Numerical Sequences and Series

3.1. Convergent Sequences

Definition 3.1. A sequence $\{p_n\}$ in a metric space X is said to **converge** in X if there is a point $p \in X$ with the following property: For every number $\epsilon > 0$, there exists an integer $N \in \mathbf{N}$ such that whenever $n \in \mathbf{N}$ and $n \geq N$ it follows that $d(p_n, p) < \epsilon$. That is, $\{p_n\}$ is said to converge in X if the following is true:

$$\exists p \in X \ \forall \epsilon > 0 \ \exists N \in \mathbf{N} \ \forall n \in \mathbf{N} \ (n \geq N \Longrightarrow d(p_n, p) < \epsilon).$$

In this case, we also say that $\{p_n\}$ **converges to** p, or p is a **limit** of $\{p_n\}$, and we write $p_n \to p$, or

$$\lim_{n\to\infty} p_n = p, \quad \textit{or simply,} \quad \lim p_n = p.$$

Note that convergence is a concept not only depending on the given sequence but also on the metric space X in which the sequence and its limit are considered.

If a sequence $\{p_n\}$ does not converge in X, then we say that it **diverges** in X.

A sequence $\{p_n\}$ in a metric space X is said to be **bounded** if the set $E = \{p_n : n \in \mathbf{N}\}$ (i.e., the range of $\{p_n\}$) is bounded in X; that is, for some $q \in X$ and number $M > 0$,

$$d(p_n, q) \leq M \quad \forall\, n \in \mathbf{N}.$$

Theorem 3.1. *Let $\{p_n\}$ be a sequence in a metric space X.*

(a) *$\{p_n\}$ converges to a limit $p \in X$ if and only if every neighborhood of p contains p_n for all but finitely many $n \in \mathbf{N}$.*

(b) *If $\{p_n\}$ converges to $p \in X$ and to $q \in X$, then $p = q$.*

(c) *If $\{p_n\}$ converges, then $\{p_n\}$ is bounded.*

Proof. (a) Note that $d(p_n, p) < \epsilon \Longleftrightarrow p_n \in N_\epsilon(p)$.

(b) Suppose, for the contrary, $p \neq q$. Then $\delta = d(p, q) > 0$. Since $p_n \to p$ and $p_n \to q$, there exist integers $N_1, N_2 \in \mathbf{N}$ such that

$$d(p_n, p) < \frac{1}{2}\delta \ \ (\forall\, n \geq N_1), \quad d(p_n, q) < \frac{1}{2}\delta \ \ (\forall\, n \geq N_2).$$

Let $N = \max\{N_1, N_2\}$, or for the same purpose, one could let $N = N_1 + N_2$. Then $d(p_N, p) < \frac{1}{2}\delta$ and $d(p_N, q) < \frac{1}{2}\delta$. Hence, by the triangle inequality, we have

$$d(p,q) \le d(p,p_N) + d(p_N,q) < \frac{1}{2}\delta + \frac{1}{2}\delta = \delta = d(p,q),$$

which is a contradiction.

(c) Suppose $p_n \to p$. Then there exists an $N \in \mathbf{N}$ such that

$$d(p_n, p) < 1 \quad \forall\, n \ge N.$$

Let $M = \max\{1,\, d(p_1,p), \cdots, d(p_N,p)\}$. Then $d(p_n,p) \le M$ for all $n \in \mathbf{N}$; this proves $\{p_n\}$ is bounded. □

Theorem 3.2. *Let X be a metric space, $E \subseteq X$, and $p \in X$. Then p is a limit point of E if and only if there is a sequence $\{p_n\}$ in E such that $p_n \ne p$ and $p_n \to p$.*

Proof. First suppose $p \in E'$. Then, for each $n \in \mathbf{N}$, there is a point $p_n \in N_{1/n}(p)$ such that $p_n \ne p$ and $p_n \in E$. Given each $\epsilon > 0$, let $N \in \mathbf{N}$ be such that $\frac{1}{N} < \epsilon$. Then for $n \in \mathbf{N}$ if $n \ge N$ then

$$d(p_n, p) < \frac{1}{n} \le \frac{1}{N} < \epsilon.$$

By definition, we have $p_n \to p$.

Conversely, suppose there is a sequence $\{p_n\}$ in E such that $p_n \ne p$ and $p_n \to p$. Let $N_r(p)$ be any neighborhood of p, where $r > 0$. Since $p_n \to p$, there is $N \in \mathbf{N}$ such that $d(p_n, p) < r$ for all $n \ge N$. Hence $p_N \in N_r(p)$, $p_N \in E$, and $p_N \ne p$. By definition, $p \in E'$. □

Example 3.1. Show

$$\lim\left(\frac{n+1}{n}\right) = 1.$$

Proof. Let $a_n = \frac{n+1}{n}$ and $a = 1$. Then the inequality

$$|a_n - a| = \left|\frac{n+1}{n} - 1\right| = \frac{1}{n} < \epsilon$$

is the same as $n > \frac{1}{\epsilon}$. The existence of $N \in \mathbf{N}$ is guaranteed by the Archimedean property: there always exists an $N \in \mathbf{N}$ such that $N > \frac{1}{\epsilon}$. The actual proof goes as follows.

Let $\epsilon > 0$ be arbitrary. By the Archimedean property, there exists an $N \in \mathbf{N}$ such that $N > \frac{1}{\epsilon}$. Then whenever $n \in N$ we have $1/n \le 1/N < \epsilon$ and hence

$$|a_n - a| = \left|\frac{n+1}{n} - 1\right| = \frac{1}{n} < \epsilon.$$

Therefore, by definition, $\lim a_n = 1$. □

Theorem 3.3 (**Algebraic Limit Theorem**). *Suppose $\{a_n\}, \{b_n\}$ are sequences of real numbers, and $\lim a_n = a$, $\lim b_n = b$ exist in $\mathbf{R}$. Then*

(i) $\lim(ca_n) = ca$ *for all* $c \in \mathbf{R}$;

(ii) $\lim(a_n + b_n) = a + b$;

(iii) $\lim(a_n b_n) = ab$;

(iv) $\lim(a_n/b_n) = a/b$, *provided* $b_n \ne 0$ *and* $b \ne 0$.

Warning: *We can use these formulas only when both the limits* $\lim a_n$ *and* $\lim b_n$ *exist.*

Proof. We only include the proof for the product and quotient theorem.
Proof of (iii): Note that

$$a_nb_n - ab = a_nb_n - a_nb + a_nb - ab = a_n(b_n - b) + (a_n - a)b.$$

Therefore, by the Triangle Inequality,

$$|a_nb_n - ab| \le |a_n(b_n - b)| + |(a_n - a)b| = |a_n||b_n - b| + |a_n - a||b|.$$

Given $\epsilon > 0$, in order to make $|a_nb_n - ab| < \epsilon$, it suffices to make each of the two terms on the right hand side $< \epsilon/2$. Since (a_n) converges, it is bounded and so $|a_n| \le M$ $(\forall\, n \in \mathbf{N})$ for some number $M > 0$. Hence the two terms are bounded as follows:

$$|a_n||b_n - b| \le M|b_n - b|, \quad |a_n - a||b| \le |a_n - a|(|b| + 1)$$

(here we change $|b| \ge 0$ to $|b| + 1 > 0$ for the division later). Now, given arbitrary $\epsilon > 0$, since $(a_n) \to a$, we have $N_1 \in \mathbf{N}$ such that

$$|a_n - a| < \frac{\epsilon}{2(|b| + 1)} \quad \forall\, n \ge N_1.$$

Since $(b_n) \to b$, we have $N_2 \in \mathbf{N}$ such that

$$|b_n - b| < \frac{\epsilon}{2M} \quad \forall\, n \ge N_2.$$

Let $N = \max\{N_1, N_2\}$ (or $N = N_1 + N_2$). Then, for this N, whenever $n \ge N$, it follows that

$$|a_n - a| < \frac{\epsilon}{2(|b| + 1)}, \quad |b_n - b| < \frac{\epsilon}{2M};$$

hence

$$|a_n - a||b| \le \frac{\epsilon|b|}{2(|b| + 1)} < \frac{\epsilon}{2}, \quad |a_n||b_n - b| \le M|b_n - b| < \frac{\epsilon}{2},$$

and finally, it follows that, whenever $n \ge N$,

$$|a_nb_n - ab| \le |a_n||b_n - b| + |a_n - a||b| < \epsilon/2 + \epsilon/2 = \epsilon.$$

By definition, $(a_nb_n) \to ab$.

Proof of (iv): Note that

$$\frac{a_n}{b_n} - \frac{a}{b} = \frac{ba_n - ab_n}{b_nb} = \frac{b(a_n - a) + a(b - b_n)}{b_nb}.$$

Hence

$$\left|\frac{a_n}{b_n} - \frac{a}{b}\right| \le \frac{|b||a_n - a|}{|b_nb|} + \frac{|a||b - b_n|}{|b_nb|}. \tag{3.1}$$

Since $(b_n) \to b \ne 0$, with $\epsilon = |b|/2 > 0$, there exists an $N_1 \in \mathbf{N}$ such that $|b_n - b| < |b|/2$ for all $n \ge N_1$. Hence, by the triangle inequality, $|b_n| \ge |b| - |b_n - b| \ge |b|/2$ for all $n \ge N_1$. So, for all $n \ge N_1$, we have $|b_nb| \ge |b|^2/2$ and hence

$$\left|\frac{a_n}{b_n} - \frac{a}{b}\right| \le \frac{|b||a_n - a|}{|b_nb|} + \frac{|a||b - b_n|}{|b_nb|}$$

$$\le \frac{2}{|b|}|a_n - a| + \frac{2|a|}{|b|^2}|b_n - b| \le \frac{2}{|b|}|a_n - a| + \frac{2|a| + 1}{|b|^2}|b_n - b|.$$

We then use the convergences as before to select N_2 and N_3 in $\mathbf{N}$ such that

$$|a_n - a| < \frac{\epsilon|b|}{4} \quad \text{whenever } n \ge N_2$$

and

$$|b_n - b| < \frac{\epsilon|b|^2}{2(2|a|+1)} \quad \text{whenever } n \geq N_3.$$

Finally, let $N = \max\{N_1, N_2, N_3\}$. Then, whenever $n \geq N$, it follows that

$$\left|\frac{a_n}{b_n} - \frac{a}{b}\right| \leq \frac{2}{|b|}|a_n - a| + \frac{2|a|+1}{|b|^2}|b_n - b| < \epsilon/2 + \epsilon/2 = \epsilon.$$

□

Theorem 3.4 (**Order Limit Theorem**). *Suppose $\{a_n\}, \{b_n\}$ are sequences of real numbers. Assume $\lim a_n = a$ and $\lim b_n = b$ both exist. If $a_n \leq b_n$ for all $n \geq N_0$, where $N_0 \in \mathbf{N}$ is some integer, then $a \leq b$.*

Proof. Suppose, for the contrary, $a > b$. Then $\lim(a_n - b_n) = a - b > 0$. Using $\epsilon = \frac{a-b}{2} > 0$, we have an $N \in \mathbf{N}$ such that

$$|(a_n - b_n) - (a - b)| < \epsilon = \frac{a-b}{2} \quad \forall\, n \geq N.$$

Hence $a - b - \epsilon < a_n - b_n < a - b + \epsilon$ for all $n \geq N$. But $a - b - \epsilon = \frac{a-b}{2} > 0$; this implies that $a_n - b_n > \frac{a-b}{2} > 0$ for all $n \geq N$. So $a_n > b_n$ for all $n \geq N$; in particular, $a_n > b_n$ for $n = N_0 + N > N_0$, which contradicts the assumption $a_n \leq b_n$ for all $n \geq N_0$. □

Theorem 3.5. *Consider the Euclidean space $\mathbf{R}^k$.*

(a) *Suppose $\mathbf{x}_n \in \mathbf{R}^k$ for each $n \in N$, and let $\mathbf{x}_n = (a_{1,n}, a_{2,n}, \ldots, a_{k,n})$, where $a_{j,n} \in \mathbf{R}\,(1 \leq j \leq k)$. Then $\mathbf{x}_n \to \mathbf{x} = (a_1, a_2, \ldots, a_k) \in \mathbf{R}^k$ if and only if*

$$\lim_{n\to\infty} a_{j,n} = a_j \quad (1 \leq j \leq k).$$

(b) *Suppose $\{\mathbf{x}_n\}, \{\mathbf{y}_n\}$ are sequences in $\mathbf{R}^k$, $\{\beta_n\}$ is a sequence of real numbers, and $\mathbf{x}_n \to \mathbf{x}$, $\mathbf{y}_n \to \mathbf{y}$, $\beta_n \to \beta$. Then*

$$\lim(\mathbf{x}_n + \mathbf{y}_n) = \mathbf{x} + \mathbf{y}, \quad \lim(\mathbf{x}_n \cdot \mathbf{y}_n) = \mathbf{x} \cdot \mathbf{y}, \quad \lim(\beta_n \mathbf{x}_n) = \beta\mathbf{x}.$$

3.2. Subsequences

Definition 3.2. Let $\{p_n\}$ be a sequence in a metric space X, and let $n_1 < n_2 < n_3 < \cdots$ be an increasing sequence of natural numbers. Then the sequence

$$\{p_{n_k}\}_{k=1}^{\infty} = \{p_{n_1}, p_{n_2}, p_{n_3}, \cdots\}$$

is called a **subsequence** of $\{p_n\}$. Note that the order of the terms in a subsequence is kept unchanged as in the original sequence.

If a subsequence $\{p_{n_k}\}$ converges in X, then its limit is called a **subsequential limit** of $\{p_n\}$ in X.

Theorem 3.6. *A sequence $\{p_n\}$ in a metric space X converges to $p \in X$ if and only if every subsequence of $\{p_n\}$ converges to p.*

Proof. Clearly, a sequence is also a subsequence of itself. Thus, to prove the theorem, we need to prove that if $p_n \to p$ then every subsequence $\{p_{n_k}\}$ also converges to p. Since $1 \leq n_1 < n_2 < \cdots$ are integers, clearly, $n_k \geq k$ for $k \in \mathbf{N}$. Given $\epsilon > 0$, let $N \in \mathbf{N}$ be such that $d(p_n, p) < \epsilon$ for all $n \geq N$. Then, for all $k \geq N$, since $n_k \geq k \geq N$, it follows that $d(p_{n_k}, p) < \epsilon$. By definition, $\lim\limits_{k\to\infty} p_{n_k} = p$. □

Theorem 3.7. *Every sequence in a compact metric space contains a convergent subsequence.*

Proof. Let X be a compact metric space and $\{p_n\}$ be a sequence in X. Let E be the range of $\{p_n\}$.

If E is finite, then there is a $p \in E$ such that $p_n = p$ for infinitely many $n \in \mathbf{N}$. Hence, there is a sequence $\{n_i\}$ in $\mathbf{N}$ with $n_1 < n_2 < \cdots$, such that $p_{n_i} = p$ for all $i \in \mathbf{N}$. The subsequence so obtained is constant and hence converges to p.

If E is infinite, then, as an infinite subset of a compact set X, E has a limit point $p \in X$. Choose $n_1 \in \mathbf{N}$ so that $d(p_{n_1}, p) < 1$. Having chosen $n_1 < n_2 < \cdots < n_{i-1}$ in $\mathbf{N}$, since every neighborhood of p contains infinitely many points of E, there exists an $n_i > n_{i-1}$ in $\mathbf{N}$ such that $d(p_{n_i}, p) < 1/i$. Then, the subsequence $\{p_{n_i}\}$ converges to p. □

Corollary 3.8 (Bolzano-Weierstrass Theorem). *Every bounded sequence in $\mathbf{R}^k$ contains a convergent subsequence.*

Proof. The closure of the range of every bounded sequence is a compact set of $\mathbf{R}^k$ which contains the given sequence. Then use the theorem above. □

Theorem 3.9. *The set of all subsequential limits of a sequence in a metric space X is a closed subset of X.*

Proof. Let $\{p_n\}$ be a sequence in X and let E^* be the set of all subsequential limits of $\{p_n\}$ in X. To show E^* is closed, let q be a limit point of E^* (if E^* has no limit points then E^* is closed). We need to show $q \in E^*$.

Take $n_1 = 1$ and let $d(p_1, q) \leq M$ for some $M > 0$. Now assume $1 = n_1 < n_2 < \cdots < n_{i-1}$ are chosen in $\mathbf{N}$. Since q is a limit point of E^*, there is an $x \in E^*$ such that $d(x, q) < M/i$. Since $x \in E^*$, there exists a subsequence of $\{p_n\}$ converging to x; hence there is an integer $n_i > n_{i-1}$ such that $d(p_{n_i}, x) < M/i$. Thus, we obtain a subsequence $\{p_{n_i}\}$ that satisfies

$$d(p_{n_i}, q) \leq d(p_{n_i}, x) + d(x, q) < 2M/i$$

for all $i \in \mathbf{N}$. Hence $\{p_{n_i}\}$ converges to q, proving $q \in E^*$. □

3.3. Cauchy Sequences

Definition 3.3. A sequence $\{p_n\}$ of a metric space is called a **Cauchy sequence** if, for every $\epsilon > 0$, there exists an $N \in \mathbf{N}$ such that whenever $m, n \geq N$ in $\mathbf{N}$ it follows that $d(p_n, p_m) < \epsilon$; that is,

$$\forall\, \epsilon > 0\ \exists\, N \in \mathbf{N}\ \forall\, m, n \in \mathbf{N}\ (m, n \geq N \Longrightarrow d(p_n, p_m) < \epsilon).$$

Definition 3.4. Let E be a nonempty subset of a metric space X, and let S be the set of all numbers of the form $d(p, q)$, with $p \in E$ and $q \in E$. The number $\sup S$ (may be $+\infty$) is called the **diameter** of E, and is denoted by $\operatorname{diam} E$.

If $\{p_n\}$ is a sequence in X and E_n is the set consisting of $\{p_k \colon k \geq n\}$, then $\{p_n\}$ is a Cauchy sequence if and only if

$$\lim_{n\to\infty} \operatorname{diam} E_n = 0.$$

Theorem 3.10. *Let X be a metric space.*

(a) *For any subset E of X,*

$$\operatorname{diam}\bar{E} = \operatorname{diam}E.$$

(b) *If $\{K_n\}$ is a sequence of compact sets in X such that $K_{n+1} \subseteq K_n$ for all $n \in \mathbf{N}$ and satisfies*

$$\lim_{n\to\infty} \operatorname{diam}K_n = 0,$$

then $\cap_{n=1}^{\infty} K_n$ consists of exactly one point.

Proof. (a) Since $E \subseteq \bar{E}$, it is clear that $\operatorname{diam}E \le \operatorname{diam}\bar{E}$. To prove the other direction, let $\epsilon > 0$, and take any p, q in $\bar{E}$. Then, there exist p', q' in E such that $d(p', p) < \epsilon$, $d(q', q) < \epsilon$. (For example, if $p \in E$ then take $p' = p$; if $p \notin E$ then $p \in E'$ and hence $p' \in \hat{N}_\epsilon(p) \cap E$ exists.) Hence

$$d(p, q) \le d(p, p') + d(p', q') + d(q', q) < 2\epsilon + d(p', q') \le 2\epsilon + \operatorname{diam}E.$$

It follows that

$$\operatorname{diam}\bar{E} = \sup_{p,q\in\bar{E}} d(p, q) \le 2\epsilon + \operatorname{diam}E.$$

As $\epsilon > 0$ is arbitrary, this implies $\operatorname{diam}\bar{E} \le \operatorname{diam}E$; (a) is proved.

(b) Let $K = \cap_{n=1}^{\infty} K_n$. Then by the Nested Compact Set Theorem, $K \ne \emptyset$. Since $K \subseteq K_n$ for all $n \in \mathbf{N}$, we have

$$0 \le \operatorname{diam}K \le \operatorname{diam}K_n \to 0.$$

Hence $\operatorname{diam}K = 0$, which shows that K consists of exactly one point. □

Theorem 3.11. *We have the following.*

(a) *In a metric space, every convergent sequence is a Cauchy sequence.*

(b) *In a compact metric space, every Cauchy sequence converges.*

(c) *In $\mathbf{R}^k$, a sequence converges if and only if it is a Cauchy sequence.*

Proof. (a) Assume X is a metric space and sequence $\{p_n\}$ in X converges to $p \in X$. Then, for every $\epsilon > 0$, there exists an $N \in \mathbf{N}$ such that whenever $n \ge N$ in $\mathbf{N}$ it follows that $d(p_n, p) < \epsilon/2$. Hence, whenever $n, m \ge N$ in $\mathbf{N}$, it follows by the triangle inequality that

$$d(p_n, p_m) \le d(p_n, p) + d(p_m, p) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

By definition, $\{p_n\}$ is a Cauchy sequence.

(b) Let $\{p_n\}$ be a Cauchy sequence in a compact metric space X. For each $n \in \mathbf{N}$ let $E_n = \{p_k \colon k \ge n\}$. Then

$$\lim_{n\to\infty} \operatorname{diam}\bar{E}_n = 0.$$

Also, since $E_{n+1} \subseteq E_n$, it follows that $\bar{E}_{n+1} \subseteq \bar{E}_n$ for all $n \in \mathbf{N}$. Also, as a closed subset of a compact set X, each $\bar{E}_n$ is compact. Hence $\{\bar{E}_n\}$ is a *nested* sequence of compacts with $\operatorname{diam}\bar{E}_n \to 0$. Hence $\cap_{n=1}^{\infty} \bar{E}_n = \{p\}$ for a unique $p \in X$. We show that $p_n \to p$.

Let $\epsilon > 0$ be given. There is an integer $N \in \mathbf{N}$ such that $\operatorname{diam}\bar{E}_n < \epsilon$ for all $n \ge N$. Since $p \in \bar{E}_n$ for all $n \in \mathbf{N}$, it follows that $d(p_n, p) \le \operatorname{diam}\bar{E}_n < \epsilon$ for all $n \ge N$. Hence $p_n \to p$.

(c) From (a), we only have to show that a Cauchy sequence in $\mathbf{R}^k$ converges. Let $\{p_n\}$ be a Cauchy sequence in $\mathbf{R}^k$. We first prove $\{p_n\}$ is bounded. Since there exists an $N \in \mathbf{N}$

such that $d(p_n, p_m) < 1$ for all $n, m \in \mathbf{N}$ and $n, m \geq N$, with $m = N$, we have $d(p_n, p_N) < 1$ for all $n \geq N$. Let

$$M = \max\{1,\, d(p_1, p_N), \cdots, d(p_{N-1}, p_N)\}.$$

Then $d(p_n, p_N) \leq M$ for all $n \in \mathbf{N}$; thus $\{p_n\}$ is bounded. Let $E = \{p_n \colon n \in \mathbf{N}\}$; then E and $\bar{E}$ are both bounded subsets of $\mathbf{R}^k$, and hence $\bar{E}$ is compact. Then (c) follows from (b). □

Definition 3.5. A metric space in which every Cauchy sequence converges is called a **complete metric space**.

Theorem 3.11 says that *all compact metric spaces and all Euclidean spaces are a complete metric space.* However, the set $\mathbf{Q}$ viewed as a sub-metric space of $\mathbf{R}$ is not complete.

3.4. Monotonic Convergence in R

Definition 3.6. A sequence $\{a_n\}$ in $\mathbf{R}$ is said to be **monotonically increasing** if $a_n \leq a_{n+1}$ for all $n \in \mathbf{N}$, and is said to be **monotonically decreasing** if $a_n \geq a_{n+1}$ for all $n \in \mathbf{N}$.

A sequence in $\mathbf{R}$ is said to be **monotonic** if it is either monotonically increasing or monotonically decreasing.

A sequence $\{a_n\}$ in $\mathbf{R}$ is said to be **bounded above** if there exists $M \in \mathbf{R}$ such that $a_n \leq M$ for all $n \in \mathbf{N}$. Similarly, $\{a_n\}$ is said to be **bounded below** if there exists $M \in \mathbf{R}$ such that $a_n \geq M$ for all $n \in \mathbf{N}$.

Theorem 3.12 (**Monotonic Convergence Theorem**). *Suppose $\{a_n\}$ is a monotonic sequence in $\mathbf{R}$. Then $\{a_n\}$ converges if and only if $\{a_n\}$ is bounded.*

Proof. We only prove the theorem when $\{a_n\}$ is monotonically increasing in $\mathbf{R}$. Since every convergent sequence in a metric space is bounded, it suffices to show that if $\{a_n\}$ is *bounded above* then $\{a_n\}$ converges in $\mathbf{R}$. Thus assume, for some real number $M > 0$, $a_n \leq M$ for all $n \in \mathbf{N}$. Consider the set $S = \{a_n \colon n \in \mathbf{N}\}$. Then S is nonempty and bounded above (with M being an upper-bound). So $a = \sup S$ exists in $\mathbf{R}$. We prove $a_n \to a$. Since a is an upper-bound for S, we have

$$a_n \leq a \quad \forall\, n \in \mathbf{N}.$$

On the other hand, given arbitrary $\epsilon > 0$, since $a = \sup S$, there exists an $a_N \in S$ such that $a - \epsilon < a_N$. Then, by the monotonicity of a_n,

$$a_n \geq a_N > a - \epsilon \quad \forall\, n \geq N.$$

Combining above inequalities, we have $a - \epsilon < a_n \leq a < a_n + \epsilon$; that is, $|a_n - a| < \epsilon$ for all $n \geq N$. Hence, by definition, $\lim a_n = a$. □

The MCT is useful for determining the convergence of a real sequence without explicitly knowing the actual limit and checking the ϵ-N definition.

Example 3.2. Show that the sequence

$$\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \cdots$$

converges and find the limit.

Solution. Let a_n be the n-th term of this sequence; that is, $a_1 = \sqrt{2}, a_2 = \sqrt{2\sqrt{2}}, \cdots$. We have

$$a_{n+1} = \sqrt{2a_n}; \quad \text{hence} \quad a_{n+1}^2 = 2a_n, \quad \forall\, n = 1, 2, 3, \cdots .$$

Using induction we easily show that

$$\sqrt{2} \le a_n \le 2, \quad a_n \le a_{n+1} \quad \forall\, n \in \mathbf{N}.$$

Hence $\{a_n\}$ is bounded and monotonically increasing. Therefore, by the MCT, $\lim a_n = a$ exists. Moreover. the order limit theorem says $\sqrt{2} \le a \le 2$. Since $\{a_{n+1}\}$ is a subsequence of $\{a_n\}$, we have that $\lim a_{n+1} = a$. So, taking the limit on both sides of $a_{n+1}^2 = 2a_n$, we have $a^2 = 2a$. Since $a \ne 0$, it follows that $a = 2$; that is, $\lim a_n = 2$. □

3.5. Upper and Lower Limits

Definition 3.7. A sequence $\{a_n\}$ in $\mathbf{R}$ is said to **converge to** $+\infty$, written

$$a_n \to +\infty \quad \text{or} \quad \lim_{n\to\infty} a_n = +\infty,$$

if, for every real M there is an $N \in \mathbf{N}$ such that $a_n \ge M$ for all $n \ge N$.

Similarly, $\{a_n\}$ is said to **converge to** $-\infty$, written

$$a_n \to -\infty \quad \text{or} \quad \lim_{n\to\infty} a_n = -\infty,$$

if, for every real M there is an $N \in \mathbf{N}$ such that $a_n \le M$ for all $n \ge N$.

Obviously, $a_n \to -\infty \iff -a_n \to +\infty$. Frequently we write $\lim a_n = \pm\infty$. Clearly if $\lim a_n = \pm\infty$, then $\lim a_{n_k} = \pm\infty$ for all subsequences $\{a_{n_k}\}$.

Lemma 3.13. *A real sequence $\{a_n\}$ is not bounded above (or below) if and only if there exists a monotonically increasing (or decreasing) subsequence $\{a_{n_k}\}$ such that $a_{n_k} \to +\infty$ (or $-\infty$).*

Proof. (i) Assume $\{a_n\}$ is bounded, say, $a_n \le M$ for all $n \in \mathbf{N}$, for some $M \in \mathbf{R}$. Then there is no subsequence $\{a_{n_k}\}$ converging to $+\infty$, since otherwise, there would be a $K \in \mathbf{N}$ such that $a_{n_k} \ge M + 1$ for all $k \ge K$, a contradiction.

(ii) Assume $\{a_n\}$ is not bounded above. Then, for every $M \in \mathbf{R}$, there is an $n \in \mathbf{N}$ such that $a_n > M$. First, with $M = 1$, we have $a_{n_1} > 1$. Suppose that $n_1 < n_2 < \cdots < n_{i-1}$ are defined, then, with $M = i + \max\{a_1, a_2, \cdots, a_{n_{i-1}}\}$, we have $n_i \in \mathbf{N}$ such that $a_{n_i} > M$; clearly, $n_i > n_{i-1}$ and $a_{n_i} \ge i + a_{n_{i-1}}$. In this way, we obtain a monotonically increasing subsequence $\{a_{n_k}\}$ such that $a_{n_k} > k$ for all $k \in \mathbf{N}$; hence $a_{n_k} \to +\infty$.

The case for bounded-below sequences is completely analogous. □

Definition 3.8. Let $\{a_n\}$ be a sequence in $\mathbf{R}$. Let E be the set of (extended) real numbers x (including $+\infty$ and $-\infty$) such that $a_{n_k} \to x$ for some subsequence $\{a_{n_k}\}$. This set E contains the set E^* of all subsequential limits of $\{a_n\}$ defined in the proof of Theorem 3.9, plus possibly the extended numbers $+\infty$ or $-\infty$. Let

$$a^* = \sup E, \quad a_* = \inf E.$$

The (extended real) numbers a^* and a_* are called the **upper limit** and **lower limit** of the sequence $\{a_n\}$, respectively; we use the notation

$$\limsup_{n\to\infty} a_n = a^*, \quad \liminf_{n\to\infty} a_n = a_*,$$

or simply, $a^* = \limsup a_n$, $a_* = \liminf a_n$.

EXAMPLE 3.3. (i) Let $a_n = (-1)^n(1+\frac{1}{n})$. Then the set E defined above consists of two numbers, 1 and -1. (Why?) Hence

$$\limsup_{n\to\infty} a_n = 1, \quad \liminf_{n\to\infty} a_n = -1.$$

(ii) Let $a_n = n^{(-1)^n}$. Then the set E defined above consists of $\{0, +\infty\}$, and hence

$$\limsup_{n\to\infty} a_n = +\infty, \quad \liminf_{n\to\infty} a_n = 0.$$

(iii) Let $\{a_n\}$ be the sequence of all rational numbers. Then the set E defined is all the extended real numbers; hence

$$\limsup_{n\to\infty} a_n = +\infty, \quad \liminf_{n\to\infty} a_n = -\infty.$$

Theorem 3.14. *Let $\{a_n\}$ be a sequence in $\mathbf{R}$. Then $\lim_{n\to\infty} a_n = a$ exists in the extended real system if and only if*

$$\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = a.$$

Proof. If $\lim a_n = a$, then the set E defined above consists of only one element a; hence $\limsup a_n = \liminf a_n = a$.

Assume $\limsup a_n = \liminf a_n = a$. Then $E = \{a\}$.

(i) Assume $a = +\infty$. Then $\{a_n\}$ must be bounded below since otherwise there would be a subsequence converging to $-\infty$. We show $\lim a_n = +\infty$; that is,

$$\forall\, M \;\; \exists\, N \in \mathbf{N} \;\; \forall\, n \in \mathbf{N} \;\; (n \geq N \Longrightarrow a_n > M). \tag{3.2}$$

Suppose (3.2) is false; then, by negating (3.2),

$$\exists\, M \;\; \forall\, N \in \mathbf{N} \;\; \exists\, k_N \in \mathbf{N} \;\; (k_N \geq N,\; a_{k_N} \leq M). \tag{3.3}$$

We use (3.3) as follows. First, choose $n_1 = k_1 \geq 1$ such that $a_{n_1} \leq M$. Once n_1 is chosen, with $N = n_1 + 1$, there exists $n_2 = k_N \geq n_1 + 1$ such that $a_{n_2} \leq M$. Continue in this way, and we get a subsequence $\{a_{n_k}\}$ such that $a_{n_k} \leq M$. Since $\{a_n\}$ is bounded below, we have the sequence $\{a_{n_k}\}$ is bounded; hence, by the Bolzano-Weierstrass Theorem, this sequence has a convergent subsequence converging to a finite number, and this subsequence is also a subsequence of the original sequence $\{a_n\}$, showing E has a finite number in it, a contradiction.

Similarly, if $a = -\infty$, then it follows that $\lim a_n = -\infty$.

(ii) Now assume a is finite. In this case, by Lemma 3.13, $\{a_n\}$ is bounded. Suppose $\{a_n\}$ does not converge to a. Then, by negating the definition of $a_n \to a$,

$$\exists\, \epsilon_0 > 0\, \forall\, N \in \mathbf{N}\, \exists\, k_N \in \mathbf{N}\, (k_N \geq N,\; |a_{k_N} - a| \geq \epsilon_0). \tag{3.4}$$

We use (3.4) as follows. First, choose $n_1 = k_1 \geq 1$ such that $|a_{n_1} - a| \geq \epsilon_0$. Once n_1 is chosen, with $N = n_1 + 1$, there exists $n_2 = k_N \geq n_1 + 1$ such that $|a_{n_2} - a| \geq \epsilon_0$. Continue in this way and we get a subsequence $\{a_{n_k}\}$ such that $|a_{n_k} - a| \geq \epsilon_0$. Since $\{a_{n_k}\}$ is bounded, by the Bolzano-Weierstrass Theorem, it has a convergent subsequence converging to a finite number b which, by the **order limit theorem**, must satisfy that $|b - a| \geq \epsilon_0$; this subsequence is also a subsequence of the original $\{a_n\}$, showing $b \in E$, but $b \neq a$, a contradiction since $E = \{a\}$.

This completes the proof. □

Theorem 3.15. *Let $\{a_n\}$ be a sequence in $\mathbf{R}$. Let E and a^* be defined as above. Then a^* has the following properties:*

(a) $a^* \in E$.

(b) *If $x > a^*$, then there is an $N \in \mathbf{N}$ such that $a_n < x$ for all $n \geq N$.*

Moreover, a^* is the only (extended real) number satisfying the properties (a) *and* (b).

An analogous result is true for a_*.

Proof. (a) If $a^* = +\infty$, then E is not bounded above; hence $\{a_n\}$ is not bounded above, and, by Lemma 3.13, there is a subsequence $\{a_{n_k}\}$ converging to $+\infty$.

If a^* is finite, then, for every $\epsilon > 0$, there is $x \in E$ such that $a^* - \epsilon < x$. Since x is finite, we have $x \in E^*$, where E^* is the set defined in the proof of Theorem 3.9; hence, $a^* - \epsilon < \sup E^*$ for all $\epsilon > 0$, which proves that $a^* \leq \sup E^* \leq \sup E = a^*$, and thus $a^* = \sup E^* \in \overline{E^*} = E^* \subseteq E$ since E^* is closed.

If $a^* = -\infty$, then E contains only one element $-\infty$; in this case, in fact, the whole sequence $a_n \to -\infty$. (See the proof of Theorem 3.14.)

This proves (a) in all cases.

(b) Nothing is to prove if $a^* = +\infty$. So let $a^* < +\infty$, and hence $\{a_n\}$ is bounded above. Suppose that there is a number $x > a^*$ such that $a_n \geq x$ for infinitely many $n \in \mathbf{N}$. These terms determine a subsequence of $\{a_n\}$ which is bounded and thus, by the **Bolzano-Weierstrass theorem**, has a convergent subsequence with limit $y \geq x$. Since a subsequence of a subsequence of $\{a_n\}$ is also a subsequence of $\{a_n\}$, we have $y \in E$, but $y \geq x > a^*$, contradicting the definition of $a^* = \sup E$.

Thus a^* satisfies (a) and (b).

To show the uniqueness, let p and q both satisfy (a) and (b), and suppose $p < q$. Choose x such that $p < x < q$. Since p satisfies (b), there exists an $N \in \mathbf{N}$ such that $a_n < x$ for all $n \geq N$. So the limit of any convergent subsequence of $\{a_n\}$ must be less than or equal to x; hence, $\forall\, y \in E,\ y \leq x < q$, but then q cannot be in E, contradicting (a) for q. ☐

The following result is useful.

Theorem 3.16. *If $a_n \leq b_n$ for all $n \geq N_0$, where $N_0 \in \mathbf{N}$ is a fixed integer, then*

$$\liminf_{n\to\infty} a_n \leq \liminf_{n\to\infty} b_n, \quad \limsup_{n\to\infty} a_n \leq \limsup_{n\to\infty} b_n.$$

Remark 3.9. There is another way to define upper and lower limits. Let $\{a_n\}$ be a sequence of real numbers. Define

$$x_n = \inf\{a_k \,|\, k \geq n\} = \inf\{a_n, a_{n+1}, a_{n+2}, \cdots\},$$
$$y_n = \sup\{a_k \,|\, k \geq n\} = \sup\{a_n, a_{n+1}, a_{n+2}, \cdots\}.$$

If $\{a_n\}$ is not bounded below, then $x_n = -\infty$ for all $n \in \mathbf{N}$; if $\{a_n\}$ is bounded below, then $\{x_n\}$ is monotonically increasing, and hence, by Theorem 3.12, $\{x_n\}$ converges.

Similarly, if $\{a_n\}$ is not bounded above, then $y_n = +\infty$ for all $n \in \mathbf{N}$; if $\{a_n\}$ is bounded above, then $\{y_n\}$ is monotonically decreasing, and hence again, by Theorem 3.12, $\{y_n\}$ converges.

Therefore, the limits $\lim\limits_{n\to\infty} x_n$ and $\lim\limits_{n\to\infty} y_n$ are *always* well-defined in the extended real number system. In fact, we have the following.

Theorem 3.17.

$$\lim_{n\to\infty} x_n = \liminf_{n\to\infty} a_n, \quad \lim_{n\to\infty} y_n = \limsup_{n\to\infty} a_n.$$

3.6. Some Special Sequences

We shall use the following simple result, known as the **Squeezing Theorem**.

Lemma 3.18. *Suppose $y_n \le x_n \le z_n$ for all $n \ge N$, where $N \in \mathbf{N}$ is some fixed number. If $y_n \to l$ and $z_n \to l$, then $x_n \to l$.*

Proof. Given $\epsilon > 0$, let $N \in \mathbf{N}$ be such that, for all $n \ge N$, $|y_n - l| < \epsilon$ and $|z_n - l| < \epsilon$; thus $l - \epsilon < y_n \le x_n \le z_n < l + \epsilon$, that is, $|x_n - l| < \epsilon$ for all $n \ge N$. Hence $x_n \to l$. □

Theorem 3.19. *We have*

(a) *If $p > 0$, then $\lim \frac{1}{n^p} = 0$.*
(b) *If $p > 0$, then $\lim \sqrt[n]{p} = 1$.*
(c) $\lim \sqrt[n]{n} = 1$.
(d) *If $p > 0$ and α is real, then $\lim \frac{n^\alpha}{(1+p)^n} = 0$.*
(e) *If $x \in \mathbf{R}$ with $|x| < 1$, then $\lim x^n = 0$.*

Proof. (a) Take $N > (1/\epsilon)^{1/p}$ in the ϵ-N definition of the convergence.

(b) If $p > 1$, let $x_n = \sqrt[n]{p} - 1$. Then $x_n > 0$, and, by the binomial theorem,

$$1 + nx_n \le (1 + x_n)^n = p,$$

so that

$$0 < x_n \le \frac{p-1}{n} \to 0.$$

If $p = 1$, (b) is trivial; if $0 < p < 1$, then the result follows by taking reciprocals.

(c) Let $x_n = \sqrt[n]{n} - 1$. Then $x_n \ge 0$, and by the binomial theorem, if $n \ge 2$,

$$n = (1 + x_n)^n \ge \frac{n(n-1)}{2} x_n^2.$$

Hence

$$0 \le x_n \le \sqrt{\frac{2}{n-1}} \quad (n \ge 2).$$

(d) Let k be an integer such that $k > \alpha$, $k > 0$. For $n > 2k$,

$$(1+p)^n > \frac{n(n-1)\cdots(n-k+1)}{k!} p^k > \frac{n^k p^k}{2^k k!}.$$

Hence

$$0 < \frac{n^\alpha}{(1+p)^n} < \frac{2^k k!}{p^k} n^{\alpha - k} \quad (n > 2k).$$

Since $\alpha - k < 0$, it follows that $n^{\alpha-k} \to 0$, by (a).

(e) If $x = 0$, then (e) is trivial. Assume $0 < |x| < 1$. Then

$$|x^n| = |x|^n = \frac{1}{(1+p)^n} \to 0,$$

by (d) with $\alpha = 0$ and $p = \frac{1}{|x|} - 1 > 0$. □

3.7. Infinite Series

Definition 3.10. Let $\{b_n\}$ be a sequence of real numbers. An **infinite series**, or just a **series**, of terms b_n is a formal expression of the form

$$\sum_{n=1}^{\infty} b_n = b_1 + b_2 + b_3 + \cdots .$$

The corresponding **sequence of partial sums** $\{s_n\}$ is defined by

$$s_n = b_1 + b_2 + \cdots + b_n$$

for all $n \in \mathbf{N}$.

If $\{s_n\}$ converges to a number $s \in \mathbf{R}$, then we say that the series $\sum_{n=1}^{\infty} b_n$ **converges** (to $s \in \mathbf{R}$), and write

$$\sum_{n=1}^{\infty} b_n = s.$$

The number s is called the **sum of the series**.

If the partial sum sequence $\{s_n\}$ diverges, then we say that the infinite series $\sum_{n=1}^{\infty} b_n$ **diverges**.

Sometimes, for convenience of notation, we shall consider series of the form

$$\sum_{n=0}^{\infty} b_n = b_0 + b_1 + b_2 + \cdots .$$

Frequently, when there is no possible ambiguity, we shall write a series as $\sum b_n$, whether n starts with 0 or 1.

By studying the partial sum sequences, we easily obtain the following result.

Theorem 3.20 (Algebraic Theorem for Series). *If* $\sum_{k=1}^{\infty} a_k = A$ *and* $\sum_{k=1}^{\infty} b_k = B$, *then*

$$\sum_{k=1}^{\infty} (ca_k + db_k) = cA + dB \quad \textit{for all } c, d \in \mathbf{R}.$$

Note that there is no similar rule for the product series $\sum_{k=1}^{\infty} (a_k b_k)$ or the quotient series $\sum_{k=1}^{\infty} (a_k/b_k)$.

Theorem 3.21 (Cauchy Criterion for Series). *The series* $\sum_{k=1}^{\infty} a_k$ *converges if and only if, given any* $\epsilon > 0$, *there exists an* $N \in \mathbf{N}$ *such that whenever* $n > m \geq N$ *it follows that*

$$|a_{m+1} + a_{m+2} + \cdots + a_n| < \epsilon.$$

Proof. Note that $s_n - s_m = a_{m+1} + a_{m+2} + \cdots + a_n$. Hence the criterion is equivalent to the Cauchy criterion for the converegnce of partial sum sequence $\{s_n\}$. □

Theorem 3.22. *If* $\sum_{n=1}^{\infty} a_n$ *converges, then* $a_n \to 0$.

Proof. Let $\{s_n\}$ be the sequence of partial sums of series $\sum a_n$, and let $s_0 = 0$. If $s_n \to s$ for some $s \in \mathbf{R}$, then, since $a_n = s_n - s_{n-1}$, it follows that $a_n = s_n - s_{n-1} \to s - s = 0$. □

This easy result is often used to show a series *diverges* by showing the sequence of its terms does not converge to 0. However, it can not be used to show the convergence simply from the limit $a_n \to 0$, as seen from the examples later.

Theorem 3.23 (**Comparison Test**). *Assume $\{a_k\}$ and $\{b_k\}$ are sequences of real numbers.*

(a) *If $|a_k| \le b_k$ for all $k \ge N_0$, where $N_0 \in \mathbf{N}$ is some integer, and if $\sum_{k=1}^\infty b_k$ converges, then $\sum_{k=1}^\infty a_k$ converges.*

(b) *If $a_k \ge b_k \ge 0$ for all $k \ge N_0$, where $N_0 \in \mathbf{N}$ is some integer, and if $\sum_{k=1}^\infty b_k$ diverges, then $\sum_{k=1}^\infty a_k$ diverges.*

Proof. (a) Given $\epsilon > 0$, by the Cauchy criterion, there exists $N \in \mathbf{N}$ with $N \ge N_0$ such that for all $n, m \in \mathbf{N}$ with $n > m \ge N$ it follows that

$$b_{m+1} + b_{m+2} + \cdots + b_n < \epsilon.$$

Hence, by the triangle inequality,

$$|a_{m+1} + a_{m+2} + \cdots + a_n| \le |a_{m+1}| + |a_{m+2}| + \cdots + |a_n| \le b_{m+1} + b_{m+2} + \cdots + b_n < \epsilon,$$

which, again by the Cauchy criterion, shows that $\sum a_n$ converges.

(b) If $\sum a_n$ converges, then by (a), $\sum b_n$ must converge.

□

Remark 3.11. The comparison test (b) gives **no information** on convergence of the larger series $\sum_{k=1}^\infty a_k$ from the convergence of smaller series $\sum_{k=1}^\infty b_k$. In fact, when the smaller series $\sum_{k=1}^\infty b_k$ converges, the larger series $\sum_{k=1}^\infty a_k$ could either converge or diverge. For example, let $b_n = \frac{1}{n^4}$ and consider either sequence $a_n = \frac{1}{n}$ or sequence $a_n = \frac{1}{n^2}$. (See details later.)

Corollary 3.24 (**Absolute Convergence Test**). *If the series $\sum_{k=1}^\infty |a_k|$ converges, then $\sum_{k=1}^\infty a_k$ converges as well.*

Proof. This follows directly from (a) of Theorem 3.23 with $b_k = |a_k|$. □

Definition 3.12. We say that the series $\sum_{k=1}^\infty a_k$ **converges absolutely**, or is **absolutely convergent**, if $\sum_{k=1}^\infty |a_k|$ converges. The previous result says that an absolutely convergent series always converges.

We say that the series $\sum_{k=1}^\infty a_k$ **converges conditionally** if $\sum_{k=1}^\infty a_k$ converges but $\sum_{k=1}^\infty |a_k|$ diverges.

Geometric Series.

Theorem 3.25 (**Geometric series**). *Let $x \in \mathbf{R}$. If $|x| < 1$, then*

$$\sum_{n=0}^\infty x^n = 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}. \tag{3.5}$$

If $|x| \ge 1$, then the series $\sum\limits_{n=0}^\infty x^n$ diverges. The series $\sum\limits_{n=0}^\infty x^n$ is called the **geometric series of ratio** *x.*

Proof. If $x \ne 1$, then the partial sum sequence $\{s_n\}$ of the series $\sum_{n=0}^\infty x^n$ is given by

$$s_n = 1 + x + x^2 + \cdots + x^{n-1} = \frac{1-x^n}{1-x} \quad (n \in \mathbf{N}).$$

The sequence $\{s_n\}$ converges to $\frac{1}{1-x}$ if $|x| < 1$. If $|x| \ge 1$, then clearly $|x^n| = |x|^n \ge 1$, and hence $\sum x^n$ diverges. □

3.8. Series of Nonnegative Terms

Theorem 3.26. *A series of nonnegative terms converges if and only if its partial sum sequence is bounded above.*

Proof. Note that if $b_n \geq 0$ then the partial sum sequence $\{s_n\}$ of series $\sum b_n$ is monotonically increasing. Therefore, in this case, by the (proof of MCT), the partial sum sequence $\{s_n\}$ converges if and only if it is *bounded above.* □

Theorem 3.27 (**Cauchy's Theorem**). *Suppose $a_1 \geq a_2 \geq a_3 \geq \cdots \geq 0$. Then the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the series*

$$\sum_{k=0}^{\infty} 2^k a_{2^k} = a_1 + 2a_2 + 4a_4 + 8a_8 + 16a_{16} + \cdots$$

converges.

Proof. By Theorem 3.26, it suffices to prove that the partial sum sequence of $\sum a_n$ is bounded above if and only if the partial sum sequence of $\sum 2^k a_{2^k}$ is bounded above.

Let $s_n = a_1 + a_2 + \cdots + a_n$ and

$$t_k = \sum_{j=0}^{k} 2^j a_{2^j} = a_1 + 2a_2 + 4a_4 + \cdots + 2^k a_{2^k}.$$

For any $n \in \mathbf{N}$, if $2^k > n$, then

$$s_n \leq a_1 + (a_2 + a_3) + \cdots + (a_{2^k} + \cdots + a_{2^{k+1}-1}) \leq a_1 + 2a_2 + \cdots + 2^k a_{2^k} = t_k.$$

Hence, if $\{t_k\}$ is bounded above, then so is $\{s_n\}$.

For any $k \in \mathbf{N}$, if $n > 2^k$, then

$$\begin{aligned} s_n &\geq a_1 + a_2 + (a_3 + a_4) + \cdots + (a_{2^{k-1}+1} + \cdots + a_{2^k}) \\ &\geq \frac{1}{2}a_1 + a_2 + 2a_4 + \cdots + 2^{k-1}a_{2^k} = \frac{1}{2}t_k, \end{aligned}$$

so that $t_k \leq 2s_n$, and hence if $\{s_n\}$ is bounded above, then so is $\{t_k\}$. □

Theorem 3.28. *The series $\sum \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.*

Proof. Clearly, if $p \leq 0$, then $\frac{1}{n^p} \geq 1$ and hence $\sum \frac{1}{n^p}$ diverges. If $p > 0$ then Theorem 3.27 is applicable, and we are led to the series

$$\sum_{k=0}^{\infty} 2^k \frac{1}{2^{kp}} = \sum_{k=0}^{\infty} 2^{(1-p)k},$$

which is a geometric series with ratio $x = 2^{1-p}$; hence the result follows. □

Theorem 3.29. *Let $b > 1$. Then the series $\sum_{n=2}^{\infty} \frac{1}{n(\log_b n)^p}$ converges if $p > 1$ and diverges if $p \leq 1$.*

Proof. If $p \leq 0$, the series diverges by comparison with $\sum \frac{1}{n}$. Hence, assume $p > 0$. By the previous theorem, the series

$$\sum_{k=1}^{\infty} 2^k \frac{1}{2^k (\log_b 2^k)^p} = \frac{1}{(\log_b 2)^p} \sum_{k=1}^{\infty} \frac{1}{k^p}$$

converges if $p > 1$ and diverges if $0 < p \leq 1$. The result then follows from Cauchy's theorem. $\square$

Remark 3.13. Given $p > 1$ and $q > 1$, there exists no integer $N_0 \in \mathbf{N}$ such that

$$\frac{1}{n(\log_b n)^p} \leq \frac{1}{n^q} \quad \forall\, n \geq N_0.$$

Hence the convergence result in Theorem 3.29 cannot be deduced from Theorem 3.28 by using the comparison theorem.

3.9. The Root and Ratio Tests

Theorem 3.30. *For any sequence $\{c_n\}$ of positive numbers, it follows that*

$$\liminf_{n\to\infty} \frac{c_{n+1}}{c_n} \leq \liminf_{n\to\infty} \sqrt[n]{c_n} \leq \limsup_{n\to\infty} \sqrt[n]{c_n} \leq \limsup_{n\to\infty} \frac{c_{n+1}}{c_n}.$$

Proof. We prove $\limsup \sqrt[n]{c_n} \leq \limsup \frac{c_{n+1}}{c_n}$; the proof concerning liminf is similar.

Let $\alpha = \limsup \frac{c_{n+1}}{c_n}$. If $\alpha = +\infty$, there is nothing to prove. Assume $\alpha < +\infty$. Let $\beta > \alpha$ be any number, and choose an integer $N \in \mathbf{N}$ such that

$$\frac{c_{n+1}}{c_n} \leq \beta \quad \forall\, n \geq N;$$

that is, $c_{n+1} \leq \beta c_n$ for all $n \geq N$. Hence,

$$c_{N+p} \leq \beta^p c_N \quad \forall\, p \in \mathbf{N},$$

or $c_n \leq c_N \beta^{-N} \beta^n$ for all $n \geq N$. Hence

$$\sqrt[n]{c_n} \leq (c_N \beta^{-N})^{1/n} \beta \quad \forall\, n \geq N.$$

Thus

$$\limsup_{n\to\infty} \sqrt[n]{c_n} \leq \beta,$$

for all $\beta > \alpha$. Consequently, $\limsup \sqrt[n]{c_n} \leq \alpha$. This completes the proof. $\square$

Theorem 3.31 (Root Test). *Given a series $\sum a_n$, let $\alpha = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$ (including $\alpha = +\infty$). Then*

(a) *if $\alpha < 1$, $\sum a_n$ converges absolutely;*

(b) *if $\alpha > 1$, $\sum a_n$ diverges;*

(c) *if $\alpha = 1$, the test gives no information.*

Proof. (a) Assume $\alpha < 1$. We choose β so that $\alpha < \beta < 1$, and $N \in \mathbf{N}$ such that

$$\sqrt[n]{|a_n|} < \beta \quad \forall\, n \geq N.$$

That is, $|a_n| < \beta^n$ for all $n \geq N$. Since $0 < \beta < 1$, $\sum \beta^n$ converges; hence, by the comparison test, $\sum |a_n|$ converges, and thus $\sum a_n$ converges absolutely, by Corollary 3.24.

(b) Assume $\alpha > 1$. Let $\{a_{n_k}\}$ be a subsequence such that

$$\lim_{k\to\infty} \sqrt[n_k]{|a_{n_k}|} = \alpha.$$

Hence $|a_{n_k}| > 1$ for $k \geq K$, where $K \in \mathbf{N}$ is some integer. Therefore, $\{a_n\}$ cannot converge to 0, and hence $\sum a_n$ diverges.

(c) Consider two series $\sum \frac{1}{n}$ and $\sum \frac{1}{n^2}$; for both, we have $\alpha = 1$, but the first diverges, while the second one converges. □

From Theorem 3.30 and the **root test**, we easily have the following theorem, which sometimes may be easier to use than the root test.

Corollary 3.32 (**Ratio Test**). *The series* $\sum a_n$ *converges absolutely if* $\limsup_{n\to\infty} |\frac{a_{n+1}}{a_n}| < 1$.

EXAMPLE 3.4. Sometimes the ratio may fail, but the root test still works. Consider the series

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots ;$$

that is, $\sum a_n$, where

$$a_n = \begin{cases} (\frac{1}{2})^{\frac{n+1}{2}} & (n = 1, 3, 5, \cdots), \\ (\frac{1}{3})^{\frac{n}{2}} & (n = 2, 4, 6, \cdots), \end{cases}$$

or $a_{2k-1} = (\frac{1}{2})^k$, $a_{2k} = (\frac{1}{3})^k$ for all $k \in \mathbf{N}$. Then we have

$$\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{k\to\infty} \frac{a_{2k}}{a_{2k-1}} = \lim_{k\to\infty} (\frac{2}{3})^k = 0,$$
$$\liminf_{n\to\infty} \sqrt[n]{a_n} = \lim_{k\to\infty} \sqrt[2k]{a_{2k}} = \lim_{k\to\infty} \sqrt[2k]{3^{-k}} = \sqrt{1/3},$$
$$\limsup_{n\to\infty} \sqrt[n]{a_n} = \lim_{k\to\infty} \sqrt[2k-1]{a_{2k-1}} = \lim_{k\to\infty} \sqrt[2k-1]{2^{-k}} = \sqrt{1/2},$$
$$\limsup_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{k\to\infty} \frac{a_{2k+1}}{a_{2k}} = \lim_{k\to\infty} \frac{1}{2}(\frac{3}{2})^k = +\infty.$$

Hence, the series converges by the root test, which cannot be concluded by using the ratio test.

3.10. Summation by Parts

Lemma 3.33. *Given two sequences* $\{a_n\}$ *and* $\{b_n\}$, *let*

$$A_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n$$

if $n \in \mathbf{N}$, *and* $A_0 = 0$. *Then, if* $1 \le p \le q$, *we have*

$$\sum_{k=p}^{q} a_k b_k = A_q b_q - A_{p-1} b_p + \sum_{k=p}^{q-1} A_k (b_k - b_{k+1}).$$

Proof.

$$\sum_{k=p}^{q} a_k b_k = \sum_{k=p}^{q} (A_k - A_{k-1}) b_k = \sum_{k=p}^{q} A_k b_k - \sum_{k=p}^{q} A_{k-1} b_k = \sum_{k=p}^{q} A_k b_k - \sum_{k=p-1}^{q-1} A_k b_{k+1}$$
$$= \left(\sum_{k=p}^{q-1} A_k b_k + A_q b_q \right) - \left(A_{p-1} b_p + \sum_{k=p}^{q-1} A_k b_{k+1} \right) = A_q b_q - A_{p-1} b_p + \sum_{k=p}^{q-1} A_k (b_k - b_{k+1}).$$

□

Theorem 3.34. *Given two sequences* $\{a_n\}$ *and* $\{b_n\}$, *suppose that*

(a) *the partial sum sequence* $\{A_n\}$ *of* $\{a_n\}$ *is bounded;*

(b) $b_1 \geq b_2 \geq b_3 \geq \cdots \geq 0$ *and* $b_n \to 0$.

Then $\sum a_n b_n$ *converges.*

Proof. Let $|A_n| \leq M$ for all $n \in \mathbf{N}$, where $M > 0$ is a fixed finite number. Given $\epsilon > 0$, there is an integer $N \in \mathbf{N}$ such that $b_N < \frac{\epsilon}{2M}$. Then, for all $p, q \in \mathbf{N}$ with $q \geq p \geq N$,

$$\left|\sum_{k=p}^{q} a_k b_k\right| = \left|\sum_{k=p}^{q-1} A_k(b_k - b_{k+1}) + A_q b_q - A_{p-1} b_p\right|$$

$$\leq \sum_{k=p}^{q-1} |A_k||b_k - b_{k+1}| + |A_q||b_q| + |A_{p-1}||b_p|$$

$$\leq M\left[\sum_{k=p}^{q-1}(b_k - b_{k+1}) + b_q + b_p\right] = 2Mb_p \leq 2Mb_N < \epsilon.$$

Hence, by the Cauchy criterion, $\sum a_n b_n$ converges. □

Corollary 3.35 (**Alternating Series Test**). *Let* $\{b_n\}$ *satisfy*

$$b_1 \geq b_2 \geq b_3 \geq \cdots \geq 0, \quad b_n \to 0.$$

Then the **alternating series**

$$\sum_{n=1}^{\infty}(-1)^{n+1} b_n = b_1 - b_2 + b_3 - b_4 + \cdots$$

converges.

Proof. Apply the theorem with $a_n = (-1)^{n+1}$ and b_n. □

Example 3.5. (i) The **alternating harmonic series:**

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

converges by the **alternating series test.** But since the series of absolute values $\sum \frac{1}{n}$ diverges, this series converges **conditionally**.

(ii) The series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$$

is also a convergent alternating series, but converges **absolutely** because the absolute series $\sum \frac{1}{n^2}$ converges. So there are two tests we can use to deduce the convergence of this series; however, the **alternating series test** only asserts the convergence and does not tell whether the convergence is conditional or absolute.

(iii) Often, to determine whether a series converges or not, you should first try to use the **absolute convergence test**; if it does not work, then try to use other tests.

3.11. The Number e

Definition 3.14. Define

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots,$$

where $0! = 1$ and $n! = 1 \cdot 2 \cdot 3 \cdots n$ for $n \geq 1$. Since

$$s_n = \sum_{k=0}^{n} \frac{1}{k!} = 1 + 1 + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \cdots + \frac{1}{1 \cdot 2 \cdots n}$$
$$< 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} < 3,$$

it follows from Theorem 3.26 that the series $\sum_{n=0}^{\infty} \frac{1}{n!}$ converges, and the definition makes sense.

Theorem 3.36.

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e.$$

Proof. Let

$$s_n = \sum_{k=0}^{n} \frac{1}{k!}, \quad t_n = \left(1 + \frac{1}{n}\right)^n \tag{3.6}$$

By the binomial theorem,

$$t_n = \sum_{k=0}^{n} \frac{n(n-1)\cdots(n-k+1)}{k!n^k} = \sum_{k=0}^{n} \frac{1}{k!}(1 - \frac{1}{n})(1 - \frac{2}{n})\cdots(1 - \frac{k-1}{n}).$$

(Note that in this expression the first two terms when $k = 0, 1$ are both taken to be 1.) Hence $t_n \leq s_n$, so that

$$\limsup t_n \leq e.$$

Next, if $n \geq m \geq 2$, then

$$t_n \geq 1 + 1 + \frac{1}{2!}(1 - \frac{1}{n}) + \cdots + \frac{1}{m!}(1 - \frac{1}{n})(1 - \frac{2}{n})\cdots(1 - \frac{m-1}{n}).$$

Keep m fixed and let $n \to \infty$, and we get

$$\liminf t_n \geq 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{m!},$$

so that $s_m \leq \liminf t_n$, for all $m \geq 2$, which gives

$$e \leq \liminf t_n.$$

Therefore, by Theorem 3.14, $\lim t_n = e$. □

Lemma 3.37. *Let s_n be defined by* (3.6)*. Then*

$$0 < e - s_n < \frac{1}{n!n}.$$

Proof. Clearly,

$$e - s_n = \sum_{k=n+1}^{\infty} \frac{1}{k!} < \frac{1}{(n+1)!} \sum_{k=0}^{\infty} \frac{1}{(n+1)^k} = \frac{1}{n!n}.$$

□

Theorem 3.38. *e is irrational.*

Proof. Suppose e is rational. Then $e = p/q$, where $p, q \in \mathbf{N}$. By Lemma 3.37,

$$0 < q!(e - s_q) < \frac{1}{q}.$$

Clearly,

$$q!s_q = q!(1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{q!})$$

is an integer, and by assumption, $q!e = (q-1)!p \in \mathbf{N}$; hence, $q!(e - s_q)$ is an integer, which contradicts with $0 < q!(e - s_q) < \frac{1}{q}$. ☐

3.12. Power Series

Definition 3.15. Given a sequence $\{a_n\}_{n=0}^{\infty}$ of real numbers, the series

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

is called a **power series**. The numbers a_n are called the **coefficients** of the power series, and x is a real number.

In general, the convergence of a power series depends on the choice of variable x. On the set of all x such that the power converges, the sum $\sum a_n x^n$ defines a function. The following theorem characterizes this set.

Theorem 3.39. *Given a power series $\sum a_n x^n$, let*

$$\alpha = \limsup_{n\to\infty} \sqrt[n]{|a_n|}, \quad R = \frac{1}{\alpha},$$

(if $\alpha = 0$, then $R = +\infty$; if $\alpha = +\infty$, then $R = 0$). This number R is called the **radius of convergence** of the power series.

Then, the series $\sum a_n x^n$ converges absolutely if $|x| < R$, and diverges if $|x| > R$. The series may converge or diverge if $|x| = R$.

Proof. Let $b_n = a_n x^n$. Then $\sqrt[n]{|b_n|} = |x| \sqrt[n]{|a_n|}$ and hence

$$\limsup_{n\to\infty} \sqrt[n]{|b_n|} = |x| \limsup_{n\to\infty} \sqrt[n]{|a_n|} = |x|/R.$$

Consequently, the conclusion follows from the root test. ☐

Corollary 3.40. *If a power series $\sum a_n x^n$ converges at some $x = x_0 \neq 0$, then $\sum a_n x^n$ converges absolutely for all x with $|x| < |x_0|$.*

Proof. If $\sum a_n x^n$ converges at $x = x_0 \neq 0$, then its radius of convergence $R \geq |x_0|$; hence $\sum a_n x^n$ converges absolutely for all x with $|x| < |x_0| \leq R$. ☐

EXAMPLE 3.6.
(1) The power series $\sum n^n x^n$ has $R = 0$, and only converges when $x = 0$.
(2) The power series $\sum \frac{x^n}{n!}$ has $R = +\infty$ (easily seen using the ratio test), and converges for all $x \in \mathbf{R}$.
(3) The power series $\sum \frac{x^n}{n^2}$ has $R = 1$; it also converges for all $x \in \mathbf{R}$ with $|x| \leq 1$.
(4) The power series $\sum \frac{x^n}{n}$ has $R = 1$; it converges if $x = -1$, but diverges if $x = 1$.

3.13. Multiplication of Series

Definition 3.16. Given $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$, let

$$c_n = \sum_{k=0}^{n} a_k b_{n-k} \quad (n = 0, 1, 2, \cdots).$$

Then the series $\sum c_n$ is called the **product** of the two given series.

To motivate this definition, let us consider two power series

$$\sum_{n=0}^{\infty} a_n x^n, \quad \sum_{n=0}^{\infty} b_n x^n.$$

If we multiply the two series term-by-term and collect the like terms, we get

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} c_n x^n.$$

Setting $x = 1$, we arrive at the above definition.

EXAMPLE 3.7. We show that even if $\sum a_n$ and $\sum b_n$ both converge the product series $\sum c_n$ may diverge.

Let $a_n = b_n = \frac{(-1)^n}{\sqrt{n+1}}$. Then, by the alternating series test, $\sum a_n$ converges. Compute

$$c_n = \sum_{k=0}^{n} a_k a_{n-k} = \sum_{k=0}^{n} \frac{(-1)^k}{\sqrt{k+1}} \frac{(-1)^{n-k}}{\sqrt{n-k+1}} = (-1)^n \sum_{k=0}^{n} \frac{1}{\sqrt{(n-k+1)(k+1)}}.$$

Note that, for all k,

$$(n-k+1)(k+1) = (\frac{n}{2}+1)^2 - (\frac{n}{2}-k)^2 \leq (\frac{n}{2}+1)^2;$$

hence,

$$|c_n| \geq \sum_{k=0}^{n} \frac{2}{n+2} = \frac{2(n+1)}{n+2} \geq 1.$$

Hence $\sum c_n$ diverges.

However, we have the following result.

Theorem 3.41. *Suppose $\sum a_n$ converges absolutely and $\sum b_n$ converges. Let*

$$\sum_{n=0}^{\infty} a_n = A, \quad \sum_{n=0}^{\infty} b_n = B, \quad c_n = \sum_{k=0}^{n} a_k b_{n-k} \quad (n = 0, 1, 2, \cdots).$$

Then $\sum c_n$ converges, and

$$\sum_{n=0}^{\infty} c_n = AB.$$

Proof. Let

$$A_n = \sum_{k=0}^{n} a_k, \quad B_n = \sum_{k=0}^{n} b_k, \quad C_n = \sum_{k=0}^{n} c_k.$$

Let $\beta_n = B_n - B$, and let

$$\gamma_n = \sum_{k=0}^{n} a_k \beta_{n-k} = a_0\beta_n + a_1\beta_{n-1} + \cdots + a_n\beta_0.$$

Then

$C_n = a_0b_0 + (a_0b_1 + a_1b_0) + \cdots + (a_0b_n + a_1b_{n-1} + \cdots + a_nb_0) = a_0B_n + a_1B_{n-1} + \cdots + a_nB_0$
$= a_0(B+\beta_n)+a_1(B+\beta_{n-1})+\cdots+a_n(B+\beta_0) = A_nB+a_0\beta_n+a_1\beta_{n-1}+\cdots+a_n\beta_0 = A_nB+\gamma_n.$

It suffices to show that $\gamma_n \to 0$. Let $\epsilon > 0$ be given. Let $\alpha = \sum |a_n|$. Since $\beta_n \to 0$, we can choose an $N \in \mathbf{N}$ such that $|\beta_n| < \epsilon$ for all $n \geq N$. Hence, if $n \geq N$, then

$$|\gamma_n| \leq |\beta_0 a_n + \cdots + \beta_N a_{n-N}| + |\beta_{N+1}a_{n-N-1} + \cdots + \beta_n a_0|$$
$$\leq |\beta_0||a_n| + \cdots + |\beta_N||a_{n-N}| + \epsilon\alpha.$$

Keeping N fixed, and letting $n \to \infty$, we get

$$\limsup_{n\to\infty} |\gamma_n| \leq \epsilon\alpha,$$

since $a_k \to 0$ as $k \to \infty$. Since $\epsilon > 0$ is arbitrary, we have proved $\lim |\gamma_n| = 0$. □

Theorem 3.42 (Abel's theorem). *If $\sum a_n$, $\sum b_n$ and their product series $\sum c_n$ all converge, then*

$$\left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right) = \sum_{n=0}^{\infty} c_n.$$

Proof. The proof uses the continuity of power series, and will not be discussed here. □

3.14. Rearrangements

Definition 3.17. Given two series $\sum\limits_{n=1}^{\infty} a_n$ and $\sum\limits_{n=1}^{\infty} b_n$, we say that $\sum\limits_{n=1}^{\infty} b_n$ is a **rearrangement** of $\sum\limits_{n=1}^{\infty} a_n$ if there exists a *1-1 function f from* $\mathbf{N}$ *onto* $\mathbf{N}$ such that

$$b_n = a_{f(n)} \quad \forall\, n \in \mathbf{N}.$$

If $\sum b_n$ is a rearrangement of $\sum a_n$, we see that every term of $\sum b_n$ appears *exactly once* in $\sum a_n$ and, vice-versa, every term of $\sum a_n$ appears *exactly once* in $\sum b_n$. However, their partial sum sequences may differ greatly.

Now, if a series $\sum a_n$ and one of its rearrangements $\sum b_n$ both converge, do we have that $\sum b_n = \sum a_n$? The answer is no, in general.

Example 3.8. Consider the convergent alternating harmonic series

$$S = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots.$$

Let s_n be the partial sum. Then $s_{2k} > \frac{1}{2}$ and hence $S \geq \frac{1}{2}$. Also

$$\frac{1}{2}S = \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \cdots + (-1)^{n+1}\frac{1}{2n} + \cdots$$
$$= 0 + \frac{1}{2} + 0 - \frac{1}{4} + 0 + \frac{1}{6} + 0 - \frac{1}{8} + \cdots + 0 + \frac{1}{2(2k-1)} + 0 - \frac{1}{4k} + 0 + \cdots,$$

where all the odd terms are 0. So

$$S + \frac{1}{2}S = \frac{3}{2}S = 1 + 0 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + 0 + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + 0 + \frac{1}{11} - \frac{1}{6} + \cdots$$
$$= 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \cdots,$$

which becomes a rearrangement of $S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$; but certainly their sums are not equal since $S \neq 0$.

In fact, Riemann proved the following theorem.

Theorem 3.43 (Riemann's theorem). *Let $\sum a_n$ converge conditionally. Suppose $-\infty \leq \alpha \leq \beta \leq +\infty$. Then there exists a rearrangement $\sum b_n$ of $\sum a_n$ whose partial sum sequence $\{B_n\}$ satisfies*

$$\liminf_{n\to\infty} B_n = \alpha, \quad \limsup_{n\to\infty} B_n = \beta. \tag{3.7}$$

In particular, for any $-\infty \leq \alpha \leq +\infty$, there exists a rearrangement $\sum b_n$ of $\sum a_n$ such that $\sum b_n = \alpha$.

Proof. Let

$$p_n = \frac{|a_n| + a_n}{2}, \quad q_n = \frac{|a_n| - a_n}{2} \quad (n \in \mathbf{N}).$$

Then $p_n - q_n = a_n, p_n + q_n = |a_n|, p_n \geq 0, q_n \geq 0$. Since $\sum a_n$ converges but $\sum |a_n|$ diverges, it follows that both $\sum p_n$ and $\sum q_n$ must diverge.

Now let $P_1, P_2, P_3, \cdots$ denote the nonnegative terms in the sequence $\{a_n\}$ in the order they occur, and let $Q_1, Q_2, Q_3, \cdots$ denote the absolute values of the negative terms in $\{a_n\}$, also in their original order.

The series $\sum P_n, \sum Q_n$ differ from $\sum p_n$, $\sum q_n$ only by zero terms, and hence both diverge.

We shall construct sequences $\{m_n\}, \{k_n\}$ of increasing positive integers, such that the series

$$P_1 + P_2 + \cdots + P_{m_1} - Q_1 - Q_2 - \cdots - Q_{k_1} + P_{m_1+1} + \cdots + P_{m_2} - Q_{k_1+1} - \cdots - Q_{k_2} + \cdots, \tag{3.8}$$

which clearly is a rearrangement of $\sum a_n$, satisfies (3.7).

Let $\{\alpha_n\}, \{\beta_n\}$ be real sequences, such that $\alpha_n \to \alpha, \beta_n \to \beta, \alpha_n < \beta_n$ and $\beta_1 > 0$.

Let m_1, k_1 be the smallest positive integers such that

$$P_1 + P_2 + \cdots + P_{m_1} > \beta_1, \quad P_1 + P_2 + \cdots + P_{m_1} - Q_1 - \cdots - Q_{k_1} < \alpha_1;$$

let $m_2 > m_1, k_2 > k_1$ be the smallest positive integers such that

$$P_1 + P_2 + \cdots + P_{m_1} - Q_1 - \cdots - Q_{k_1} + P_{m_1+1} + \cdots + P_{m_2} > \beta_2,$$

$$P_1 + P_2 + \cdots + P_{m_1} - Q_1 - \cdots - Q_{k_1} + P_{m_1+1} + \cdots + P_{m_2} - Q_{k_1+1} - \cdots - Q_{k_2} < \alpha_2;$$

this process continues without stopping in a finite number of steps since $P_n \to 0, Q_n \to 0$ and both $\sum P_n$, $\sum Q_n$ diverge.

If x_n, y_n denote the partial sums of (3.8) whose last terms are P_{m_n}, $-Q_{k_n}$, respectively, then $x_n - P_{m_n} \leq \beta_n < x_n$ and $y_n < \alpha_n \leq y_n + Q_{k_n}$; hence

$$|x_n - \beta_n| \leq P_{m_n} \to 0, \quad |y_n - \alpha_n| \leq Q_{k_n} \to 0.$$

Hence $x_n \to \beta$, $y_n \to \alpha$.

For every $k \in \mathbf{N}$, let S_k be the partial of (3.8) of first k terms. The last term of S_k is either P_j for some $m_{i-1} + 1 \leq j \leq m_i$, where $i = i(k) \in \mathbf{N}$ is such that $i(k) \to \infty$ as $k \to \infty$, or Q_p for some $k_{q-1} + 1 \leq p \leq k_q$, where $q = q(k) \in \mathbf{N}$ is such that $q(k) \to \infty$ as $k \to \infty$.

In the first case, if $j = m_i$ then $S_k = x_i$; if $m_{i-1} + 1 \leq j < m_i$ then $\alpha_i \leq S_k \leq \beta_i$. In the second case, if $p = k_q$ then $S_k = y_q$; if $k_{q-1} \leq p < k_q$ then $\alpha_q \leq S_k \leq \beta_q$. Therefore, any convergent subsequence of $\{S_k\}$ can only have a limit in $[\alpha, \beta]$. This proves (3.7). □

However, the situation is totally different if $\sum a_n$ converges absolutely.

Theorem 3.44. *If* $\sum_{n=1}^{\infty} |a_n|$ *converges, then, for any rearrangement function (i.e., 1-1 correspondence)* $f\colon \mathbf{N} \to \mathbf{N}$, *it follows that*

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a_{f(n)}.$$

Proof. Let $b_k = a_{f(k)}$ for $k \in \mathbf{N}$. For $n, m \in \mathbf{N}$, define

$$s_n = a_1 + a_2 + \cdots + a_n; \quad t_m = b_1 + b_2 + \cdots + b_m.$$

Let $s_n \to A$ as $n \to \infty$. We show that $t_m \to A$ as $m \to \infty$. Given any $\epsilon > 0$, we find an $N \in \mathbf{N}$ such that

$$|s_N - A| < \epsilon/2, \qquad \sum_{k=m+1}^{n} |a_k| < \epsilon/2 \quad \forall\ n > m \geq N.$$

Since $f\colon \mathbf{N} \to \mathbf{N}$ is 1-1 and onto, let $\{i_1, i_2, \cdots, i_N\} \subseteq \mathbf{N}$ be such that $f(i_k) = k$ for each $k = 1, 2, \cdots, N$. Let

$$M = \max\{i_1, i_2, \cdots, i_N\}.$$

Then $M \geq N$. Let $m \in \mathbf{N}$ be such that $m \geq M$. Then, since $\{i_1, i_2, \cdots, i_N\} \subseteq \{1, 2, 3, \cdots, m\}$, it follows that

$$\begin{aligned} t_m = b_1 + b_2 + \cdots + b_m &= a_{f(1)} + a_{f(2)} + \cdots + a_{f(m)} \\ &= a_{f(i_1)} + a_{f(i_2)} + \cdots + a_{f(i_N)} + \sum_{j \in J} a_{f(j)} \\ &= a_1 + a_2 + \cdots + a_N + \sum_{j \in J} a_{f(j)} \\ &= s_N + \sum_{j \in J} a_{f(j)}, \end{aligned}$$

where $J = \{1, 2, 3, \cdots, m\} \setminus \{i_1, i_2, \cdots, i_N\}$. Since $J \cap \{i_1, i_2, \cdots, i_N\} = \emptyset$, we have $f(j) \geq N + 1$ for all $j \in J$. Let $K = \max\{f(j) \,:\, j \in J\} \geq N + 1$. Then $N + 1 \leq f(j) \leq K$ for all $j \in J$ and hence

$$\left| \sum_{j \in J} a_{f(j)} \right| \leq \sum_{j \in J} |a_{f(j)}| \leq \sum_{k=N+1}^{K} |a_k| < \epsilon/2.$$

Finally, it follows that, for all $m \geq M$,

$$\begin{aligned} |t_m - A| &\leq |s_N - A| + \left| \sum_{j \in J} a_{f(j)} \right| \\ &< \epsilon/2 + \sum_{j \in J} |a_{f(j)}| < \epsilon/2 + \epsilon/2 = \epsilon. \end{aligned}$$

This proves $t_m \to A$ as $m \to \infty$. □

Suggested Homework Problems

Pages 79 – 82

Problems: 2–8, 11, 12, 14 (assume $\{s_n\}$ is real), 16(a), 17(a-c), 23–25