(1) (8 points) Assume that \((a_n)\) is a bounded sequence with the property that every convergent subsequence of \((a_n)\) converges to the same limit \(a \in \mathbb{R}\). Show that \((a_n)\) must converge to \(a\).

**Proof.** Suppose \((a_n)\) does not converge to \(a\). So \(\exists \epsilon_0 > 0\) such that for each integer \(N\) there is an integer \(n = n(N) \geq N\) with
\[
|a_n - a| \geq \epsilon_0.
\]
For \(N = 1\) we obtain \(n_1 = n(1) \geq 1\) such that \(|a_{n_1} - a| \geq \epsilon_0\). Once \(n_k\) is chosen, we use \(N = n_k + 1\) to obtain \(n_{k+1} = n(N) \geq n_k + 1\) such that \(|a_{n_{k+1}} - a| \geq \epsilon_0\). In this way, we obtain a subsequence \((a_{n_k})\) \((k = 1, 2, \ldots, n_k \to \infty)\) such that
\[
|a_{n_k} - a| \geq \epsilon_0 \quad \forall k = 1, 2, \ldots.
\]
Since \((a_{n_k})\) is bounded, by the Bolzano-Weierstrass Theorem, there exists a subsequence of \((a_{n_k})\) which is convergent. We denote this subsequence by \((a_{n_{kj}})\) with \(k_j \to \infty\) as \(j \to \infty\). This convergent subsequence is also a subsequence of the original sequence \((a_n)\) and, by assumption, it converges to \(a\). However \(|a_{n_{kj}} - a| \geq \epsilon_0\) for all \(j = 1, 2, \ldots\); so \(\lim(a_{n_{kj}}) \neq a\), a contradiction. This proves that \(\lim(a_n) = a\).

(2) (8 points) Give an example of each of the following, or argue that such a request is impossible.
(a) A sequence that is Cauchy but is not monotone.
(b) A sequence that is monotone, but is not Cauchy.
(c) A sequence that is unbounded and contains a subsequence that is Cauchy.

**Answers:**
(a) The sequence \((\frac{(-1)^n}{n})\) converges to 0 and hence is Cauchy, but is not monotone.
(b) The sequence \((n) = (1, 2, 3, \ldots)\) is monotone (increasing), but is not Cauchy since it is unbounded.
(c) The sequence \((n^{(-1)n})\) is unbounded and contains a convergent (thus Cauchy) subsequence.
Let \((a_n)\) be a given sequence. For each \(n \in \mathbb{N}\), define \(p_n = a_n\) if \(a_n > 0\), and \(p_n = 0\) if \(a_n \leq 0\). In a similar manner, define \(q_n = 0\) if \(a_n > 0\), and \(q_n = a_n\) if \(a_n \leq 0\).

(a) Show that, if \(\sum a_n\) diverges, then at least one of \(\sum p_n\) or \(\sum q_n\) diverges.
(b) Show that if \(\sum a_n\) converges conditionally, then both \(\sum p_n\) and \(\sum q_n\) diverges.

**Proof.** By the definitions of \(p_n\) and \(q_n\), we have
\[
a_n = p_n + q_n, \quad |a_n| = p_n - q_n.
\]
(a) Assume \(\sum a_n\) diverges. We show, by contradiction, that at least one of \(\sum p_n\) or \(\sum q_n\) diverges.
Suppose both \(\sum p_n\) and \(\sum q_n\) converge. Then, since \(a_n = p_n + q_n\), we have that \(\sum a_n\) also converges, a contradiction.
(b) Assume \(\sum a_n\) converges conditionally; that is, \(\sum a_n\) converges but \(\sum |a_n|\) diverges.
If \(\sum p_n\) converges then, since \(q_n = a_n - p_n\), \(\sum q_n\) also converges and, since \(|a_n| = p_n - q_n\), we have \(\sum |a_n|\) converges, a contradiction; so \(\sum p_n\) must diverge.
In a totally similar way, \(\sum q_n\) must also diverge.

(4) (9 points) Let \((a_n)\) be a sequence.
(a) Show that if the series \(\sum a_n\) converges absolutely, then the series \(\sum a_n^2\) converges.
(b) If \(\sum a_n\) converges, is it true that the series \(\sum a_n^2\) must converge? Justify your answer.
(c) If \(\sum a_n\) converges and \(a_n \geq 0\), then the series \(\sum \sqrt{a_n}\) may converge or may diverge. Provide example to both cases.

(a) **Proof.** If \(\sum |a_n|\) converges then \((a_n) \to 0\) and hence is bounded: \(|a_n| \leq M\) for all \(n\). Hence \(a_n^2 = |a_n|^2 \leq M|a_n|\). Therefore, by the **Comparison Test**, \(\sum a_n^2\) also converges absolutely.
(b) **Answer.** \(\sum a_n^2\) may not converge if \(\sum a_n\) converges only conditionally. For example, \(a_n = (-1)^n/\sqrt{n}\). Then \(\sum a_n\) converges conditionally by the **Alternating Series Test**, but \(\sum a_n^2 = \sum 1/n\) is the famous divergent harmonic series.
(c) **Examples:**
(i) \(\sum 1/n^2\) is convergent, but \(\sum 1/n\) is divergent.
(ii) \(\sum 1/n^2\) is convergent, but \(\sum 1/n^4\) is also convergent.
(5) (8 points) Let \( B = \left\{ \frac{(-1)^n}{n+1} \mid n \in \mathbb{N} \right\} \).

(a) Find all the limit points of \( B \). Make sure that there are no other limit points.
(b) Show that every point of \( B \) is an isolated point of \( B \).
(c) Find the closure of \( B \).

**Solution:**

(a) Consider two sequences \( (a_k) = (-2k-1) \) and \( (b_k) = (2k+1) \) in \( B \). Then
\[
\lim(a_k) = -1, \quad \lim(b_k) = 1.
\]
Since \( \pm 1 \notin B \), we have that \( \pm 1 \) are limit points of \( B \). We show that they are the only limit points of \( B \).

Let \( a \) be any limit point of \( B \). Then \( \exists x_n \in B, x_n \neq a \) such that \( (x_n) \to a \). Consider the set of positive terms of \( (x_n) \) and the set of negative terms of \( (x_n) \). Then at least one of the two sets is infinite set (in fact, exactly one set is infinite). This infinite set with the same order as in the sequence \( (x_n) \) becomes a subsequence of \( (x_n) \) and hence converges to \( a \). But this subsequence is also a subsequence of either \( (a_k) \) or \( (b_k) \) defined above and hence must converge to either \(-1 \) or \( 1 \). Therefore, either \( a = -1 \) or \( a = 1 \); hence, \( \pm 1 \) are the only limit points of \( B \).

(b) Every point of \( B \) is an isolated point of \( B \) since it is not equal to either \(-1 \) or \( 1 \) and hence is not a limit point of \( B \).

(c) The closure of \( B \) is \( \bar{B} = B \cup \{-1, 1\} \).

(6) (6 points) Let \( A \subseteq \mathbb{R} \) be a nonempty and bounded set. Show that \( \sup A \in \bar{A} \).

**Solution.** We first prove \( \sup A \in \bar{A} \). Let \( s = \sup A \). Then \( s \geq x \) for all \( x \in A \). Also, there exists \( x_n \in A \) such that \( x_n > s - \frac{1}{n} \). Hence \( s - \frac{1}{n} < x_n \leq s \) for all \( n \). This implies \( |x_n - s| < \frac{1}{n} \) and hence \( (x_n) \to s \).

If \( s \in A \), then \( s \in \bar{A} \). Now assume \( s \notin A \). In this case, \( x_n \neq s \) because \( x_n \in A \). Hence \( s \) is a limit point of \( A \); thus \( s \in \bar{A} \). Therefore, in all cases, \( s = \sup A \in \bar{A} \).

However, \( s = \sup A \) may not be a limit point of \( A \).

**Example:** Let \( A = \{1, 2, 3\} \). Then \( \sup A = 3 \) and 3 is not a limit point of \( A \); in fact, \( A \) has no limit point.
(7) (6 points) Determine whether the following statements are true or false. If the statement is true, supply a short proof, and if the statement is false, provide a counterexample.

(a) If \( F_1 \supseteq F_2 \supseteq F_3 \supseteq F_4 \supseteq \cdots \) is a nested sequence of nonempty closed sets, then the intersection \( \bigcap_{n=1}^{\infty} F_n \neq \emptyset \).

(b) An arbitrary intersection of compact sets is compact.

Answers:

(a) The statement is False. The intersection \( \bigcap_{n=1}^{\infty} F_n \) could be empty.

Example: Let \( F_n = [n, \infty) \) for \( n \in \mathbb{N} \). Then \( F_1 \supseteq F_2 \supseteq F_3 \supseteq F_4 \supseteq \cdots \) is a nested sequence of nonempty closed sets, but \( \bigcap_{n=1}^{\infty} F_n = \emptyset \).

Proof: If \( a \in \bigcap_{n=1}^{\infty} F_n \), then \( a \geq n \) for all \( n \in \mathbb{N} \), which contradicts with AP(i).

(b) The statement is True.

Proof: Let \( F_\alpha \) be compact for all \( \alpha \in \Lambda \). Let \( F = \bigcap_{\alpha \in \Lambda} F_\alpha \). By Heine-Borel Theorem, each \( F_\alpha \) is bounded and closed and thus the intersection \( F \) is also bounded and closed; hence \( F \) is compact.

(8) (6 points) Determine whether the following statements are true or false. If the statement is true, supply a short proof, and if the statement is false, provide a counterexample.

(a) \( A_1 \cup A_2 \cup \cdots \cup A_n = \overline{A_1} \cup \overline{A_2} \cup \cdots \cup \overline{A_n} \).

(b) \( \overline{A \cap B} = \overline{A} \cap \overline{B} \).

Answers:

(a) The statement is True.

Proof: Since \( A_1 \cup A_2 \cup \cdots \cup A_n \subseteq \overline{A_1} \cup \overline{A_2} \cup \cdots \cup \overline{A_n} \) and \( \overline{A_1} \cup \overline{A_2} \cup \cdots \cup \overline{A_n} \) is closed, we have

\[
\overline{A_1} \cup \overline{A_2} \cup \cdots \cup \overline{A_n} \subseteq \overline{A_1 \cup A_2 \cup \cdots \cup A_n}.
\]

Since each \( \overline{A_k} \subseteq \overline{A_1 \cup A_2 \cup \cdots \cup A_n} \), we have

\[
\overline{A_1} \cup \overline{A_2} \cup \cdots \cup \overline{A_n} \subseteq \overline{A_1 \cup A_2 \cup \cdots \cup A_n}.
\]

Therefore,

\[
\overline{A_1 \cup A_2 \cup \cdots \cup A_n} = \overline{A_1} \cup \overline{A_2} \cup \cdots \cup \overline{A_n}.
\]

(In this proof, we repeatedly use the fact that if \( A \subseteq B \) and \( B \) is closed then \( \overline{A} \subseteq B \).)

(b) The statement is False. It is possible that \( \overline{A \cap B} \neq \overline{A} \cap \overline{B} \).

Example: Let \( A = (-1, 0) \), \( B = (0, 1) \). Then \( A \cap B = \emptyset \) and hence \( \overline{A \cap B} = \emptyset \). But, \( \overline{A} = [-1, 0] \), \( \overline{B} = [0, 1] \) and \( \overline{A \cap B} = \{0\} \).