $\rho > a + \frac{2M_a a}{\epsilon\sqrt{2\pi e}}$. Then

$$\begin{cases} v^a(x,t) \le \epsilon \le \epsilon + u(x,t), & |x| = \rho,\ 0 < t < T, \\ v^a(x,0) \le g(x) \le \epsilon + u(x,0), & |x| \le \rho. \end{cases}$$

By the maximum principle,

$$v^a(x,t) \le \epsilon + u(x,t) \quad \text{for } |x| \le \rho, \quad 0 \le t < T.$$

Let $\rho \to \infty$ and we find the same inequality for all $x \in \mathbb{R}$, $0 \le t < T$. Setting $\epsilon \to 0$ yields that

$$v^a(x,t) \le u(x,t), \quad x \in \mathbb{R},\ 0 \le t < T.$$

Since ζ^a is a non-decreasing bounded functions of a, we find that

$$v(x,t) = \lim_{a\to\infty} v^a(x,t) = \int_{\mathbb{R}} K(x,y,t)g(y)dy$$

exists for $x \in \mathbb{R},\ 0 \le t < T$ and

$$0 \le v(x,t) \le u(x,t).$$

Regularity of $v(x,t)$ can be obtained from the analyticity of $v^a(x,t)$.

2. Let $w = u - v$. Then w is continuous for $x \in \mathbb{R},\ 0 \le t < T,\ w_t = w_{xx},\ w(x,t) \ge 0$ and $w(x,0) = 0$. It remains to show that $w \equiv 0$. (That is, we reduced the problem to the case $g = 0$.) We introduce the new function

$$W(x,t) = \int_0^t w(x,s)ds.$$

Then $W_{xx}(x,t) = w(x,t) = W_t(x,t) \ge 0$. So $W(x,t)$ is convex in x; hence,

$$2W(x,t) \le W(x+H,t) + W(x-H,t)$$

for any $H > 0$. Given $x > 0$, integrating the inequality with respect to H from 0 to x we find

$$2xW(x,t) \le \int_0^x W(x+H,t)dH + \int_0^x W(x-H,t)dH = \int_0^{2x} W(y,t)dy.$$

From Step 1, for all $t > s$, $x > 0$,

$$\begin{aligned} W(0,t) &\ge \int_{\mathbb{R}} K(0,y,t-s)W(y,s)dy \ge \int_0^{2x} K(0,y,t-s)W(y,s)dy \\ &\ge \frac{e^{-x^2/(t-s)}}{\sqrt{4\pi(t-s)}} \int_0^{2x} W(y,s)dy \ge \frac{e^{-x^2/(t-s)}}{\sqrt{4\pi(t-s)}} 2xW(x,s). \end{aligned}$$

The similar argument can also apply to the case $x < 0$. Therefore, we deduce that

$$W(x,s) \le \sqrt{\frac{\pi(t-s)}{x^2}} e^{x^2/(t-s)} W(0,t) \quad (x \in \mathbb{R},\ t > s \ge 0). \tag{3.12}$$

3. Now for $T > \epsilon > 0$, consider $W(x,s)$ for $x \in \mathbb{R}$, $0 \le s \le T - 2\epsilon$. Then $W(x,s)$ is bounded for $|x| \le \sqrt{\pi T}$ and $0 \le s \le T - 2\epsilon$. Using (3.12) above with $t = T - \epsilon$, we have, for $|x| \ge \sqrt{\pi T}$ and $0 \le s \le T - 2\epsilon$,

$$W(x,s) \le e^{x^2/\epsilon} W(0, T-\epsilon).$$

Hence W satisfies the assumption of Theorem 3.8 for all $x \in \mathbb{R}$ and $0 \le s \le T - 2\epsilon$, with some constants $a, A > 0$. Since $W(x,0) = 0$, it follows that $W(x,s) = 0$ for $s \in [0, T-\epsilon)$. Since $\epsilon > 0$ is arbitrary, $W(x,s) = 0$ for $s \in [0,T)$. That is $u(x,t) = v(x,t)$. □

3.5. Mean value property and the strong maximum principle

In this section we derive for the heat equation some kind of analogue of the mean value property for harmonic functions.

Definition 3.6. Fix $x \in \mathbb{R}^n,\ t \in \mathbb{R},\ r > 0$. We define the **heat ball**

$$E(x,t;r) = \{(y,s) \mid \Phi(x-y,t-s) \geq \frac{1}{r^n}\}.$$

This is a *bounded* region in space-time, whose boundary is a level set of $\Phi(x-y,t-s)$.

3.5.1. Mean-value property on the heat ball.

Theorem 3.16 (Mean-value property for the heat equation)**.** *Let $u \in C^2_1(\Omega_T)$ satisfy $u_t - \Delta u \leq 0$ in Ω_T (that is, u is a subsolution to the heat equation). Then*

$$u(x,t) \leq \frac{1}{4r^n} \iint_{E(x,t;r)} u(y,s) \frac{|x-y|^2}{(t-s)^2} dyds$$

for all $E(x,t;r) \subset \Omega_T$.

Proof. 1. We may assume that $x = 0,\ t = 0$ and write $E(r) = E(0,0;r)$. Set

$$\begin{aligned} \phi(r) &= \frac{1}{r^n} \iint_{E(r)} u(y,s) \frac{|y|^2}{s^2} dyds \\ &= \iint_{E(1)} u(ry, r^2 s) \frac{|y|^2}{s^2} dyds. \end{aligned}$$

Let $v(y,s) = u(ry, r^2 s)$ $((y,s) \in E(1))$. Then $v_s(y,s) \leq \Delta_y v(y,s)$ in $E(1)$. Let

$$H(y,s) = \Phi(y,-s) = \frac{1}{(-4\pi s)^{n/2}} e^{\frac{|y|^2}{4s}} \quad (y \in \mathbb{R}^n,\ s < 0).$$

Then $H(y,s) = 1$ for $(y,s) \in \partial E(1)$,

$$\ln H(y,s) = -\frac{n}{2} \ln(-4\pi s) + \frac{|y|^2}{4s} \geq 0 \quad ((y,s) \in E(1)),$$

and

$$4(\ln H)_s + \frac{|y|^2}{s^2} = -\frac{2n}{s}, \quad \nabla(\ln H) = \frac{y}{2s}. \tag{3.13}$$

2. Note that $\nabla u(ry, r^2 s) = \frac{1}{r}\nabla v(y,s),\ u_t(ry, r^2 s) = \frac{1}{r^2} v_s(y,s)$. We calculate $\phi'(r)$ as follows.

$$\begin{aligned}
\phi'(r) &= \iint_{E(1)} \left(\nabla u(ry, r^2 s)\cdot y + 2rsu_t(ry, r^2 s)\right)\frac{|y|^2}{s^2}dyds \\
&= \frac{1}{r}\iint_{E(1)} \left(y\cdot\nabla v(y,s) + 2sv_s(y,s)\right)\frac{|y|^2}{s^2}dyds \\
&= \frac{1}{r}\iint_{E(1)} \left(y\cdot\nabla v\frac{|y|^2}{s^2} + 4v_s y\cdot\nabla(\ln H)\right)dyds \\
&= \frac{1}{r}\left[\iint_{E(1)} y\cdot\nabla v\frac{|y|^2}{s^2}dyds - 4\iint_{E(1)} (nv_s + y\cdot(\nabla v)_s)\ln H dyds\right] \\
&= \frac{1}{r}\left[\iint_{E(1)} y\cdot\nabla v\frac{|y|^2}{s^2}dyds - 4\iint_{E(1)} (nv_s\ln H - y\cdot\nabla v(\ln H)_s)\, dyds\right] \\
&= -\frac{1}{r}\left[\iint_{E(1)} y\cdot\nabla v\frac{2n}{s}dyds + 4n\iint_{E(1)} v_s\ln H dyds\right] \quad \text{(using (3.13a))} \\
&\geq -\frac{1}{r}\left[\iint_{E(1)} y\cdot\nabla v\frac{2n}{s}dyds + 4n\iint_{E(1)} \Delta v\ln H dyds\right] \quad (\text{as } v_s \leq \Delta v,\ \ \ln H \geq 0 \text{ in } E(1)) \\
&= -\frac{2n}{r}\iint_{E(1)} \left(\frac{y}{s}\cdot\nabla v - 2\nabla v\cdot\nabla\ln H\right)dyds = 0 \quad \text{(using (3.13b))}.
\end{aligned}$$

Consequently, ϕ is an increasing function of r, and hence

$$\phi(r) \geq \phi(0^+) = u(0,0)\iint_{E(1)} \frac{|y|^2}{s^2}dyds.$$

3. It remains to show that

$$\iint_{E(1)} \frac{|y|^2}{s^2}dyds = 4.$$

Note that $E(1) = \{(y,s) \mid -\frac{1}{4\pi} \leq s < 0,\ |y|^2 \leq 2ns\ln(-4\pi s)\}$. Under the change of variables $(y,s)\to(y,\tau)$, where $\tau = \ln(-4\pi s)$ and hence $s = -\frac{e^\tau}{4\pi}$, the set $E(1)$ is mapped one-to-one and onto a set $\tilde{E}(1)$ in (y,τ) space given by $\tilde{E}(1) = \{(y,\tau) \mid \tau\in(-\infty,0),\ |y|^2 \leq -\frac{n}{2\pi}\tau e^\tau\}$. Therefore,

$$\begin{aligned}
\iint_{E(1)} \frac{|y|^2}{s^2}dyds &= \int_{-\infty}^0 \int_{|y|^2\leq -\frac{n}{2\pi}\tau e^\tau} 4\pi|y|^2 e^{-\tau}dyd\tau \\
&= 4\pi n\alpha_n \int_{-\infty}^0 \int_0^{\sqrt{-\frac{n}{2\pi}\tau e^\tau}} r^{n+1}e^{-\tau}drd\tau \\
&= \frac{4\pi n\alpha_n}{n+2}\left(\frac{n}{2\pi}\right)^{\frac{n+2}{2}}\int_{-\infty}^0 (-\tau)^{\frac{n+2}{2}}e^{\frac{n}{2}\tau}d\tau \\
&= \frac{4\pi n\alpha_n}{n+2}\left(\frac{n}{2\pi}\right)^{\frac{n+2}{2}}\left(\frac{2}{n}\right)^{\frac{n+4}{2}}\int_0^\infty t^{\frac{n+2}{2}}e^{-t}dt \quad \left(t = -\frac{n}{2}\tau\right) \\
&= \frac{4n\alpha_n}{n+2}\frac{2}{n}\left(\frac{1}{\pi}\right)^{n/2}\Gamma\left(\frac{n}{2}+2\right) \quad \text{(here } \Gamma(s) \text{ is the Gamma-function)} \\
&= 2n\alpha_n\left(\frac{1}{\pi}\right)^{n/2}\Gamma\left(\frac{n}{2}\right) = 4 \quad \left(\text{noting that } n\alpha_n = \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})}\right).
\end{aligned}$$

This completes the proof. □

3.5.2. Strong maximum principle.

Theorem 3.17 (Strong Maximum Principle)**.** *Let Ω be a bounded domain in $\mathbb{R}^n$. Assume that $u \in C_1^2(\Omega_T) \cap C^0(\overline{\Omega_T})$ satisfies $u_t - \Delta u \le 0$ in Ω_T* (that is, u is a subsolution to the heat equation) *and there exists a point $(x_0, t_0) \in \Omega_T$ such that*

$$u(x_0, t_0) = \max_{\overline{\Omega_T}} u.$$

Then u is constant in $\overline{\Omega_{t_0}}$.

Proof. 1. Let $M = u(x_0, t_0) = \max_{\overline{\Omega_T}} u$. Then for all sufficiently small $r > 0$, $E(x_0, t_0; r) \subset \Omega_T$, and we employ the mean value theorem to get

$$M = u(x_0, t_0) \le \frac{1}{4r^n} \iint_{E(x_0,t_0,r)} u(y,s) \frac{|x_0 - y|^2}{(t_0 - s)^2} dyds \le \frac{M}{4r^n} \iint_{E(x_0,t_0,r)} \frac{|x_0 - y|^2}{(t_0 - s)^2} dyds = M.$$

As equality holds only if $u(y, s) \equiv M$ on $E(x_0, t_0, r)$, we see that

$$u(y, s) \equiv M \quad ((y, s) \in E(x_0, t_0; r)).$$

Draw any line segment L in Ω_T connecting (x_0, t_0) to another point (y_0, s_0) in Ω_T with $s_0 < t_0$. Consider

$$r_0 = \inf\{s \in [s_0, t_0] \mid u(x, t) = M \text{ for all points } (x, t) \in L,\ s \le t \le t_0\}.$$

Since u is continuous, the infimum r_0 is attained. We prove $r_0 = s_0$. Suppose for the contrary that $r_0 > s_0$. Then $u(z_0, r_0) = M$ for some point $(z_0, r_0) \in L \cap \Omega_T$. From the previous argument, $u(x, t) = M$ on $E(z_0, r_0; r)$ for all sufficiently small r. Note that $E(z_0, r_0; r)$ contains $L \cap \{r_0 - \sigma < t \le r_0\}$ for some $\sigma > 0$; this is a contradiction to r_0 being the infimum. Hence $r_0 = s_0$ and so $u \equiv M$ on L.

2. Now fix any $x \in \Omega$ and $0 \le t < t_0$. (The following argument is not needed if Ω is convex.) There exist points $\{x_0, x_1, \cdots, x_m = x\}$ such that the line segments connecting x_{i-1} to x_i lie in Ω for all $i = 1, 2, \cdots, m$. Select times $t_0 > t_1 > \cdots > t_m = t$ such that the line segments L_i connecting (x_{i-1}, t_{i-1}) to (x_i, t_i) lie in Ω_T. According to Step 2, $u \equiv M$ on L_i and hence $u(x, t) = M$ for all $(x, t) \in \Omega_T$. ☐

Remark 3.7. (a) The weak maximum principle follows from the strong maximum principle above.

(b) If a solution u to the heat equation attains its maximum (or minimum) value at an interior point (x_0, t_0) then u is constant at *all earlier times* $t \le t_0$. However, the solution may change at later times $t > t_0$ if the boundary conditions alter after t_0. The solution will not respond to changes of the boundary data until these changes happen.

(c) Suppose u solves the heat equation in Ω_T and equals zero on $\partial\Omega \times [0, T]$. If the initial data $u(x, 0) = g(x)$ is nonnegative and is positive somewhere, then u is positive everywhere within Ω_T. This is another illustration of **infinite propagation speed** of the disturbances of initial data for the heat equation. (A positive initial local source of heat will generate the positive temperature immediately afterwards *everywhere* away from the boundary.)

3.6. Maximum principles for second-order linear parabolic equations

(*This material is from Section 7.1 of the textbook.*)

We consider the second-order linear differential operator $\partial_t + L$ defined by

$$u_t + Lu = u_t - \sum_{i,j=1}^{n} a^{ij}(x,t)u_{x_ix_j} + \sum_{i=1}^{n} b^i(x,t)u_{x_i} + c(x,t)u,$$

where a^{ij}, b^i and c are all bounded functions in $\Omega_T = \Omega \times (0,T]$, where Ω is a bounded open set in $\mathbb{R}^n$.

Without the loss of generality, we assume that $a^{ij}(x,t) = a^{ji}(x,t)$. The operator $\partial_t + L$ is called **parabolic** in Ω_T if there exists $\lambda(x,t) > 0$ such that

$$\sum_{i,j=1}^{n} a^{ij}(x,t)\xi_i\xi_j \geq \lambda(x,t)\sum_{k=1}^{n} \xi_k^2, \quad \forall\,(x,t) \in \Omega_T,\ \xi \in \mathbb{R}^n.$$

If $\lambda(x,t) \geq \lambda_0 > 0$ for all $(x,t) \in \Omega_T$, we say the operator $\partial_t + L$ is **uniformly parabolic** in Ω_T.

3.6.1. Weak maximum principle.

Theorem 3.18 (Weak maximum principle). *Assume $\partial_t + L$ is parabolic in Ω_T. Let $u \in C_1^2(\Omega_T) \cap C(\overline{\Omega_T})$ satisfy $u_t + Lu \leq 0$ in Ω_T (in this case, we say that u is a* **subsolution of** *$u_t + Lu = 0$). Then*

$$(a) \qquad \max_{\overline{\Omega_T}} u = \max_{\partial'\Omega_T} u \quad \text{if } c(x,t) = 0 \text{ in } \Omega_T,$$

$$(b) \qquad \max_{\overline{\Omega_T}} u \leq \max_{\partial'\Omega_T} u^+ \quad \text{if } c(x,t) \geq 0 \text{ in } \Omega_T.$$

Proof. 1. Fix $\epsilon > 0$ and let $v = u - \epsilon t$. Note that in both cases of (a) and (b) above,

$$v_t + Lv = u_t + Lu - [\epsilon + c(x,t)\epsilon t] \leq -\epsilon < 0 \quad ((x,t) \in \Omega_T).$$

We show that

$$(3.14) \qquad \begin{aligned} &(a) \quad \max_{\overline{\Omega_T}} v = \max_{\partial'\Omega_T} v \quad \text{if } c(x,t) = 0 \text{ in } \Omega_T, \\ &(b) \quad \max_{\overline{\Omega_T}} v \leq \max_{\partial'\Omega_T} u^+ \quad \text{if } c(x,t) \geq 0 \text{ in } \Omega_T. \end{aligned}$$

To prove (3.14)(a), let $v(x^0,t^0) = \max_{\overline{\Omega_T}} v$ for some $(x^0,t^0) \in \overline{\Omega_T}$. If $(x^0,t^0) \in \Omega_T$ then $x^0 \in \Omega$ and $t^0 \in (0,T]$; consequently at maximum point (x^0,t^0) we have $Dv(x^0,t^0) = 0$, $(v_{x_ix_j}(x^0,t^0)) \leq 0$ (in matrix sense) and $v_t(x^0,t^0) \geq 0$, and hence

$$v_t + Lv = v_t - \sum_{i,j=1}^{n} a^{ij} v_{x_ix_j} \geq 0 \quad \text{at } (x^0,t^0),$$

which is a contradiction, as $v_t + Lv \leq -\epsilon$ on Ω_T. Therefore, we must have $(x^0,t^0) \in \partial'\Omega_T$; this proves (3.14)(a). Note that (3.14)(b) is obviously valid if $\max_{\overline{\Omega_T}} v \leq 0$. So we assume $\max_{\overline{\Omega_T}} v > 0$. Let $v(x^0,t^0) = \max_{\overline{\Omega_T}} v > 0$ for some $(x^0,t^0) \in \overline{\Omega_T}$. If $(x^0,t^0) \in \Omega_T$ then $x^0 \in \Omega$ and $t^0 \in (0,T]$; consequently at maximum point (x^0,t^0) we have $Dv(x^0,t^0) = 0$, $(v_{x_ix_j}(x^0,t^0)) \leq 0$ (in matrix sense) and $v_t(x^0,t^0) \geq 0$, and hence

$$v_t + Lv = v_t - \sum_{i,j=1}^{n} a^{ij} v_{x_ix_j} + cv \geq cv \geq 0 \quad \text{at } (x^0,t^0),$$

which is also a contradiction, as $v_t + Lv \le -\epsilon$ on Ω_T. Therefore, we must have $(x^0, t^0) \in \partial'\Omega_T$; hence

$$\max_{\overline{\Omega_T}} v = \max_{\partial'\Omega_T} v \le \max_{\partial'\Omega_T} u \le \max_{\partial'\Omega_T} u^+.$$

2. By (3.14), we deduce that

$$\max_{\overline{\Omega_T}} u = \max_{\overline{\Omega_T}}(v + \epsilon t) \le \max_{\overline{\Omega_T}} v + \epsilon T = \max_{\partial'\Omega_T} v + \epsilon T \le \max_{\partial'\Omega_T} u + \epsilon T \quad \text{if } c(x,t) = 0 \text{ in } \Omega_T,$$

$$\max_{\overline{\Omega_T}} u = \max_{\overline{\Omega_T}}(v + \epsilon t) \le \max_{\overline{\Omega_T}} v + \epsilon T \le \max_{\partial'\Omega_T} u^+ + \epsilon T \quad \text{if } c(x,t) \ge 0 \text{ in } \Omega_T.$$

Setting $\epsilon \to 0^+$ in both cases, we deduce

$$(a) \quad \max_{\overline{\Omega_T}} u \le \max_{\partial'\Omega_T} u \quad \text{if } c(x,t) = 0 \text{ in } \Omega_T,$$

$$(b) \quad \max_{\overline{\Omega_T}} u \le \max_{\partial'\Omega_T} u^+ \quad \text{if } c(x,t) \ge 0 \text{ in } \Omega_T.$$

Clearly the equality holds in the case of (a) above. □

We have the following result for general $c(x,t)$.

Theorem 3.19 (Weak maximum principle for general c). *Assume $\partial_t + L$ is parabolic in Ω_T. Let $u \in C_1^2(\Omega_T) \cap C(\overline{\Omega_T})$ satisfy $u_t + Lu \le 0$ in Ω_T (that is, u is a* subsolution of *$u_t + Lu = 0$). Then*

$$\max_{\overline{\Omega_T}} u \le e^{CT} \max_{\partial'\Omega_T} u^+,$$

where C is any constant such that $c(x,t) + C \ge 0$ in Ω_T. For example, $C = -\inf_{\Omega_T} c^-(x,t)$.

Proof. The inequality is obviously valid if $\max_{\overline{\Omega_T}} u \le 0$. So we assume $\max_{\overline{\Omega_T}} u > 0$. Consider $w(x,t) = e^{-Ct} u(x,t)$. Then

$$w_t + Lw = e^{-Ct}(u_t + Lu) - Ce^{-Ct} u = e^{-Ct}(u_t + Lu) - Cw$$

and hence

$$w_t + (Lw + Cw) = e^{-Ct}(u_t + Lu) \le 0 \quad \text{in } \Omega_T.$$

The operator $\partial_t + \tilde{L}$, where $\tilde{L}w = Lw + Cw$, is parabolic and has the zeroth order coefficient $\tilde{c}(x,t) = c(x,t) + C \ge 0$ in Ω_T. Hence, by the previous theorem,

$$\max_{\overline{\Omega_T}} w \le \max_{\partial'\Omega_T} w^+.$$

Consequently, since $\max_{\overline{\Omega_T}} u > 0$,

$$\max_{\overline{\Omega_T}} u = \max_{\overline{\Omega_T}}(e^{Ct} w) \le e^{CT} \max_{\overline{\Omega_T}} w \le e^{CT} \max_{\partial'\Omega_T} w^+ \le e^{CT} \max_{\partial'\Omega_T} u^+.$$

□

The weak maximum principle for general $c(x,t)$ implies the uniqueness of **mixed-value problem** for parabolic equations *regardless of the sign of $c(x,t)$.*

Theorem 3.20. *The mixed-value problem*

$$\begin{cases} u_t + Lu = f(x,t), & (x,t) \in \Omega_T, \\ u(x,0) = g(x), & x \in \Omega, \\ u(x,t) = h(x,t), & x \in \partial\Omega,\ t \in [0,T] \end{cases}$$

can have at most one solution u in $C_1^2(\Omega_T) \cap C(\overline{\Omega_T})$.

Proof. Let u_1, u_2 be two solutions of the problem in $C_1^2(\Omega_T) \cap C(\overline{\Omega_T})$. Define $u = u_1 - u_2$. Then $u \in C_1^2(\Omega_T) \cap C(\overline{\Omega_T})$ solves $u_t + Lu = 0$ in Ω_T and satisfies $u = 0$ on $\partial'\Omega_T$. Hence

$$\max_{\Omega_T} u \le e^{CT} \max_{\partial'\Omega_T} u^+ = 0.$$

Hence $u \le 0$ in Ω_T; that is, $u_1 \le u_2$ in Ω_T. By symmetry (or applying the maximum principle to $-u$), we have $u_2 \le u_1$ in Ω_T. This proves $u_1 = u_2$ in Ω_T. □

3.6.2. Harnack's inequality and the strong maximum principle. Let Ω be a smooth and bounded domain of $\mathbb{R}^n$. We consider the special case when the operator $\partial_t + L$ is given simply by

$$u_t + Lu = u_t - \sum_{ij=1}^n a_{ij}(x,t) D_{ij} u$$

and is uniformly parabolic in Ω_T. i.e., there are positive constants λ and Λ such that

$$\lambda|\xi|^2 \le \sum_{ij} a_{ij}(x,t)\xi_i\xi_j \le \Lambda|\xi|^2 \quad ((x,t) \in \Omega_T,\ \xi \in \mathbb{R}^n).$$

We also assume a_{ij} is smooth on $\overline{\Omega_T}$. Then we have the following

Theorem 3.21. *Assume $V \subset\subset \Omega$ is connected. Then, for $0 < t_1 < t_2 \le T$, there is a constant C depending only on V, t_1, t_2 and the coefficients of L such that*

$$\sup_V u(.,t_1) \le C \inf_V u(.,t_2)$$

whenever $u \ge 0$ is a smooth solution of $u_t + Lu = 0$ in Ω_T.

Proof. 1. Without loss of generality, we may assume that $u > 0$ (otherwise consider $u + \epsilon$ and let $\epsilon \to 0$) and assume that V is a ball since in general V is covered by a finite number of balls.

Let $v(x,t) = \ln u(x,t)$. The idea is to show that for some $\gamma > 0$

$$v(x_2,t_2) - v(x_1,t_1) \ge -\gamma \quad (x_1,\ x_2 \in V,\ 0 < t_1 < t_2 \le T). \tag{3.15}$$

This will imply $u(x_1,t_1) \le e^\gamma u(x_2,t_2)$, proving the theorem.

Note that

$$\begin{aligned} v(x_2,t_2) - v(x_1,t_1) &= \int_0^1 \frac{d}{ds} v(sx_2 + (1-s)x_1, st_2 + (1-s)t_1) ds \\ &= \int_0^1 [(x_2 - x_1) \cdot Dv + (t_2 - t_1) v_t] ds \\ &\ge \int_0^1 [(t_2 - t_1) v_t - \epsilon |Dv|^2]\, ds - C(\epsilon). \end{aligned}$$

Therefore to show (3.15), it suffices to show that

$$v_t \ge \nu |Dv|^2 - \gamma \quad \text{in } V \times [t_1, t_2]. \tag{3.16}$$

2. A direct computation shows that

$$D_j v = \frac{D_j u}{u}, \quad D_{ij} v = \frac{D_{ij} u}{u} - \frac{D_i u D_j u}{u^2} = \frac{D_{ij} u}{u} - D_i v D_j v,$$

$$v_t = \frac{u_t}{u} = \frac{\sum_{ij} a_{ij} D_{ij} u}{u} = \sum_{ij} (a_{ij} D_{ij} v + a_{ij} D_i v D_j v) := \alpha + \beta,$$

where

$$\alpha(x,t)=\sum_{ij}a_{ij}D_{ij}v,\quad \beta(x,t)=\sum_{ij}a_{ij}D_ivD_jv.$$

Note that $\beta\geq\lambda|Dv|^2$. Hence, to show $v_t=\alpha+\beta\geq\nu|Dv|^2-\gamma$, we only need to show that $w:=\alpha+\kappa\beta\geq-\gamma$ for some $\kappa\in(0,1/2)$ to be determined later. To this end, we calculate

$$\begin{aligned}D_i\alpha&=\sum_{kl}a_{kl}D_{ikl}v+\sum_{kl}D_{kl}vD_ia_{kl},\\ D_k\beta&=2\sum_{ij}a_{ij}D_{ik}vD_jv+\sum_{ij}D_ka_{ij}D_ivD_jv,\\ D_{kl}\beta&=2\sum_{ij}a_{ij}D_{ikl}vD_jv+2\sum_{ij}a_{ij}D_{ik}vD_{jl}v+R,\end{aligned}$$

where R is a term satisfying

$$|R|\leq\epsilon|D^2v|^2+C(\epsilon)|Dv|^2+C\quad(\epsilon>0).\tag{3.17}$$

Hence

$$\begin{aligned}\sum_{kl}a_{kl}D_{kl}\beta&=2\sum_{ijkl}a_{kl}a_{ij}D_{ikl}vD_jv+2\sum_{ijkl}a_{ij}a_{kl}D_{ik}vD_{jl}v+R\\ &=2\sum_{ij}a_{ij}D_i\alpha D_jv+2\sum_{ijkl}a_{ij}a_{kl}D_{ik}vD_{jl}v+R,\end{aligned}\tag{3.18}$$

where R is also a term satisfying (3.17).

3. Notice that $D_{ij}v_t=D_{ij}\alpha+D_{ij}\beta$ and hence

$$\begin{aligned}\alpha_t&=\sum_{ij}[a_{ij}D_{ij}v_t+a_{ij,t}D_{ij}v]\\ &=\sum_{ij}a_{ij}D_{ij}\alpha+\sum_{ij}a_{ij}D_{ij}\beta+\sum_{ij}a_{ij,t}D_{ij}v\\ &=\sum_{ij}a_{ij}D_{ij}\alpha+2\sum_{ij}a_{ij}D_jvD_i\alpha+2\sum_{ijkl}a_{ij}a_{kl}D_{ik}vD_{jl}v+R,\end{aligned}$$

where R is also a term satisfying (3.17). Note that, by the uniform parabolicity and linear algebra,

$$\sum_{ijkl}a_{ij}a_{kl}D_{ik}vD_{jl}v\geq\lambda^2|D^2v|^2.$$

Hence, with suitable choice of $\epsilon>0$ in (3.17), there exists a constant $C>0$ such that

$$\alpha_t-\sum_{ij}a_{ij}D_{ij}\alpha+\sum_{i=1}^nb_iD_i\alpha\geq\lambda^2|D^2v|^2-C|Dv|^2-C,\tag{3.19}$$

where

$$b_i=-2\sum_{j=1}^na_{ij}D_jv.$$

To derive a similar inequality for $\beta(x,t)$, we compute by (3.18)

$$\begin{aligned}\beta_t - \sum_{kl} a_{kl} D_{kl}\beta &= \sum_{ij} a_{ij,t} D_i v D_j v + 2\sum_{ij} a_{ij} D_i v_t D_j v - \sum_{kl} a_{kl} D_{kl}\beta \\ &= 2\sum_{ij} a_{ij} D_j v (D_i v_t - D_i\alpha) - 2\sum_{ijkl} a_{ij} a_{kl} D_{ik} v D_{jl} v + R \\ &= 2\sum_{ij} a_{ij} D_j v D_i \beta - 2\sum_{ijkl} a_{ij} a_{kl} D_{ik} v D_{jl} v + R,\end{aligned}$$

where R is a term satisfying (3.17). Hence, with the same b_i defined above,

$$\beta_t - \sum_{kl} a_{kl} D_{kl}\beta + \sum_i b_i D_i \beta \geq -C(|D^2 v|^2 + |Dv|^2 + 1). \tag{3.20}$$

By (3.19) and (3.20), for the function $w = \alpha + \kappa\beta$, we can choose $\kappa \in (0, 1/2)$ sufficiently small such that

$$w_t + Lw + \sum_{k=1}^n b_k D_k w \geq \frac{\lambda^2}{2} |D^2 v|^2 - C|Dv|^2 - C. \tag{3.21}$$

4. Choose a cutoff function $\zeta \in C^\infty(\Omega_T)$ such that $0 \leq \zeta \leq 1$ and $\zeta \equiv 1$ on $V \times [t_1, t_2]$ and $\zeta = 0$ on $\partial'\Omega_T$. Let μ be a large positive constant.

Claim: $H(x,t) = \zeta^4 w + \mu t \geq 0$ in Ω_T, which implies that $w + \mu t \geq 0$ in $V \times [t_1, t_2]$. So, $w \geq -\mu t_2 > -\gamma$. This proves the theorem.

We now prove this claim. Suppose not; then there is a point $(x_0, t_0) \in \Omega_T$ such that $H(x_0, t_0) < 0$ is the minimum of H on $\overline{\Omega_T}$. At this minimum point, $\zeta \neq 0$,

$$D_k H = \zeta^3 (4w D_k \zeta + \zeta D_k w) = 0, \quad H_t - \sum_{kl} a_{kl} D_{kl} H + \sum_k b_k D_k H \leq 0.$$

Note that

$$\begin{aligned} H_t = \mu + \zeta^4 w_t + w(\zeta^4)_t, \quad D_k H = \zeta^4 D_k w + w D_k(\zeta^4), \\ D_{kl} H = \zeta^4 D_{kl} w + D_l(\zeta^4) D_k w + D_k(\zeta^4) D_l w + w D_{kl}(\zeta^4).\end{aligned}$$

Hence, at (x_0, t_0),

$$\zeta D_k w = -4w D_k \zeta \quad (k = 1, 2, \cdots, n)$$

and

$$0 \geq \mu + \zeta^4 \left(w_t - \sum_{kl} a_{kl} D_{kl} w + \sum_k b_k D_k w \right) + \sum_k b_k w D_k(\zeta^4) - 2\sum_{kl} a_{kl} D_k w D_l(\zeta^4) + R_1,$$

where $|R_1| \leq C\zeta^2 |w|$. Note also that $D_k w D_l(\zeta^4) = -4w\zeta^2 D_k \zeta D_l \zeta$ and $|b_k| \leq C|Dv|$. Therefore, by (3.21), at (x_0, t_0),

$$0 \geq \mu + \zeta^4 \left(\frac{\lambda^2}{2} |D^2 v|^2 - C|Dv|^2 - C \right) + R_2, \tag{3.22}$$

where $|R_2| \leq C(\zeta^2 |w| + \zeta^3 |w||Dv|)$.

5. Since $H(x_0, t_0) < 0$ and hence $w = \alpha + \kappa\beta < 0$, we have

$$|Dv|^2(x_0, t_0) \leq C|D^2 v|(x_0, t_0), \quad |w(x_0, t_0)| \leq C|D^2 v|(x_0, t_0),$$

and so by (3.22)

$$0 \geq \mu + \zeta^4 \left(\frac{\lambda^2}{4} |D^2 v|^2 - C \right) + R_2, \tag{3.23}$$

where $|R_2| \leq C(\zeta^2|w| + \zeta^3|w||Dv|)$. At (x_0, t_0), using Young's inequality with ϵ, we have

$$|R_2| \leq C\zeta^2|D^2v| + C\zeta^3|D^2v|^{3/2} \leq \epsilon\zeta^4|D^2v|^2 + C(\epsilon),$$

which is a contradiction to (3.23), provided $\mu > 0$ is sufficiently large. □

From this Harnack's inequality, we can derive the Harnack's inequality for elliptic equations. Suppose, in the operator L above, that $a_{ij}(x,t) = a_{ij}(x)$ is independent of t. Then any solution $u(x)$ to $Lu = 0$ is a time-independent solution of parabolic equation $u_t + Lu = 0$. Therefore, we have the following theorem.

Corollary 3.22 (Harnack's inequality for elliptic equations). *Let*

$$Lu = -\sum_{i,j=1}^{n} a_{ij}(x) D_{ij} u$$

be uniformly elliptic in Ω with smooth a_{ij}. Then, for any connected subdomain $V \subset\subset \Omega$, there exists a constant $C(L, V) > 0$ such that

$$\sup_{x \in V} u(x) \leq C(L, V) \inf_{x \in V} u(x)$$

whenever $u \geq 0$ is a smooth solution to $Lu = 0$ in Ω.

And we can also deduce the following stronger version of maximum principle.

Theorem 3.23 (Strong maximum principle). *Assume that $u \in C^2_1(\Omega_T) \cap C^0(\overline{\Omega}_T)$.*

(i) If $u_t + Lu \leq 0$ in Ω_T and u attains its maximum over Ω_T at a point $(x_0, t_0) \in \Omega_T$, then u must be a constant on Ω_{t_0}.

(ii) Similarly if $u_t + Lu \geq 0$ in Ω_T and u attains its minimum over Ω_T at a point $(x_0, t_0) \in \Omega_T$, then u must be a constant on Ω_{t_0}.

Proof. We prove (i) as (ii) can be shown by replacing u with $-u$.

1. Select any smooth, open set $W \subset\subset \Omega$, with $x_0 \in W$. Let v be the solution of $v_t + Lv = 0$ in W_T and $v|_{\partial' W_T} = u|_{\partial' W_T}$. (We assume the existence of such a solution.) Then by the weak maximum principle, we have

$$u(x,t) \leq v(x,t) \leq M$$

for all $(x,t) \in W_T$, where $M := \max_{\overline{\Omega_T}} u = u(x_0, t_0)$. From this we deduce that $v(x_0, t_0) = u(x_0, t_0) = M$ is maximum of v.

2. Now let $w = M - v(x,t)$. Then

$$w_t + Lw = 0, \quad w(x,t) \geq 0 \quad \text{in } W_{t_0}.$$

Choose any connected $V \subset\subset W$ with $x_0 \in V$. Let $0 < t < t_0$. Using Harnack's inequality we have a constant $C = C(L, V, t, t_0)$ such that

$$0 \leq w(x,t) \leq C \inf_{y \in V} w(y, t_0) = Cw(x_0, t_0) = 0.$$

This implies that $w \equiv 0$ in V_{t_0}. Since V is arbitrary, this implies $w \equiv 0$ in W_{t_0}. But therefore $v \equiv M = u(x_0, t_0)$ in W_{t_0}. Since $v = u$ on $\partial' W_T$, we have $u = M$ on $\partial' W_{t_0}$.

3. The above conclusion holds for arbitrary open sets $W \subset\subset \Omega$, and therefore $u \equiv M$ on Ω_{t_0}. □

Suggested exercises

Materials covered are from Chapters 2 and 7 of the textbook with some new materials added. So complete the arguments that are left in lectures. Also try working on the following problems related to the covered materials.

Chapter 2: Problems 12–17.

Chapter 7: Problems 1, 7, 8.

Homework # 5.

(1) (10 points) Write down an explicit formula (without proof) for a solution of the Cauchy problem

$$\begin{cases} u_t - \Delta u + cu = f & \text{in } \mathbf{R}^n \times (0, \infty) \\ u = g & \text{on } \mathbf{R}^n \times \{t = 0\}, \end{cases}$$

where $c \in \mathbf{R}$ is a constant.

(2) (10 points) Let $g \in C(\mathbf{R}^n)$ be bounded and satisfy $g \in L^1(\mathbf{R}^n)$. Show that the Cauchy problem

$$u_t - \Delta u = 0 \quad \text{in } \mathbf{R}^n \times (0, \infty); \quad u(x, 0) = g(x) \quad (x \in \mathbf{R}^n)$$

has a unique solution u satisfying

$$|u(x, t)| \le C(1 + t)^{-n/2} \quad (x \in \mathbf{R}^n,\ t \ge 0),$$

where C is a constant.

(3) (10 points) Let $K(x, y, t)$ be the heat kernel defined by

$$K(x, y, t) = (4\pi t)^{-n/2} e^{-|x-y|^2/4t} \quad (x,\ y \in \mathbf{R}^n,\ t > 0).$$

Prove

(*a*) $$K(x, y, t) \le (n/2e\pi)^{n/2} |x - y|^{-n} \quad (x,\ y \in \mathbf{R}^n,\ t > 0).$$

(*b*) $$\int_{\mathbf{R}^n} K(x, y, t)\, K(y, z, s)\, dy = K(x, z, t + s) \quad (x,\ z \in \mathbf{R}^n,\ t > 0,\ s > 0).$$

(4) (10 points) Given a continuous function $h\colon [0, \infty) \to \mathbf{R}$ with $h(0) = 0$, derive the formula

$$u(x, t) = \frac{x}{\sqrt{4\pi}} \int_0^t \frac{1}{(t - s)^{3/2}} e^{-\frac{x^2}{4(t-s)}} h(s)\, ds$$

for a solution of the initial/boundary-value problem of the heat equation $u_t - u_{xx} = 0$ in the quadrant $x > 0$, $t > 0$ satisfying the conditions:

$$u(x, 0) = 0 \quad (x > 0), \quad u(0, t) = h(t).$$

(5) (10 points) Let $\Omega_T = \{(x, t) \mid 0 < x < 1, 0 < t \le T\}$, and let $\partial'\Omega_T$ be the parabolic boundary of Ω_T. Let $u \in C_1^2(\Omega_T) \cap C(\overline{\Omega_T})$ be a solution of

$$u_t = a(x, t)u_{xx} + 2b(x, t)u_x + c(x, t)u \quad ((x, t) \in \Omega_T),$$

where a, b, c are given bounded functions in Ω_T and $a > 0$ in Ω_T. Show directly *without quoting maximum principles* that

$$|u(x, t)| \le e^{CT} \max_{\partial'\Omega_T} |u|,$$

where $C = \max\{0,\ \sup_{\Omega_T} c(x, t)\}$.

(6) (10 points) Let Ω be a bounded open set in $\mathbf{R}^n$ and $0 < T < \infty$. Assume $B\colon \mathbf{R}^n \to \mathbf{R}$ and $\sigma\colon \mathbf{R} \to \mathbf{R}$ are smooth functions. Show that, given any functions f, g, h, the following initial/boundary value problem

$$\begin{cases} u_t - \Delta u + B(\nabla u) + \sigma(u) = f(x,t), & (x,t) \in \Omega_T, \\ u(x,0) = g(x), & x \in \Omega, \\ u(x,t) = h(x,t), & x \in \partial\Omega,\ t \in [0,T] \end{cases}$$

can have at most one solution u in $C_1^2(\Omega_T) \cap C(\overline{\Omega_T})$.
(*Hint:* You may use, *without proof,* the maximum principles and uniqueness theorem for linear parabolic equations.)

Chapter 4

Wave Equation

In this part we investigate the wave equation

$$u_{tt} - \Delta u = 0 \tag{4.1}$$

and the nonhomogeneous wave equation

$$u_{tt} - \Delta u = f(x,t) \tag{4.2}$$

subject to appropriate initial and boundary conditions. Here $x \in \Omega \subset \mathbb{R}^n$, $t > 0$; the unknown function $u : \Omega \times [0,\infty) \to \mathbb{R}$.

We shall discover that solutions to the wave equation behave quite differently from solutions of Laplace's equation or the heat equation. For example, these solutions are generally not C^∞ and exhibit **finite speed of propagation**.

4.1. Derivation of the wave equation

The wave equation is a simplified model for a vibrating string ($n = 1$), membrane ($n = 2$), or elastic solid ($n = 3$). In this physical interpretation $u(x,t)$ represents the displacement in some direction of the point at time $t \geq 0$.

Let V represent any smooth subregion of Ω. The acceleration within V is then

$$\frac{d^2}{dt^2}\int_V u dx = \int_V u_{tt} dx,$$

and the net force is

$$\int_{\partial V} \mathbf{F}\cdot \nu dS,$$

where $\mathbf{F}$ denoting the force acting on V through ∂V, ν is the unit outnormal on ∂V. Newton's law says (assume the mass is 1) that

$$\int_V u_{tt} = \int_{\partial V} \mathbf{F}\cdot \nu dS.$$

This identity is true for any region, hence the divergence theorem tells that

$$u_{tt} = \operatorname{div}\mathbf{F}.$$

For elastic bodies, $\mathbf{F}$ is a function of Du, i.e., $\mathbf{F} = F(Du)$. For small u and small Du, we use the linearization $F(Du) \approx aDu$ to approximate $\mathbf{F}$, and so we obtain

$$u_{tt} - a\Delta u = 0.$$

When $a = 1$, the resulting equation is the **wave equation**. The physical interpretation strongly suggests it will be mathematically appropriate to specify two initial conditions: *initial displacement* $u(x, 0)$ and *initial velocity* $u_t(x, 0)$.

4.2. One-dimensional wave equations and d'Alembert's formula

We first focus on the following Cauchy problem for 1-D wave equation:

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{in } \mathbb{R} \times (0, \infty), \\ u(x,0) = g(x),\ u_t(x,0) = h(x) & \text{on } \mathbb{R} \times \{t = 0\}, \end{cases} \tag{4.3}$$

where $g,\ h$ are given. We derive the solution u in terms of $g,\ h$.

Note that

$$u_{tt} - u_{xx} = (u_t - u_x)_t + (u_t - u_x)_x = v_t + v_x,$$

where $v = u_t - u_x$. So, for a classical solution u, function v solves the initial value problem for linear transport equation:

$$v_t + v_x = 0, \quad v(x,0) = u_t(x,0) - u_x(x,0) = h(x) - g'(x) := a(x).$$

Solving this problem, we have $v(x,t) = a(x - t)$, that is,

$$u_t - u_x = v(x,t) = a(x - t), \quad u(x,0) = g(x).$$

Again this is an initial value problem for a nonhomogeneous linear transport equation of u. Solving this problem gives

$$\begin{aligned} u(x,t) &= g(x+t) + \int_0^t v(x - (s - t), s)\, ds \\ &= g(x+t) + \int_0^t a(x - s + t - s)\, ds \\ &= g(x+t) + \frac{1}{2}\int_{x-t}^{x+t} a(y)\, dy \\ &= g(x+t) + \frac{1}{2}\int_{x-t}^{x+t} [h(y) - g'(y)]\, dy, \end{aligned}$$

from which we deduce the **d'Alembert's formula**:

$$u(x,t) = \frac{g(x+t) + g(x-t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} h(y)\, dy. \tag{4.4}$$

Remark 4.1. (i) From d'Alembert's formula (4.4), any solution u to the wave equation $u_{tt} - u_{xx} = 0$ has the form

$$u(x,t) = F(x+t) + G(x-t)$$

for appropriate functions F and G. Note that $F(x+t)$ and $G(x-t)$ are called **traveling waves**, *with speed 1 and travelling to right and left.* Conversely, if F and G are C^2, the function u above is indeed a C^2 solution to the wave equation; this is the case when $g \in C^2$ and $h \in C^1$.

(ii) Even when initial data g and h are not smooth, we still consider the formula (4.4) defines a (weak) solution to the Cauchy problem. Note that, if initial data $g \in C^k$ and $h \in C^{k-1}$, then $u \in C^k$ but is not in general smoother. Thus the wave equation does not have the *smoothing effect* as does the heat equation. However, solutions to the one-dimensional wave equation *do not* lose the regularity from initial data, which, as we shall see later, is not the case for the higher dimensional wave equation.

(iii) The value $u(x_0, t_0)$ depends only on the initial data from $x_0 - t_0$ to $x_0 + t_0$. Therefore, $[x_0 - t_0, x_0 + t_0]$ is called the **domain of dependence** for the point (x_0, t_0). The Cauchy data at $(x_0, 0)$ influence the value of u in the region

$$I(x_0) = \{(x, t) \mid x_0 - t < x < x_0 + t,\ t > 0\},$$

which is called the **domain of influence** of x_0.

We easily have the following property.

Lemma 4.1 (Parallelogram Property). *Any solution u of the 1-D wave equation satisfies*

$$u(A) + u(C) = u(B) + u(D), \tag{4.5}$$

where $ABCD$ is any parallelogram in $\mathbb{R} \times [0, \infty)$ of slope 1 or -1, with A and C being two opposite points. (See Figure 4.1.)

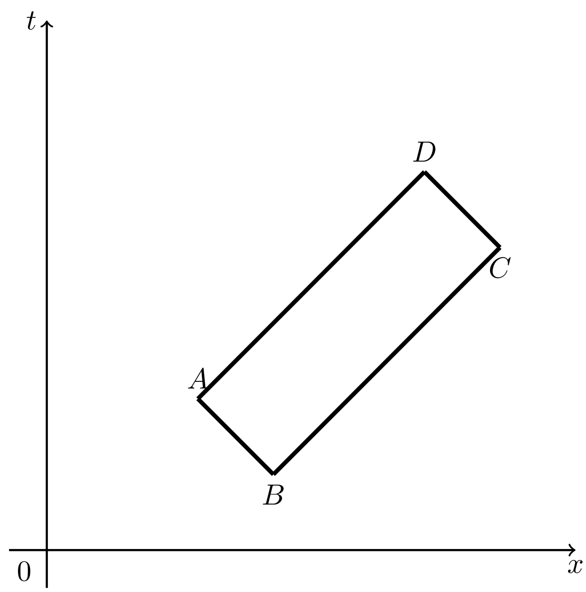

Figure 4.1. Parallelogram Property

Example 4.2. Solve the initial-boundary value problem

$$\begin{cases} u_{tt} - u_{xx} = 0 & x > 0,\ t > 0, \\ u(x,0) = g(x),\ u_t(x,0) = h(x) & x > 0, \\ u(0,t) = k(t) & t > 0, \end{cases}$$

where functions g, h, k satisfy certain smoothness and compatibility conditions.

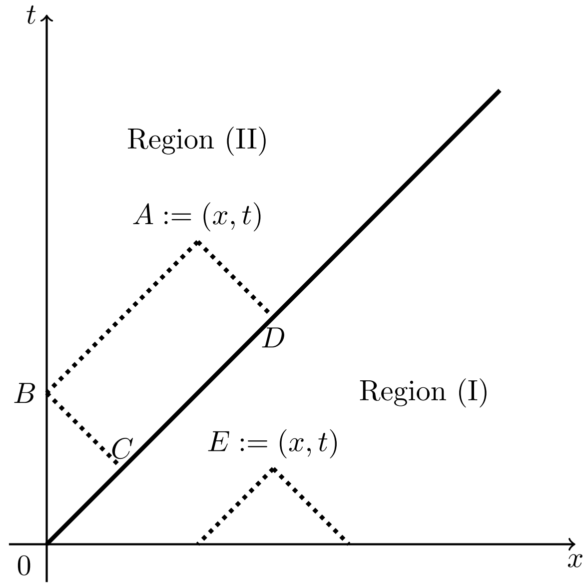

Figure 4.2. Mixed-value problem on $x > 0,\ t > 0$

Solution. (See Figure 4.2.) If $(x, t) := E$ is in the region (I), that is $x \geq t$, then $u(x, t)$ is given by (4.4):

$$u(x,t) = \frac{g(x+t) + g(x-t)}{2} + \frac{1}{2}\int_{x-t}^{t+x} h(y)dy.$$

In particular, on $x = t$,

$$u(x,x) = \frac{g(2x) + g(0)}{2} + \frac{1}{2}\int_{0}^{2x} h(y)dy.$$

If $(x, t) := A$ is in the region (II), that is, $0 \leq x < t$, then we can use the parallelogram property to obtain

$$\begin{aligned} u(x,t) &= u(B) + u(D) - u(C) \\ &= u(0, t-x) + u(\frac{x+t}{2}, \frac{x+t}{2}) - u(\frac{t-x}{2}, \frac{t-x}{2}) \\ &= k(t-x) + \frac{g(x+t) + g(0)}{2} + \frac{1}{2}\int_0^{x+t} h(y)dy - \left(\frac{g(t-x) + g(0)}{2} + \frac{1}{2}\int_0^{t-x} h(y)dy\right) \\ &= k(t-x) + \frac{g(x+t) - g(t-x)}{2} + \frac{1}{2}\int_{t-x}^{t+x} h(y)dy. \end{aligned}$$

Therefore, the solution u to the problem is given by

$$u(x,t) = \begin{cases} \dfrac{g(x+t) + g(x-t)}{2} + \frac{1}{2}\int_{x-t}^{t+x} h(y)dy & (x \geq t \geq 0), \\ k(t-x) + \dfrac{g(x+t) - g(t-x)}{2} + \frac{1}{2}\int_{t-x}^{t+x} h(y)dy & (0 \leq x < t). \end{cases} \tag{4.6}$$

Of course, we need certain conditions in order that this formula does define a smooth solution in the domain $x > 0,\ t > 0$. (**Exercise:** derive such conditions on g, h, k.) □

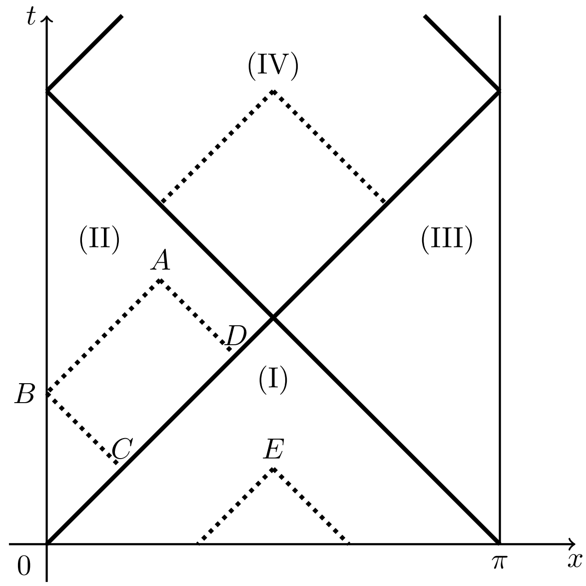

Figure 4.3. Mixed-value problem on $0 < x < \pi,\ t > 0$

EXAMPLE 4.3. Solve the initial-boundary value problem

$$\begin{cases} u_{tt} - u_{xx} = 0, & (x,t) \in (0,\pi) \times \mathbb{R}^+, \\ u(x,0) = g(x),\ u_t(x,0) = h(x), & x \in (0,\pi), \\ u(0,t) = u(\pi,t) = 0, & t > 0. \end{cases}$$

Solution. (See Figure 4.3.) We can solve u in the region (I) by formula (4.4). In all other regions we use the formula (4.5).

Another way to solve this problem is using the method of **separation of variables**. First try find the solutions of wave equation in $(0,\pi) \times (0,\infty)$ satisfying the boundary condition that are in the form of $u(x,t) = X(x)T(t)$. Then we should have

$$X''(x)T(t) = T''(t)X(x), \quad X(0) = X(\pi) = 0,$$

which implies that

$$\frac{T''(t)}{T(t)} = \frac{X''(x)}{X(x)} = \lambda,$$

where λ is a constant. From

$$X''(x) = \lambda X(x), \quad X(0) = X(\pi) = 0,$$

we find that $\lambda = -j^2$ for all $j = 1, 2, \ldots$, and

$$X_j(x) = \sin(jx), \quad T_j(t) = a_j \cos(jt) + b_j \sin(jt).$$

To make sure that u satisfies the initial condition, we consider

$$u(x,t) = \sum_{j=1}^{\infty} [a_j \cos(jt) + b_j \sin(jt)] \sin(jx). \tag{4.7}$$

Set $t = 0$ for u and u_t, and we have

$$u(x,0) = g(x) = \sum_{j=1}^{\infty} a_j \sin(jx), \quad u_t(x,0) = h(x) = \sum_{j=1}^{\infty} jb_j \sin(jx).$$

So a_j and jb_j are the Fourier coefficients of functions $g(x)$ and $h(x)$ on $[0,\pi]$; that is,

$$a_j = \frac{2}{\pi}\int_0^{\pi} g(x)\sin(jx)dx, \quad b_j = \frac{2}{j\pi}\int_0^{\pi} h(x)\sin(jx)dx.$$

Substitute these coefficients into (4.7) and we obtain the solution u in terms of trigonometric series. The issue of convergence will not be discussed here. □

4.3. Higher dimensional wave equation

In this section we solve the Cauchy problem for n-dimensional wave equation ($n \geq 2$)

$$\text{(4.8)} \qquad \begin{cases} u_{tt} - \Delta u = 0, & (x,t) \in \mathbb{R}^n \times (0,\infty), \\ u(x,0) = g(x),\ u_t(x,0) = h(x), & x \in \mathbb{R}^n. \end{cases}$$

The idea is to reduce the problem to a certain one spatial-dimensional problem that we are able to solve. This reduction requires the **spherical mean method**.

4.3.1. Spherical means and the Euler-Poisson-Darboux equation. Let $x \in \mathbb{R}^n,\ r > 0,\ t > 0$. Define the spherical means:

$$U(x;r,t) = ⨍_{\partial B(x,r)} u(y,t)\, dS_y = M_{u(\cdot,t)}(x;r),$$

$$G(x;r) = ⨍_{\partial B(x,r)} g(y)\, dS_y = M_g(x;r),$$

$$H(x;r) = ⨍_{\partial B(x,r)} h(y)\, dS_y = M_h(x;r).$$

Lemma 4.4 (Euler-Poisson-Darboux equation)**.** *Let $u \in C^m(\mathbb{R}^n \times [0,\infty))$ with $m \geq 2$ solve (4.8). Then, for any fixed $x \in \mathbb{R}^n$, $U(x;r,t) \in C^m(\bar{\mathbb{R}}_r^+ \times \bar{\mathbb{R}}_t^+)$ and solves the Cauchy problem of the* **Euler-Poisson-Darboux equation**

$$\text{(4.9)} \qquad \begin{cases} U_{tt} - U_{rr} - \frac{n-1}{r}U_r = 0 & \text{in } (0,\infty)\times(0,\infty), \\ U = G,\ U_t = H & \text{on } (0,\infty)\times\{t=0\}. \end{cases}$$

Proof. 1. Note that the regularity of U with respect to t follows easily. To show the regularity of U with respect to $(r,t) \in [0,\infty)\times[0,\infty)$, we compute, for $r > 0$,

$$U_r(x;r,t) = \frac{r}{n} ⨍_{B(x,r)} \Delta u(y,t)dy.$$

From this we deduce that $\lim_{r\to 0^+} U_r(x;r,t) = 0$ and, after some computations, that

$$U_{rr}(x;r,t) = ⨍_{\partial B(x,r)} \Delta u\, dS + \left(\frac{1}{n} - 1\right) ⨍_{B(x,r)} \Delta u\, dy.$$

Thus $\lim_{r\to 0^+} U_{rr}(x;r,t) = \frac{1}{n}\Delta u(x,t)$. We can further compute U_{rrr} and other higher-order derivatives of U up to the m-th order, and this proves that $U \in C^m(\bar{\mathbb{R}}_r^+ \times \bar{\mathbb{R}}_t^+)$.

2. Using the formula above,

$$U_r = \frac{r}{n} \fint_{B(x,r)} \Delta u(y,t)dy = \frac{1}{n\alpha_n r^{n-1}} \int_{B(x,r)} \Delta u(y,t)dy = \frac{1}{n\alpha_n r^{n-1}} \int_{B(x,r)} u_{tt}(y,t)dy. \tag{4.10}$$

Thus

$$r^{n-1}U_r = \frac{1}{n\alpha_n} \int_{B(x,r)} u_{tt}(y,t)dy = \frac{1}{n\alpha_n} \int_0^r \left(\int_{\partial B(x,\rho)} u_{tt}(y,t)\, dS_y \right) d\rho,$$

and so

$$(r^{n-1}U_r)_r = \frac{1}{n\alpha_n} \int_{\partial B(x,r)} u_{tt}(y,t)\, dS_y = r^{n-1}U_{tt}, \tag{4.11}$$

which expands into the Euler-Poisson-Darboux equation. The initial condition is satisfied easily; this completes the proof of the theorem. □

4.3.2. Solution in $\mathbb{R}^3$ and Kirchhoff's formula. For the most important case $n = 3$, one can see easily from the Euler-Poisson-Darboux equation that

$$(rU)_{rr} = rU_{rr} + 2U_r = \frac{1}{r}(r^2U_r)_r = rU_{tt} = (rU)_{tt}. \tag{4.12}$$

That is, function $\tilde{U}(r,t) = rU(x;r,t)$ satisfies the 1-D wave equation in $r > 0,\ t > 0$. Note that

$$\tilde{U}(0,t) = 0,\ \tilde{U}(r,0) = rG := \tilde{G},\ U_t(r,0) = rH := \tilde{H}.$$

Hence, by formula (4.6) with $k = 0$, we have for $0 \le r \le t$

$$\tilde{U}(r,t) = \frac{1}{2}[\tilde{G}(r+t) - \tilde{G}(t-r)] + \frac{1}{2}\int_{-r+t}^{r+t} \tilde{H}(y)\, dy.$$

We recover $u(x,t)$ as follows:

$$u(x,t) = \lim_{r\to 0^+} U(x;r,t) = \lim_{r\to 0^+} \frac{\tilde{U}(r,t)}{r}$$

$$= \lim_{r\to 0^+} \left[\frac{\tilde{G}(r+t) - \tilde{G}(t-r)}{2r} + \frac{1}{2r}\int_{-r+t}^{r+t} \tilde{H}(y)\, dy \right] = \tilde{G}'(t) + \tilde{H}(t).$$

This gives the **Kirchhoff's formula for 3-D wave equation**:

$$u(x,t) = \frac{\partial}{\partial t}\left(t \fint_{\partial B(x,t)} g(y)dS_y \right) + t \fint_{\partial B(x,t)} h(y)dS_y = \fint_{\partial B(x,t)} (th(y) + g(y) + Dg(y)\cdot(y-x))\, dS_y \quad (x \in \mathbb{R}^3,\ t > 0), \tag{4.13}$$

where we have used

$$\fint_{\partial B(x,t)} g(y)dS_y = \fint_{\partial B(0,1)} g(x+tz)dS_z$$

and so

$$\frac{\partial}{\partial t}\left(\fint_{\partial B(x,t)} g(y)dS_y \right) = \frac{\partial}{\partial t}\left(\fint_{\partial B(0,1)} g(x+tz)dS_z \right)$$

$$= \fint_{\partial B(0,1)} Dg(x+tz)\cdot z dS_z = \frac{1}{t}\fint_{\partial B(x,t)} Dg(y)\cdot (y-x) dS_y.$$

We will give some remarks about Kirchhoff's formula later.

4.3.3. Solution in $\mathbb{R}^2$ by Hadamard's method of descent. Assume $u \in C^2(\mathbb{R}^2\times[0,\infty)$ solves (4.8) with $n=2$. We would like to deduce a formula for u in terms of $g,\ h$.

The trick is to consider u as a solution of a 3-D wave equation and then apply Kirchhoff's formula to find u. This is called the **Hadamard's method of descent**. We now discuss it in more details. Define

$$\bar{u}(x_1,x_2,x_3,t) = u(x_1,x_2,t).$$

Then $\bar{u}(x,t)$ solves

$$\begin{cases} \bar{u}_{tt} - \Delta \bar{u} = 0, & (x,t)\in\mathbb{R}^3\times(0,\infty),\\ \bar{u}(x,0) = \bar{g}(x),\ \bar{u}_t(x,0) = \bar{h}(x), & x\in\mathbb{R}^3,\end{cases}$$

where $\bar{g}(x_1,x_2,x_3) = g(x_1,x_2)$ and $\bar{h}(x_1,x_2,x_3) = h(x_1,x_2)$.

If we write $x=(x_1,x_2)\in\mathbb{R}^2$ and $\bar{x} = (x_1,x_2,0) = (x,0)\in\mathbb{R}^3$, then

$$u(x,t) = \bar{u}(\bar{x},t) = \frac{\partial}{\partial t}\left(t\fint_{\partial\tilde{B}(\bar{x},t)} \bar{g}(y) dS_y\right) + t\fint_{\partial\tilde{B}(\bar{x},t)} \bar{h}(y)dS_y,$$

where $\tilde{B}(\bar{x},t)$ is the ball in $\mathbb{R}^3$ centered at $\bar{x}$ of radius t.

We parameterize $y\in\partial\tilde{B}(\bar{x},t)$ with parameter $z\in B(x,t)$ in $\mathbb{R}^2$ by

$$y=(z,y_3),\ y_3 = \pm\gamma(z),$$

where $\gamma(z) = \sqrt{t^2-|z-x|^2}$, and hence

$$D\gamma(z) = -\frac{z-x}{\sqrt{t^2-|z-x|^2}},\quad \sqrt{1+|D\gamma(z)|^2} = \frac{t}{\sqrt{t^2-|z-x|^2}}.$$

We can then compute that

$$\begin{aligned}\fint_{\partial\tilde{B}(\bar{x},t)} \bar{g}(y)dS_y &= \frac{1}{4\pi t^2}\int_{\partial\tilde{B}(\bar{x},t)} \bar{g}(y)dS_y\\ &= \frac{2}{4\pi t^2}\int_{B(x,t)} g(z)\sqrt{1+|D\gamma(z)|^2}\,dz\\ &= \frac{1}{2\pi t}\int_{B(x,t)} \frac{g(z)}{\sqrt{t^2-|z-x|^2}}dz\\ &= \frac{t}{2}\fint_{B(x,t)} \frac{g(z)}{\sqrt{t^2-|z-x|^2}}dz.\end{aligned}$$

Therefore,

$$\begin{aligned} u(x,t) =& \frac{\partial}{\partial t}\left(\frac{t^2}{2}\fint_{B(x,t)}\frac{g(z)}{\sqrt{t^2-|z-x|^2}}dz\right)\\ &+ \frac{t^2}{2}\fint_{B(x,t)}\frac{h(z)}{\sqrt{t^2-|z-x|^2}}dz \quad (x\in\mathbb{R}^2,\ t>0).\end{aligned} \tag{4.14}$$

Since

$$t^2\fint_{B(x,t)}\frac{g(z)}{\sqrt{t^2-|z-x|^2}}dz = t\fint_{B(0,1)}\frac{g(x+tw)}{\sqrt{1-|w|^2}}dw,$$

we obtain

$$\frac{\partial}{\partial t}\left(t^2 \fint_{B(x,t)} \frac{g(z)}{\sqrt{t^2-|z-x|^2}}dz\right) = \fint_{B(0,1)} \frac{g(x+tw)}{\sqrt{1-|w|^2}}dw + t\fint_{B(0,1)} \frac{Dg(x+tw)\cdot w}{\sqrt{1-|w|^2}}dw$$
$$= t\fint_{B(x,t)} \frac{g(z)}{\sqrt{t^2-|z-x|^2}}dz + t\fint_{B(x,t)} \frac{Dg(z)\cdot(z-x)}{\sqrt{t^2-|z-x|^2}}dz.$$

So we can rewrite (4.14) to deduce the so-called **Poisson's formula for 2-D wave equation**:

$$u(x,t) = \frac{1}{2}\fint_{B(x,t)} \frac{tg(z)+t^2h(z)+tDg(z)\cdot(z-x)}{\sqrt{t^2-|z-x|^2}}dz \quad (x\in\mathbb{R}^2,\ t>0). \tag{4.15}$$

Remark 4.2. (a) There is a fundamental difference for wave equations between $n=1$ and $n=2,3$. In both Kirchhoff's formula and Poisson's formula, the solution u depends on the derivative Dg of the initial data $u(x,0)=g(x)$. Therefore, if $g\in C^m,\ h\in C^{m-1}$ then $u\in C^{m-1}$, $u_t\in C^{m-2}$ and thus there is a **loss of regularity** for wave equation when $n=2,3$ (in fact, for all $n\ge 2$). This is different from the 1-D case, where u, u_t are at least as smooth as g, h.

(b) There are also fundamental differences between $n=3$ and $n=2$ for wave equations. In $\mathbb{R}^2$, we need the information of initial data f and g in the whole disk $B(x,t)$ to compute $u(x,t)$. While in $\mathbb{R}^3$ we only need the information of initial data f and g on the sphere $\partial B(x,t)$ to compute $u(x,t)$ and, in this case, a "disturbance" originating at x_0 propagates along a sharp wavefront $|x-x_0|=t$. This is called the **Strong Huygens' principle** in $\mathbb{R}^3$. But for $n=2$, a disturbance at x_0 will affect the values of the solution in the region $|x-x_0|\le t$. That is, the domain of influence of the point $(x_0,0)$ is the surface $|x-x_0|=t$ for $n=3$, and is the solid disc $|x-x_0|\le t$ for $n=2$. (Imagine you are at position x in $\mathbb{R}^n$ and there is a sharp initial disturbance at position x_0 away from you. If $n=3$, then you only feel the disturbance (hear the sound) once exactly at time $t=|x-x_0|$; however, if $n=2$, you will feel the disturbance (feel the wave) all the time $t\ge|x-x_0|$, although the effect on you will die out as $t\to\infty$.)

4.3.4. Solution for general odd dimensions n. For general odd dimensions n, we use the following result whose proof is left as an exercise.

Lemma 4.5 (Some useful identities)**.** *Let $f:\mathbb{R}\to\mathbb{R}$ be C^{k+1}. Then, for $k=1,2,\ldots,$*

$$(i)\qquad \left(\frac{d^2}{dr^2}\right)\left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}(r^{2k-1}f(r)) = \left(\frac{1}{r}\frac{d}{dr}\right)^{k}(r^{2k}f'(r)),$$

$$(ii)\qquad \left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}(r^{2k-1}f(r)) = \sum_{j=0}^{k-1}\beta_j^k r^{j+1}\frac{d^jf}{dr^j}(r),$$

where $\beta_0^k=(2k-1)!!=(2k-1)(2k-3)\cdots 3\cdot 1$ and β_j^k $(j=1,2,\cdots,k-1)$ are independent of f.

Now assume that $n=2k+1$ $(k\ge 1)$. Let $u\in C^{k+1}(\mathbb{R}^n\times[0,\infty))$ be a solution to wave equation $u_{tt}-\Delta u=0$ in $\mathbb{R}^n\times(0,\infty)$. Then the function $U(x;r,t)$ defined above is C^{k+1} in $(r,t)\in[0,\infty)\times[0,\infty)$. Set

$$V(r,t) = \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1}(r^{2k-1}U(x;r,t)) \quad (r>0,\ t\ge 0).$$

Lemma 4.6. *$V_{tt} = V_{rr}$ for $r > 0$, $t > 0$, and $V(0^+, t) = 0$.*

Proof. Using part (i) of Lemma 4.5, we have

$$\begin{aligned} V_{rr} &= (\frac{\partial^2}{\partial r^2})(\frac{1}{r}\frac{\partial}{\partial r})^{k-1}(r^{2k-1}U) = (\frac{1}{r}\frac{\partial}{\partial r})^{k}(r^{2k}U_r) \\ &= (\frac{1}{r}\frac{\partial}{\partial r})^{k-1}(\frac{1}{r}(r^{2k}U_r)_r) = (\frac{1}{r}\frac{\partial}{\partial r})^{k-1}(r^{2k-1}U_{tt}) = V_{tt}, \end{aligned}$$

where we have used the Euler-Poisson-Darboux equation (4.11): $(r^{2k}U_r)_r = r^{2k}U_{tt}$. That $V(0^+, t) = 0$ follows from part (ii) of Lemma 4.5. □

Then, by (4.6), we have for $0 \le r \le t$

$$V(r,t) = \frac{1}{2}[\tilde{G}(r+t) - \tilde{G}(t-r)] + \frac{1}{2}\int_{t-r}^{r+t} \tilde{H}(y)dy, \tag{4.16}$$

where

$$\begin{aligned} \tilde{G}(r) &= (\frac{1}{r}\frac{\partial}{\partial r})^{k-1}(r^{2k-1}U(x;r,0)), \\ \tilde{H}(r) &= (\frac{1}{r}\frac{\partial}{\partial r})^{k-1}(r^{2k-1}U_t(x;r,0)). \end{aligned} \tag{4.17}$$

But recall $u(x,t) = U(x;0^+,t)$ and, by (ii) of Lemma 4.5,

$$V(r,t) = (\frac{1}{r}\frac{\partial}{\partial r})^{k-1}(r^{2k-1}U(x;r,t)) = \sum_{j=0}^{k-1} \beta_j^k r^{j+1} \frac{\partial^j}{\partial r^j} U(x;r,t),$$

and so

$$\begin{aligned} u(x,t) &= U(x;0^+,t) = \lim_{r\to 0} \frac{V(r,t)}{\beta_0^k r} \\ &= \frac{1}{\beta_0^k} \lim_{r\to 0} \left[\frac{\tilde{G}(r+t) - \tilde{G}(t-r)}{2r} + \frac{1}{2r}\int_{t-r}^{r+t} \tilde{H}(y)dy \right] \\ &= \frac{1}{\beta_0^k}[\tilde{G}'(t) + \tilde{H}(t)]. \end{aligned}$$

Therefore, we derived the following formula for the solution of the Cauchy problem

$$\begin{cases} u_{tt} - \Delta u = 0, & (x,t) \in \mathbb{R}^n \times (0,\infty), \\ u(x,0) = f(x),\ u_t(x,0) = g(x), & x \in \mathbb{R}^n \end{cases} \tag{4.18}$$

when $n = 2k+1$. That is, for odd $n \ge 3$,

$$\begin{aligned} u(x,t) = &\frac{1}{(n-2)!!}\Big[\Big(\frac{\partial}{\partial t}\Big)\Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^{\frac{n-3}{2}} \Big(t^{n-2} ⨍_{\partial B(x,t)} g dS\Big) \\ &+ \Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^{\frac{n-3}{2}} \Big(t^{n-2} ⨍_{\partial B(x,t)} h dS\Big)\Big] \quad (x \in \mathbb{R}^n,\ t > 0). \end{aligned} \tag{4.19}$$

When $n = 3$ this formula agrees with the Kirchhoff's formula we derived earlier for wave equation in $\mathbb{R}^3$.

Theorem 4.7 (Solution of wave equation in odd dimensions)**.** *If $n \ge 3$ is odd, $g \in C^{m+1}(\mathbb{R}^n)$ and $h \in C^m(\mathbb{R}^n)$ for $m \ge \frac{n+1}{2}$, then u defined by (4.19) belongs to $C^2(\mathbb{R}^n \times$*

$(0,\infty))$, solves wave equation $u_{tt}-\Delta u=0$ in $\mathbb{R}^n\times(0,\infty)$ and satisfies the Cauchy condition in the sense that, for each $x^0\in\mathbb{R}^n$,

$$\lim_{(x,t)\to(x^0,0),\ t>0} u(x,t)=g(x^0),\qquad \lim_{(x,t)\to(x^0,0),\ t>0} u_t(x,t)=h(x^0).$$

Proof. We may separate the proof in two cases: (a) $g=0$; (b) $h=0$. The proof in case (a) is given in the text. So we give a similar proof for case (b).

1. Suppose $h\equiv 0$. Let $\gamma_n=(n-2)!!$ and $k=\frac{n-1}{2}$. Then $1\le k\le m-1$ and the function u defined by (4.19) becomes

$$u(x,t)=\frac{1}{\gamma_n}\Big(\frac{\partial}{\partial t}\Big)\Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^{k-1}\Big(t^{2k-1}G(x,t)\Big),\quad G(x,t)=⨍_{\partial B(x,t)} g\,dS.$$

By Lemma 4.5(ii),

$$u(x,t)=\frac{1}{\gamma_n}\sum_{j=0}^{k-1}\beta_j^k\left[(j+1)t^j\frac{\partial^j G}{\partial t^j}+t^{j+1}\frac{\partial^{j+1}G}{\partial t^{j+1}}\right]\to\frac{\beta_0^k}{\gamma_n}G(x^0,0)=g(x^0)$$

as $(x,t)\to(x^0,0),\ t>0$. Also, by Lemma 4.5(i),

$$u_t(x,t)=\frac{1}{\gamma_n}\Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^k\Big(t^{2k}G_t\Big),\quad u_{tt}(x,t)=\frac{1}{\gamma_n}\Big(\frac{\partial}{\partial t}\Big)\Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^k\Big(t^{2k}G_t\Big). \tag{4.20}$$

The first identity implies $u_t(x,t)\to 0$ as $(x,t)\to(x^0,0),\ t>0$.

2. In the second identity of (4.20), note that

$$G_t=\frac{t}{n}⨍_{B(x,t)}\Delta g\,dy=\frac{1}{n\alpha_n t^{n-1}}\int_{B(x,t)}\Delta g\,dy$$

and so

$$(t^{2k}G_t)_t=\frac{\partial}{\partial t}\left(\frac{1}{n\alpha_n}\int_{B(x,t)}\Delta g\,dy\right)=\frac{1}{n\alpha_n}\int_{\partial B(x,t)}\Delta g\,dS=t^{n-1}⨍_{\partial B(x,t)}\Delta g\,dS.$$

Hence

$$u_{tt}=\frac{1}{\gamma_n}\Big(\frac{\partial}{\partial t}\Big)\Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^{k-1}\Big(\frac{1}{t}(t^{2k}G_t)_t\Big)=\frac{1}{\gamma_n}\Big(\frac{\partial}{\partial t}\Big)\Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^{k-1}\Big(t^{2k-1}⨍_{\partial B(x,t)}\Delta g\,dS\Big).$$

On the other hand,

$$\Delta u(x,t)=\frac{1}{\gamma_n}\Big(\frac{\partial}{\partial t}\Big)\Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^{k-1}\Big(t^{2k-1}\Delta G(x,t)\Big)$$

and

$$\Delta G(x,t)=\Delta_x⨍_{\partial B(0,t)}g(x+z)\,dS_z=⨍_{\partial B(0,t)}\Delta g(x+z)\,dS_z=⨍_{\partial B(x,t)}\Delta g\,dS.$$

This proves $u_{tt}-\Delta u=0$ in $\mathbb{R}^n\times(0,\infty)$. □

Remark 4.3. (a) To compute $u(x,t)$ for odd $n\ge 3$, we only need the information of g, $Dg,\dots,D^{\frac{n-1}{2}}g$ and of $h,\ Dh,\dots,D^{\frac{n-3}{2}}h$ on the boundary $\partial B(x,t)$.

(b) If $n=1$, in order for u to be of C^2 we only need $g\in C^2$ and $h\in C^1$. However, if $n\ge 3$ is odd, then in order that solution u defined by (4.19) is of $C^2(\mathbb{R}^n\times(0,\infty))$, we need $g\in C^{m+1}(\mathbb{R}^n)$ and $h\in C^m(\mathbb{R}^n)$ for some $m=\frac{n+1}{2}$. For example, if $n=3$ then we need $g\in C^3$ and $h\in C^2$. Therefore there is a possible loss of smoothness for higher-dimensional wave equations.

4.3.5. Solution for even n by Hadamard's method of descent. Assume that n is even in this section. Goal is to derive a representation formula similar to (4.19). The trick is to use **Hadamard's method of descent**, used before for $n = 2$.

Note that if $u(x,t)$ is a solution of the wave equation in $\mathbb{R}^n \times (0,\infty)$, it is apparently true that $v(x, x_{n+1}, t) = u(x,t)$ is a solution to the wave equation in $\mathbb{R}^{n+1} \times (0,\infty)$ with initial data

$$v(x, x_{n+1}, 0) = \bar{g}(x, x_{n+1}) \equiv g(x), \quad v_t(x, x_{n+1}, 0) = \bar{h}(x, x_{n+1}) \equiv h(x).$$

Since $n+1$ is odd, we may use (4.19) to obtain the formula for $v(x, x_{n+1}, t)$. Like before, set $\bar{x} = (x, 0)$ and $\tilde{B}(\bar{x}, r)$ to be ball centered $\bar{x}$ of radius r in $\mathbb{R}^{n+1}$. Then

$$u(x,t) = v(\bar{x},t) = \frac{1}{(n-1)!!}\Big[\Big(\frac{\partial}{\partial t}\Big)\Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^{\frac{n-2}{2}}\Big(t^{n-1}⨍_{\partial\tilde{B}(\bar{x},t)} \bar{g}\,dS\Big) + \Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^{\frac{n-2}{2}}\Big(t^{n-1}⨍_{\partial\tilde{B}(\bar{x},t)} \bar{h}\,dS\Big)\Big].$$

As above, we parameterize $y \in \partial\tilde{B}(\bar{x},t)$ with parameter $z \in B(x,t)$ in $\mathbb{R}^n$ by

$$y = (z, y_{n+1}), \quad y_{n+1} = \pm\gamma(z), \quad \gamma(z) = \sqrt{t^2 - |z-x|^2},$$

to compute that

$$\begin{aligned}
⨍_{\partial\tilde{B}(\bar{x},t)} \bar{g}(y)\,dS_y &= \frac{1}{(n+1)\alpha_{n+1}t^n}\int_{\partial\tilde{B}(\bar{x},t)} \bar{g}(y)\,dS_y \\
&= \frac{2}{(n+1)\alpha_{n+1}t^n}\int_{B(x,t)} g(z)\sqrt{1+|D\gamma(z)|^2}\,dz \\
&= \frac{2}{(n+1)\alpha_{n+1}t^{n-1}}\int_{B(x,t)} \frac{g(z)}{\sqrt{t^2-|z-x|^2}}\,dz \\
&= \frac{2t\alpha_n}{(n+1)\alpha_{n+1}}⨍_{B(x,t)} \frac{g(z)}{\sqrt{t^2-|z-x|^2}}\,dz.
\end{aligned}$$

Note that

$$\frac{2\alpha_n}{(n+1)\alpha_{n+1}}\frac{1}{(n-1)!!} = \frac{1}{n!!} \quad \Big(\text{using } \alpha_n = \frac{\pi^{n/2}}{\Gamma(\frac{n+2}{2})}\Big).$$

We deduce that, for even n,

$$u(x,t) = \frac{1}{n!!}\Big[\Big(\frac{\partial}{\partial t}\Big)\Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^{\frac{n-2}{2}}\Big(t^n⨍_{B(x,t)} \frac{g(y)}{\sqrt{t^2-|y-x|^2}}\,dy\Big) + \Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^{\frac{n-2}{2}}\Big(t^n⨍_{B(x,t)} \frac{h(y)}{\sqrt{t^2-|y-x|^2}}\,dy\Big)\Big]. \tag{4.21}$$

When $n = 2$ this formula agrees with the Poisson's formula we derived earlier for wave equation in $\mathbb{R}^2$.

Theorem 4.8 (Solution of wave equation in even dimensions)**.** *If $n \geq 2$ is even, $g \in C^{m+1}(\mathbb{R}^n)$ and $h \in C^m(\mathbb{R}^n)$ for $m \geq \frac{n+2}{2}$, then u defined by (4.21) belongs to $C^2(\mathbb{R}^n \times (0,\infty))$, solves wave equation $u_{tt} - \Delta u = 0$ in $\mathbb{R}^n \times (0,\infty)$ and satisfies the Cauchy condition in the sense that, for each $x^0 \in \mathbb{R}^n$,*

$$\lim_{(x,t)\to(x^0,0),\ t>0} u(x,t) = g(x^0), \quad \lim_{(x,t)\to(x^0,0),\ t>0} u_t(x,t) = h(x^0).$$

This follows from the theorem in odd dimensions.

Remark 4.4. (a) There is a fundamental difference between odd n and even n. For even n we need information of f and g (and their derivatives) in the whole **solid ball** $B(x,t)$ to compute $u(x,t)$.

(b) For odd n, a "disturbance" originating at x_0 propagates along a sharp wavefront $|x-x_0|=t$. This is called the **Strong Huygens' principle**. While, for even dimensions, a disturbance at x_0 will affect the values of the solution in the region $|x-x_0|\le t$. That is, the domain of influence of the point $(x_0,0)$ is the surface $|x-x_0|=t$ for odd n, and is the solid ball $|x-x_0|\le t$ for even n.

4.3.6. Nonhomogeneous problems and Duhamel's principle. We now turn to the initial value problem for the non-homogeneous equation

$$\begin{cases} u_{tt}-\Delta u = f(x,t) & \text{in } \mathbb{R}^n\times(0,\infty),\\ u(x,0)=0,\ u_t(x,0)=0 & \text{for } x\in\mathbb{R}^n. \end{cases} \tag{4.22}$$

Motivated by the **Duhamel's principle** used above for the heat equation, we let $U(x,t;s)$ be the solution to the initial value problem

$$\begin{cases} U_{tt}(x,t;s)-\Delta U(x,t;s)=0 & \text{in } \mathbb{R}^n\times(s,\infty),\\ U(x,s;s)=0,\quad U_t(x,s;s)=f(x,s) & \text{for } x\in\mathbb{R}^n. \end{cases} \tag{4.23}$$

Theorem 4.9. *Assume $n\ge 2$ and $f\in C^{[n/2]+1}(\mathbb{R}^n\times[0,\infty))$. Then*

$$u(x,t)=\int_0^t U(x,t;s)ds \tag{4.24}$$

is in $C^2(\mathbb{R}^n\times[0,\infty))$ and is a solution to (4.22)*.*

Proof. 1. The regularity of f shows that the solution U to (4.23) is given by formula (4.19) or (4.21) above. In either case, we have $u\in C^2(\mathbb{R}^n\times[0,\infty))$.

2. A direct computation shows that

$$u_t(x,t)=U(x,t;t)+\int_0^t U_t(x,t;s)ds=\int_0^t U_t(x,t;s)ds,$$

$$u_{tt}(x,t)=U_t(x,t;t)+\int_0^t U_{tt}(x,t;s)ds=f(x,t)+\int_0^t U_{tt}(x,t;s)ds,$$

$$\Delta u(x,t)=\int_0^t \Delta U(x,t;s)ds.$$

Hence $u_{tt}-\Delta u=f(x,t)$. Clearly $u(x,0)=u_t(x,0)=0$. □

Note that if $v(x,t)$ is the solution to the initial value problem

$$\begin{cases} v_{tt}(x,t)-\Delta v(x,t)=0 & \text{in } \mathbb{R}^n\times(0,\infty),\\ v(x,0)=0,\quad v_t(x,0)=f(x,s) & \text{for } x\in\mathbb{R}^n, \end{cases}$$

then

$$U(x,t;s)=v(x,t-s)\quad x\in\mathbb{R}^n,\ t>s,$$

is the solution to (4.23). Therefore,

$$u(x,t)=\int_0^t v(x,t-s)\,ds\quad x\in\mathbb{R}^n,\ t>0,$$

is the solution to the non-homogeneous problem (4.22).

Example 4.10. Find a solution of the following problem

$$\begin{cases} u_{tt} - u_{xx} = te^x, & (x,t) \in \mathbb{R} \times (0,\infty), \\ u(x,0) = 0, & u_t(x,0) = 0. \end{cases}$$

Solution. From d'Alembert's formula, we know that the problem

$$\begin{cases} v_{tt} - v_{xx} = 0, & (x,t) \in \mathbb{R} \times (0,\infty), \\ v(x,0) = 0, & v_t(x,0) = se^x, \end{cases}$$

has solution

$$v(x,t) = \frac{1}{2}\int_{x-t}^{x+t} se^y dy = \frac{s}{2}(e^{x+t} - e^{x-t}).$$

So

$$U(x,t;s) = v(x,t-s) = \frac{s}{2}(e^{x+t-s} - e^{x-t+s}).$$

Hence

$$\begin{aligned} u(x,t) &= \int_0^t U(x,t;s)ds \\ &= \frac{1}{2}[e^{x+t}(-te^{-t} - e^{-t} + 1) - e^{x-t}(te^t - e^t + 1)] \\ &= \frac{1}{2}(-2te^x + e^{x+t} - e^{x-t}). \end{aligned}$$

□

Example 4.11. Find a solution of the following problem

$$\begin{cases} u_{tt} - \Delta u = f(x,t), & (x,t) \in \mathbb{R}^3 \times (0,\infty), \\ u(x,0) = 0, \ u_t(x,0) = 0, & x \in \mathbb{R}^3. \end{cases}$$

Solution. Using Kirchhoff's formula, the solution v to the following Cauchy problem

$$\begin{cases} v_{tt} - \Delta v = 0, & (x,t) \in \mathbb{R}^3 \times (0,\infty), \\ v(x,0) = 0, \ v_t(x,0) = f(x,s), & x \in \mathbb{R}^3, \end{cases}$$

is given by

$$v(x,t) = t \fint_{\partial B(x,t)} f(y,s)\, dS_y.$$

Hence the solution U to (4.23) is given by

$$U(x,t;s) = v(x,t-s) = (t-s)\fint_{\partial B(x,t-s)} f(y,s)\, dS_y,$$

so that

$$\begin{aligned} u(x,t) &= \int_0^t U(x,t;s)\, ds = \int_0^t (t-s)\left(\fint_{\partial B(x,t-s)} f(y,s)\, dS_y\right) ds \\ &= \frac{1}{4\pi}\int_0^t \int_{\partial B(x,t-s)} \frac{f(y,s)}{t-s}\, dS_y ds \\ &= \frac{1}{4\pi}\int_0^t \int_{\partial B(x,r)} \frac{f(y,t-r)}{r}\, dS_y dr \\ &= \frac{1}{4\pi}\int_{B(x,t)} \frac{f(y,t-|y-x|)}{|y-x|} dy. \end{aligned}$$

Note that the domain of dependence is a solid ball now. □

4.4. Energy methods and the uniqueness

There are some subtle issues about the uniqueness of wave equation using the formulas (4.19) and (4.21). These formulas only hold under more and more smoothness assumptions on the initial data g, h as dimension n gets larger and larger. For initial data that are only C^2, we cannot use such formula to claim the uniqueness of the C^2 solutions. Instead we will need the fact that certain quantities of the wave behave nicely (e.g., conserved).

Let Ω be a bounded open set with a smooth boundary $\partial\Omega$. As above, set $\Omega_T = \Omega \times (0, T]$ and let

$$\Gamma_T = \partial'\Omega = \overline{\Omega_T} \setminus \Omega_T.$$

4.4.1. Uniqueness of mixed-value problem. We are interested in the uniqueness of initial-boundary value problem or mixed-value problem:

$$\begin{cases} u_{tt} - \Delta u = f & \text{in } \Omega_T, \\ u = g & \text{on } \Gamma_T, \\ u_t = h & \text{on } \Omega \times \{t = 0\}. \end{cases} \tag{4.25}$$

Theorem 4.12 (Uniqueness for wave equation)**.** *There exists at most one solution $u \in C^2(\overline{\Omega}_T)$ of* (4.25)*.*

Proof. If $\tilde{u}$ is another solution, then $v = u - \tilde{u}$ solves

$$\begin{cases} v_{tt} - \Delta v = 0 \quad \text{in } \Omega_T, \\ v = 0 \quad \text{on } \Gamma_T, \\ v_t = 0 \quad \text{on } \Omega \times \{t = 0\}. \end{cases}$$

Define the energy

$$e(t) = \frac{1}{2} \int_\Omega (v_t^2(x,t) + |Dv(x,t)|^2) dx \quad (0 \le t \le T).$$

Divergence theorem yields (note that it seems $v \in C^3$ is needed in using the divergence theorem below; however, the end result holds for only $v \in C^2$ by a smoothing argument)

$$\begin{aligned} e'(t) &= \int_\Omega (v_t v_{tt} + Dv \cdot Dv_t) dx \\ &= \int_\Omega v_t (v_{tt} - \Delta v) dx + \int_{\partial\Omega} v_t \frac{\partial v}{\partial \nu} dS = 0 \end{aligned}$$

since $v = 0$ on $\partial\Omega$ implies that $v_t = 0$ on $\partial\Omega$. Therefore $v \equiv 0$ from $v = 0$ on Γ_T. □

4.4.2. Domain of dependence. We can use the energy method to give another proof of the domain of dependence for wave equation in whole space. Let $u \in C^2$ be a solution of

$$u_{tt} - \Delta u = 0 \quad \text{in } \mathbb{R}^n \times (0, \infty).$$

Fix $x_0 \in \mathbb{R}^n$, $t_0 > 0$ and consider the *backwards wave cone* with apex (x_0, t_0):

$$K(x_0, t_0) := \{(x,t) \mid 0 \le t \le t_0, |x - x_0| \le t_0 - t\}.$$

Theorem 4.13 (Finite propagation speed)**.** *If $u = u_t = 0$ on $B(x_0, t_0) \times \{t = 0\}$, then $u \equiv 0$ within the cone $K(x_0, t_0)$.*

Remark 4.5. In particular, we see that any "disturbance" originating outside $B(x_0,t_0)$ has no effect on the solution within the cone $K(x_0,t_0)$ and consequently has finite propagation speed. We already know this from the representation formula if $u(x,0)$ and $u_t(x,0)$ are sufficiently smooth. The point here is that energy methods provide a much simpler proof.

Proof. Define the local energy

$$e(t) := \frac{1}{2}\int_{B(x_0,t_0-t)} (u_t^2(x,t) + |Du(x,t)|^2)dx \quad (0 \le t \le t_0).$$

Then

$$e(t) = \frac{1}{2}\int_0^{t_0-t}\int_{|\xi|=1} (u_t^2(x_0+r\xi,t) + |Du(x_0+r\xi,t)|^2)r^{n-1}drdS_\xi,$$

and so, by Divergence theorem,

$$\begin{aligned}
e'(t) &= \int_0^{t_0-t}\int_{|\xi|=1} [u_t u_{tt}(x_0+r\xi,t) + Du\cdot Du_t(x_0+r\xi,t)]r^{n-1}drdS_\xi \\
&\quad -\frac{1}{2}\int_{|\xi|=1}[u_t^2(x_0+(t_0-t)\xi,t) + |Du(x_0+(t_0-t)\xi,t)|^2](t_0-t)^{n-1}dS_\xi \\
&= \int_{B(x_0,t_0-t)} [u_t u_{tt} + Du\cdot Du_t]dx - \frac{1}{2}\int_{\partial B(x_0,t_0-t)} [u_t^2(x,t)+|Du(x,t)|^2]dS \\
&= \int_{B(x_0,t_0-t)} u_t[u_{tt}-\Delta u]dx + \int_{\partial B(x_0,t_0-t)} u_t\frac{\partial u}{\partial \nu}dS \\
&\quad -\frac{1}{2}\int_{\partial B(x_0,t_0-t)}[u_t^2(x,t)+|Du(x,t)|^2]dS
\end{aligned}$$

$$= \int_{\partial B(x_0,t_0-t)} \left[u_t\frac{\partial u}{\partial \nu} - \frac{1}{2}u_t^2(x,t) - \frac{1}{2}|Du(x,t)|^2\right]dS \le 0 \tag{4.26}$$

since, from $|\frac{\partial u}{\partial \nu}| = |Du\cdot\nu|| \le |Du||\nu| = |Du|$, it follows that

$$\begin{aligned}
u_t\frac{\partial u}{\partial \nu} - \frac{1}{2}u_t^2(x,t) - \frac{1}{2}|Du(x,t)|^2 &\le |u_t||Du| - \frac{1}{2}u_t^2(x,t) - \frac{1}{2}|Du(x,t)|^2 \\
&= -\frac{1}{2}(|u_t|-|Du|)^2 \le 0.
\end{aligned}$$

Now $e'(t) \le 0$ implies that $e(t) \le e(0) = 0$ for all $0 \le t \le t_0$. Thus $u_t = Du = 0$, and consequently $u \equiv 0$ in $K(x_0,t_0)$. □

4.4.3. Examples of other initial and boundary value problems. Uniqueness of wave equation can be used to find the solutions to some mixed-value problems. Since solution is unique, any solution found in special forms will be the unique solution.

Example 4.14. Solve the Cauchy problem of the wave equation

$$\begin{cases} u_{tt} - \Delta u = 0, & \text{in } \mathbb{R}^3\times(0,\infty), \\ u(x,0) = 0,\ u_t(x,0) = \frac{1}{1+|x|^2}. \end{cases}$$

In theory, we can use Kirchhoff's formula to find the solution. However, the computation would be too complicated. Instead, we can try to find a solution in the form of $u(x,t) = v(|x|,t)$ by solving an equation for v, which becomes exactly the Euler-Poisson-Darboux equation. Details are left as exercise.

EXAMPLE 4.15. Let $\lambda \in \mathbb{R}$ and $\lambda \neq 0$. Solve

$$\begin{cases} u_{tt} - \Delta u + \lambda u = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\ u(x,0) = g(x),\ u_t(x,0) = h(x) & (x \in \mathbb{R}^n). \end{cases}$$

If $\lambda = \mu^2 > 0$, let $v(x, x_{n+1}, t) = u(x,t)\cos(\mu x_{n+1})$. If $\lambda = -\mu^2 < 0$, let $v(x, x_{n+1}, t) = u(x,t)e^{\mu x_{n+1}}$. Then, in both cases, v satisfies the wave equation in $n+1$ dimension. The initial data of v at $t = 0$ are also known from g, h. Show solution v must be of separate form: $v(x, x_{n+1}, t) = u(x,t)\rho(x_{n+1})$; this gives the solution u.

EXAMPLE 4.16. Solve

$$\begin{cases} u_{tt} - \Delta u = 0, \quad x = (\tilde{x}, x_n) \in \mathbb{R}^{n-1} \times (0,\infty),\ t \in (0,\infty) \\ u(x,0) = g(x),\ u_t(x,0) = h(x), \quad x_n > 0, \\ u(\tilde{x}, 0, t) = 0. \end{cases}$$

Extend the functions g, h to odd functions F and G in x_n. Then solve the wave equation to get a solution $\tilde{u}$ on whole space $\mathbb{R}^n \times (0,\infty)$. Hence $u = \tilde{u}|_{x_n>0}$ is the solution to the original problem.

EXAMPLE 4.17. Let Ω be a bounded domain and $\Omega_T = \Omega \times (0, T]$ where $T > 0$. Solve

$$\begin{cases} u_{tt} - \Delta u = f(x,t) & \text{in } \Omega_T, \\ u(x,0) = g(x),\ u_t(x,0) = h(x), & x \in \Omega, \\ u(x,t) = 0 & x \in \partial\Omega,\ t \in [0,T]. \end{cases}$$

Use the method of **separation of variables** to find a solution of the form

$$u = \sum_{j=1}^{\infty} u_j(x) T_j(t),$$

where

$$-\Delta u_j = \lambda_j u_j, \quad u_j|_{\partial\Omega} = 0,$$
$$T_j''(t) - \lambda_j T(t) = f_j(t), \quad T_j(0) = a_j, \quad T_j'(0) = b_j,$$

and $\{u_j(x)\}_{j=1}^{\infty}$ is an orthonormal basis in $L^2(\Omega)$, and $f_j(t), a_j, b_j$ are the Fourier coefficients of $f(x,t)$, $g(x)$ and $h(x)$ with respect to $u_j(x)$ respectively.

The question is whether there is such an orthonormal basis. We need some machinary for Laplace's equation to answer this question. In any case, the answer is yes, but we will not study this in this course.

4.5. Finite propagation speed for general second-order hyperbolic equations

(*This is from §7.2.4 in the textbook.*)

We study a class of special second-order equations of the form

$$u_{tt} + Lu = 0 \quad (x,t) \in \mathbb{R}^n \times (0,\infty), \tag{4.27}$$

where L has a simple form

$$Lu = -\sum_{i,j=1}^{n} a_{ij}(x) D_{ij} u,$$

with smooth symmetric coefficients (a_{ij}) satisfying uniform ellipticity on $\mathbb{R}^n$. In this case, we call equation (4.27) **uniformly hyperbolic**.

Let u be a smooth solution to (4.27). Let $(x_0, t_0) \in \mathbb{R}^n \times (0, \infty)$ be given. We would like to find some sort of a *curved backward cone* C, with vertex (x_0, t_0) such that $u \equiv 0$ within C if $u = u_t = 0$ on $C_0 = C \cap \{t = 0\}$.

We assume C is given in terms of a function $q(x)$ smooth positive on $\mathbb{R}^n \setminus \{x_0\}$ and $q(x_0) = 0$ by

$$C = \{(x, t) \in \mathbb{R}^n \times (0, t_0) \mid q(x) < t_0 - t\}.$$

For each $t \in [0, t_0)$ let

$$C_t = \{x \in \mathbb{R}^n \mid q(x) < t_0 - t\}.$$

Assume $\partial C_t = \{x \in \mathbb{R}^n \mid q(x) = t_0 - t\}$ be a smooth, $(n-1)$-dimensional surface for each $t \in [0, t_0)$.

We define the energy

$$e(t) = \frac{1}{2} \int_{C_t} \Big(u_t^2 + \sum_{i,j=1}^n a_{ij} u_{x_i} u_{x_j} \Big) dx \quad (0 \le t \le t_0).$$

In order to compute $e'(t)$, we need the following result.

Lemma 4.18. *Let $\beta(x, t)$ be a smooth function and define*

$$\alpha(t) = \int_{C_t} \beta(x, t)\, dx \quad (0 < t < t_0).$$

Then

$$\alpha'(t) = \int_{C_t} \beta_t(x, t) dx - \int_{\partial C_t} \frac{\beta(x, t)}{|Dq(x)|}\, dS.$$

Proof. Let $\phi(x) = t_0 - q(x)$. Then C_r is the level set $\{\phi(x) = r\}$. By Coarea Formula,

$$\alpha(t) = \int_{C_t} \beta(x, t) dx = \int_t^{t_0} \left(\int_{\{\phi(x)=r\}} \frac{\beta(x, t)}{|D\phi(x)|} dS \right) dr.$$

From this, we have

$$\begin{aligned}
\alpha'(t) &= \int_t^{t_0} \left(\int_{\{\phi(x)=r\}} \frac{\beta_t(x, t)}{|D\phi(x)|} dS \right) dr - \int_{\{\phi(x)=t\}} \frac{\beta(x, t)}{|D\phi(x)|} dS \\
&= \int_{C_t} \beta_t(x, t) dx - \int_{\partial C_t} \frac{\beta(x, t)}{|Dq(x)|}\, dS.
\end{aligned}$$

$\square$

Apply this lemma, we compute

$$\begin{aligned}
e'(t) &= \int_{C_t} \Big(u_t u_{tt} + \sum_{i,j=1}^n a_{ij} u_{x_i} u_{x_j t} \Big) dx \\
&\quad - \frac{1}{2} \int_{\partial C_t} \Big(u_t^2 + \sum_{i,j=1}^n a_{ij} u_{x_i} u_{x_j} \Big) \frac{1}{|Dq|}\, dS \\
&=: A - B.
\end{aligned}$$

Note that $a_{ij}u_{x_i}u_{x_jt} = (a_{ij}u_{x_i}u_t)_{x_j} - (a_{ij}u_{x_i})_{x_j}u_t$. Then, integrating by parts, we deduce that

$$\begin{aligned} A &= \int_{C_t} u_t\Big(u_{tt} - \sum_{i,j=1}^n (a_{ij}u_{x_i})_{x_j}\Big)dx + \int_{\partial C_t}\sum_{i,j=1}^n a_{ij}u_{x_i}\nu_j u_t dS \\ &= -\int_{C_t} u_t \sum_{i,j=1}^n (a_{ij})_{x_j}u_{x_i}dx + \int_{\partial C_t}\sum_{i,j=1}^n a_{ij}u_{x_i}\nu_j u_t dS, \end{aligned}$$

where $\nu = (\nu_1, \nu_2, \cdots, \nu_n)$ is the usual outer unit normal to ∂C_t. In fact,

$$\nu = \frac{Dq}{|Dq|} \quad \text{on } \partial C_t = \{q(x) = t_0 - t\}.$$

Since $\langle v, w\rangle = \sum_{i,j=1}^n a_{ij}v_iw_j$ defines an inner product on $\mathbb{R}^n$, by Cauchy-Schwarz's inequality

$$\Big|\sum_{i,j=1}^n a_{ij}u_{x_i}\nu_j\Big| \le \Big(\sum_{i,j=1}^n a_{ij}u_{x_i}u_{x_j}\Big)^{1/2}\Big(\sum_{i,j=1}^n a_{ij}\nu_i\nu_j\Big)^{1/2}.$$

Therefore,

$$\begin{aligned} |A| &\le Ce(t) + \int_{\partial C_t}\Big(\sum_{i,j=1}^n a_{ij}u_{x_i}u_{x_j}\Big)^{1/2}|u_t|\Big(\sum_{i,j=1}^n a_{ij}\nu_i\nu_j\Big)^{1/2} dS \\ &\le Ce(t) + \frac{1}{2}\int_{\partial C_t}\Big(u_t^2 + \sum_{i,j=1}^n a_{ij}u_{x_i}u_{x_j}\Big)\Big(\sum_{i,j=1}^n a_{ij}\nu_i\nu_j\Big)^{1/2} dS. \end{aligned}$$

If we have

$$\Big(\sum_{i,j=1}^n a_{ij}\nu_i\nu_j\Big)^{1/2} \le \frac{1}{|Dq|}, \tag{4.28}$$

then we obtain that $|A| \le Ce(t) + B$ and so

$$e'(t) \le Ce(t).$$

This proves that $e \equiv 0$ if $e(0) = 0$.

Note that condition (4.28) is equivalent to

$$\sum_{i,j=1}^n a_{ij}q_{x_i}q_{x_j} \le 1. \tag{4.29}$$

In particular, this condition is satisfied if q solves the **generalized Eikonal equation**:

$$\sum_{i,j=1}^n a_{ij}q_{x_i}q_{x_j} = 1. \tag{4.30}$$

Therefore, we have proved the following theorem

Theorem 4.19 (Finite propagation speed)**.** *Assume q is a function positive, smooth and satisfying* (4.30) *for all $x \neq x_0$ and $q(x_0) = 0$. Let C and C_t be defined as above. If u is a smooth solution to* (4.27) *and satisfies $u = u_t = 0$ on C_0, then $u \equiv 0$ within the cone C.*

Suggested exercises

Materials covered are from Chapter 2 of the textbook with some new material (from Chapter 7) added. So complete the arguments that are left in lectures. Also try working on the following problems related to the covered materials, some problems being from Chapter 12 of the textbook.

Chapter 2: Problems 21-24.

Chapter 12: Problems 2, 3, 6, 7.

Homework # 6.

(1) (10 points) Let $f : \mathbf{R} \to \mathbf{R}$ be C^{m+1}. Prove that for $k = 1, 2, \ldots, m$,
  (a) $(\frac{d^2}{dr^2})(\frac{1}{r}\frac{d}{dr})^{k-1}(r^{2k-1}f(r)) = (\frac{1}{r}\frac{d}{dr})^k(r^{2k}f'(r))$;
  (b) $(\frac{1}{r}\frac{d}{dr})^{k-1}(r^{2k-1}f(r)) = \sum_{j=0}^{k-1}\beta_j^k r^{j+1}\frac{d^j f}{dr^j}(r)$, where β_j^k are constants;
  (c) $\beta_0^k = (2k-1)!! = (2k-1)(2k-3)\cdots 3 \cdot 1$.

(2) (15 points) Solve the mixed value problem for 1-D wave equation

$$\begin{cases} u_{tt} = u_{xx}, & x > 0,\ t > 0; \\ u(x,0) = g(x),\ u_t(x,0) = 0, & x > 0; \\ u_t(0,t) = \alpha u_x(0,t) & t > 0, \end{cases}$$

where $\alpha \neq -1$ is a constant and $g \in C^2(\mathbf{R}^+)$ vanishes near $x = 0$.

Moreover, show that if $\alpha = -1$ the problem has no solution unless $g \equiv 0$.

(3) Let $h \in C^2(\mathbf{R}^3)$ and let $u(x,t)$ solve the wave equation in $\mathbf{R}^3 \times \mathbf{R}$

$$u_{tt} = \Delta u, \quad u(x,0) = 0, \quad u_t(x,0) = h(x).$$

  (a) (5 points) If $0 \le h(x) \le 1$, show that $0 \le u(x,t) \le t$ for all $x \in \mathbf{R}^3$ and $t \ge 0$.
  (b) (10 points) Assume $h(x) = \frac{1}{1+|x|^2}$. Find the explicit solution to the given problem in the form of $u(x,t) = U(|x|,t)$. What is $u(0,t)$?

(4) (10 points) Show that there exists a constant K such that

$$|u(x,t)| \le \frac{K}{t}U(0) \quad \forall\, t > 0$$

whenever u is a smooth solution to the Cauchy problem of 3-D wave equation

$$u_{tt} - \Delta u = 0 \text{ in } \mathbf{R}^3 \times (0,\infty); \quad u(x,0) = g(x),\ u_t(x,0) = 0$$

and $U(0) = \int_{\mathbf{R}^3}(|g| + |Dg| + |D^2 g|)\,dy < \infty$.

(5) (10 points) Let $f(p,m,u)$ be a smooth function on $(p,m,u) \in \mathbf{R}^n \times \mathbf{R} \times \mathbf{R}$ and $f(0,0,0) = 0$. Assume $u(x,t)$ is a smooth solution to the nonlinear wave equation

$$u_{tt} - \Delta u + f(Du, u_t, u) = 0 \quad \text{on } \mathbf{R}^n \times [0,\infty).$$

Given any $(x_0,t_0) \in \mathbf{R}^n \times (0,\infty)$, let

$$K(x_0,t_0) = \{(x,t) \mid 0 \le t \le t_0,\ |x - x_0| \le t_0 - t\}.$$

Show that $u \equiv 0$ on $K(x_0,t_0)$ if $u = u_t = 0$ in $B(x_0,t_0)$ when $t = 0$.
Hint: *Use the energy method. Let*

$$e(t) = \frac{1}{2}\int_{B(x_0,t_0-t)}(u_t^2 + |Du|^2 + u^2)\,dx \quad (0 \le t \le t_0).$$

Compute $e'(t)$ and show $e'(t) \le Ce(t)$.

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