(1) Let $f : (a, b) \to \mathbb{R}$ be an increasing function and $c \in (a, b)$. We have used the following equalities in lecture:

\[ \lim_{x \to c^+} f(x) = \inf \{ f(x) : x \in (c, b) \}, \quad \lim_{x \to c^-} f(x) = \sup \{ f(x) : x \in (a, c) \}. \]

(a) (10 points) Prove the *second equality* above.
(b) (5 points) For $c, d \in (a, b)$ with $c < d$, use the given two equalities to show that

\[ \lim_{x \to c^+} f(x) \leq \lim_{x \to d^-} f(x). \]
(2) (10 points) Let $h$ be a differentiable function defined on the interval $[0, 3]$, and assume that $h(0) = 1$, $h(1) = 2$, and $h(3) = 2$.

(a) Argue that there exists a point $d \in [0, 3]$ where $h(d) = d$.

(b) Argue that at some point $c$ we have $h'(c) = 1/3$.

(c) Argue that $h'(x) = 1/2$ at some point in the domain.

(3) (5 points) A fixed point of a function $f$ is a number $x$ with $f(x) = x$. If $f$ is differentiable on an interval with $f'(x) \neq 1$, show that $f$ can have at most one fixed point in the interval.
(4) (10 points) Let \( a = p/q \) be a rational number with \( p, q \in \mathbb{N} \) and \( q \) odd, and let

\[
g(x) = \begin{cases} 
  x^a \sin(1/x) & \text{if } x \neq 0 \\
  0 & \text{if } x = 0.
\end{cases}
\]

(a) Find a number \( a \) so that function \( g \) is differentiable on \( \mathbb{R} \) but \( g' \) is unbounded on \([0, 1]\).

(b) Find a number \( a \) so that function \( g \) is differentiable on \( \mathbb{R} \) with \( g' \) continuous but not differentiable at zero.

(c) Find a number \( a \) so that function \( g \) is differentiable on \( \mathbb{R} \) and \( g' \) is differentiable on \( \mathbb{R} \), but \( g'' \) is not continuous at zero.