Throughout this homework, unless specifically stated, the set $\Omega$ is always assumed to be a bounded open set in $\mathbb{R}^n$.

(1) (10 points) Recall that $u : \Omega \to \mathbb{R}$ is subharmonic in $\Omega$ if $u \in C(\Omega)$ and if for every $\xi \in \Omega$ the inequality

$$u(\xi) \leq \int_{\partial B(\xi, \rho)} u(x) dS := M_u(\xi, \rho)$$

holds for all sufficiently small $\rho > 0$. We denote by $\sigma(\Omega)$ the set of all subharmonic functions in $\Omega$.

(a) For $u \in C(\overline{\Omega}) \cap \sigma(\Omega)$, show that $\max_{\Omega} u = \max_{\partial \Omega} u$.

(b) If $u_1, u_2, \cdots, u_k \in \sigma(\Omega)$, show that $u = \max\{u_1, \cdots, u_k\} \in \sigma(\Omega)$.

(c) Let $u \in C^2(\Omega)$. Show that $u \in \sigma(\Omega)$ if and only if $-\Delta u \leq 0$ in $\Omega$. 
(2) (10 points) Let $\Omega$ be the ellipsoid $\frac{x_1^2}{2} + \frac{x_2^2}{3} + \frac{x_3^2}{4} < 1$ in $\mathbb{R}^3$. Show that for all $u \in C^2(\Omega) \cap C(\bar{\Omega})$

$$\max_{\Omega} |u| \leq \max_{\partial \Omega} |u| + \frac{6}{13} \sup_{\Omega} |\Delta u|.$$
(3) (10 points) Let $n \geq 3$ and $\Omega \subset \mathbb{R}^n$ be an open set with $0 \in \Omega$. Suppose that $u$ is harmonic in $\Omega \setminus \{0\}$ and $\lim_{x \to 0} |x|^{n-2} u(x) = 0$. Prove that there exists a harmonic function $\tilde{u}$ in $\Omega$ such that $\tilde{u} = u$ on $\Omega \setminus \{0\}$. (That is 0 is a removable singularity.)
(4) (10 points) Let \( Lu = -\sum_{i,j=1}^n a^{ij}(x)u_{x_i x_j} + \sum_{i=1}^n b^i(x)u_{x_i} \) be a uniformly elliptic operator in \( \Omega \subset \mathbb{R}^n \). Suppose \( \sigma(s) \) is a \( C^1 \) function with \( \sigma'(s) \geq 0 \) for all \( s \in \mathbb{R} \). Show that, given any \( f \) and \( g \), the Dirichlet problem

\[
\begin{cases}
Lu + \sigma(u) = f & \text{in } \Omega \\
u = g & \text{on } \partial \Omega
\end{cases}
\]

can have at most one solution \( u \in C^2(\Omega) \cap C(\overline{\Omega}) \).
(5) (10 points) Let $a^{ij} \in C^1(\bar{\Omega})$ and $Lu = -\sum_{i,j=1}^{n} a^{ij}(x)u_{x_ix_j}$ be a uniformly elliptic operator in $\Omega \subset \mathbb{R}^n$. Let $u \in C^3(\bar{\Omega})$ be a solution of $Lu = 0$ in $\Omega$. Set $v = |Du|^2 + \lambda u^2$. Show that $Lv \leq 0$ in $\Omega$ if $\lambda$ is large enough. From this deduce

$$\|Du\|_{L^\infty(\Omega)} \leq \|Du\|_{L^\infty(\partial\Omega)} + C\|u\|_{L^\infty(\partial\Omega)}$$

for some constant $C$. 