UNIQUENESS RESULTS ON SURFACES WITH BOUNDARY

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1. Introduction

The following theorem is proved by Bidaut-Veron and Veron [BVV] .

Theorem 1. *Let (M^n, g) be a compact Riemannian manifold and $u \in C^\infty(M)$ a positive solution of the following equation*

$$-\Delta u + \lambda u = u^q,$$

where $\lambda > 0$ is a constant and $1 < q \leq (n+2)/(n-2)$. If $Ric \geq \frac{(n-1)(q-1)\lambda}{n} g$, then u must be constant unless $q = (n+2)/(n-2)$ and (M^n, g) is isometric to $\left(\mathbb{S}^n, \frac{4\lambda}{n(n-2)} g_0\right)$. In the latter case u is given on $\mathbb{S}^n$ by the following formula

$$u = \frac{1}{(a + x \cdot \xi)^{(n-2)/2}}.$$

for some $\xi \in \mathbb{R}^{n+1}$ and some constant $a > |\xi|$.

This uniqueness result has important corollaries on sharp Sobolev inequalities and the Yamabe invariant. Ilias [I] showed that the method of Bidaut-Veron and Veron works for the same equation on a compact Riemannian manifold with convex boundary, provided that u satisfies the boundary condition $\frac{\partial u}{\partial \nu} = 0$. This corresponds to the 1st type of Yamabe problem on manifolds with boundary, namely finding a conformal metric with constant scalar curvature and zero mean curvature on the boundary. It is natural to wonder if similar results hold for the other type of Yamabe problem which is about a conformal metric with zero scalar curvature and constant mean curvature on the boundary. We hope to address this problem elsewhere and confine our study in this paper to an analogous problem in dimension two.

Let (Σ, g) be a compact surface with nonnegative curvature and strictly convex boundary, i.e. Gaussian curvature $K \geq 0$ and on the boundary the geodesic curvature $\kappa > 0$. By scaling we will always assume $\kappa \geq 1$. Consider the following equation

$$\begin{aligned} \Delta u &= 0 \quad \text{on} \quad \Sigma, \\ \frac{\partial u}{\partial \nu} + \lambda &= e^u \quad \text{on} \quad \partial\Sigma, \end{aligned} \tag{1.1}$$

where λ is a constant and $\frac{\partial u}{\partial \nu}$ is the derivative w.r.t. the outer unit normal ν on the boundary. When $\lambda \leq 0$ there is no solution, so we will always assume $\lambda > 0$.

Theorem 2. *If $\lambda < 1$ then u is constant; if $\lambda = 1$ and u is not constant, then Σ is isometric to the unit disc $\overline{\mathbb{B}^2}$ and u is given by*

$$u(z) = \log \frac{1 - |a|^2}{1 + |a|^2 |z|^2 - 2\,\mathrm{Re}\,(z\overline{a})},$$

for some $a \in \mathbb{B}^2$.

Instead of an integral method, our proof is based on a pointwise analysis using the strong maximum principle. It is inspired by Payne [P] and Escobar [E] in which a sharp estimate for the 1st Stekloff eigenvalue was established by a similar method. In high dimensions this method does not seem to work, but see [HW1] where this type of method is applied successfully in a special situation.

The equation (1.1) for $\lambda = 1$ on $\overline{\mathbb{B}^2}$ has a special structure and has been studied by many authors by various different methods, c.f. [OPS, Zh, HW2]. One may also wonder what happens if $\lambda > 1$. In [OPS] it is proved that the equation (1.1) has only a constant solution if $\lambda \notin \mathbb{Z}$. Using their method we are able to resolve the case when $\lambda \in \mathbb{Z}$. We state the result for a slightly different equation which is equivalent to (1.1) by a simple translation $u \to u + c$.

Theorem 3. *Suppose u is a smooth function on $\overline{\mathbb{B}^2}$ satisfying the following equation*

$$\begin{aligned} \Delta u &= 0 && on \quad \overline{\mathbb{B}^2}, \\ \tfrac{\partial u}{\partial \nu} &= \lambda \left(e^u - 1\right) && on \quad \mathbb{S}^1, \end{aligned}$$

where $\lambda > 0$ is a constant (the case $\lambda \leq 0$ is trivial by the maximal principle). Then

(1) *If $\lambda \notin \mathbb{N}$ then $u \equiv 1$.*
(2) *If $\lambda = N \in \mathbb{N}$ then either $u \equiv 1$ or*

$$u(z) = \log \frac{|\xi|^2 - 1}{|\xi - z^N|^2},$$

for some $\xi \in \mathbb{C}$ with $|\xi| > 1$.

Finally from the uniqueness result we deduce the following geometric inequality.

Theorem 4. *For $u \in C^\infty(\Sigma)$*

$$\frac{1}{2L}\int_\Sigma |\nabla u|^2 + \frac{1}{L}\int_{\partial\Sigma} u - \log\left(\frac{1}{L}\int_{\partial\Sigma} e^u\right) \geq 0,$$

where L is the perimeter of $\partial\Sigma$. Moreover if there exists a nonconstant extremal function then Σ is isometric to $\overline{\mathbb{B}^2}$ and all extremal functions are of the following form

$$u(x) = \log \frac{1 - |a|^2}{1 + |a|^2 |x|^2 - 2x \cdot a} + c,$$

for some $a \in \mathbb{B}^2$ and $c \in \mathbb{R}$.

The paper is organized as follows. In Section 2 we prove Theorem 1. In Section 3 we discuss (1.1) for all λ on $\overline{\mathbb{B}^2}$. In the last section the geometric inequality (Theorem 4) is proved.

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2. Uniqueness result for a nonlinear PDE

We first discuss the direct analogue of the result of Bidaut-Veron and Veron in dimension 2.

Theorem 5. *Let $\left(\Sigma^2, g\right)$ be a compact surface with Gaussian curvature $K \geq 1$ with a possibly empty convex boundary (convex in the sense that on the boundary the geodesic curvature $\kappa \geq 0$). Let u be a smooth function satisfying*

$$\begin{array}{lll} -\Delta u+\lambda=e^{2u} & on & \Sigma, \\ \frac{\partial u}{\partial \nu}=0 & on & \partial\Sigma, \end{array}$$

where $\lambda \leq 1$ is a constant. Then either u is constant or $\lambda = 1$ and $\left(\Sigma^2, g\right)$ is isometric to either the standard sphere $\mathbb{S}^2$ or the hemisphere $\mathbb{S}^2_+$ and u is given by

$$u\left(x\right)=-\log\left[\cosh t+\sinh t x\cdot\xi\right],$$

for some $t \geq 0$ and $\xi \in \mathbb{S}^2$.

We sketch the proof. Let $v = e^{-u/\beta}$, where $\beta \neq 0$ is to be determined. Then v is positive and satisfies

$$\Delta v=v^{-1}\left|\nabla v\right|^2+\frac{1}{\beta}\left(v^{1-2\beta}-\lambda v\right).$$

By the Bochner formula

$$\begin{aligned} \frac{1}{2}\Delta\left|\nabla v\right|^2 &= \left|D^2 v\right|^2+\langle\nabla v,\nabla\Delta v\rangle+K\left|\nabla v\right|^2 \\ &\geq \frac{1}{2}\left(\Delta v\right)^2+\langle\nabla v,\nabla\Delta v\rangle+\left|\nabla v\right|^2. \end{aligned}$$

Multiplying both sides by v^γ, with γ a nonzero constant and integrating over Σ yields

$$\frac{1}{2}\int_\Sigma v^\gamma\Delta\left|\nabla v\right|^2 \geq \frac{1}{2}\int_\Sigma v^\gamma\left(\Delta v\right)^2+\int_\Sigma v^\gamma\langle\nabla v,\nabla\Delta v\rangle+\int_\Sigma v^\gamma\left|\nabla v\right|^2.$$

Integrating by parts yields

$$\int_\Sigma\left(1+\frac{\lambda\gamma}{\beta}\right)v^\gamma\left|\nabla v\right|^2-\left(\frac{\gamma}{\beta}+1\right)v^{\gamma-2\beta}\left|\nabla v\right|^2-\frac{\left(\gamma+1\right)^2}{2}v^{\gamma-2}\left|\nabla v\right|^4\leq 0.$$

Choosing $\gamma = -1$ and $\beta = 1$ leads to $(1-\lambda)\int_\Sigma v^\gamma\left|\nabla v\right|^2 \leq 0$. Therefore if $\lambda < 1$ we conclude that v is constant. If $\lambda = 1$ and v is not constant, we must have $D^2 v = -fg$ for some function f by inspecting the above argument. From this overdetermined system one can deduce that $\left(\Sigma^2, g\right)$ is isometric to either the standard sphere $\mathbb{S}^2$ or the hemisphere $\mathbb{S}^2_+$ and then further determine u.

Remark 1. *The above method does not yield any interesting result when $\lambda > 1$. Recently Gui and Moradifam* [GM] *has proved that on $\mathbb{S}^2$ the only solution to $-\Delta u + \lambda = e^{2u}$ is constant when $1 < \lambda \leq 2$.*

From now on (Σ, g) is a compact surface such that the Gaussian curvature $K \geq 0$ and on the boundary the geodesic curvature $\kappa \geq 1$. Suppose $u \in C^\infty(\Sigma)$ satisfies the equation (1.1). Let $v = e^{-u}$. By simple calculation v satisfies the following equation

$$\begin{array}{lll} \Delta v=v^{-1}\left|\nabla v\right|^2 & \text{on} & \Sigma, \\ -\frac{\partial v}{\partial \nu}+\lambda v=1 & \text{on} & \partial\Sigma. \end{array}$$

Therefore Theorem 2 follows from the following

Theorem 6. *Suppose $v>0$ satisfies the above equation. If $\lambda<1$ then v is constant; if $\lambda=1$ and v is not constant, then Σ is isometric to the unit disc $\overline{\mathbb{B}^2}$ and v is given by*

$$v(z)=\frac{1+|a|^2|z|^2-2\operatorname{Re}(z\overline{a})}{1-|a|^2},$$

for some $a\in\mathbb{B}^2$.

Consider $\phi=v^{-1}|\nabla v|^2$.

Lemma 1. *ϕ is subharmonic, i.e. $\Delta\phi\geq 0$.*

Proof. By the Bochner formula

$$\begin{aligned}\frac{1}{2}\Delta|\nabla v|^2&=\left|D^2 v\right|^2+\langle\nabla v,\nabla\Delta v\rangle+K|\nabla v|^2\\&\geq\frac{1}{2}(\Delta v)^2+\langle\nabla v,\nabla\Delta v\rangle\\&=\frac{1}{2}v^{-2}|\nabla v|^4+\left\langle\nabla v,\nabla\left(v^{-1}|\nabla v|^2\right)\right\rangle.\end{aligned}$$

Thus

$$\begin{aligned}\frac{1}{2}\Delta\phi&=\frac{1}{2}v^{-1}\Delta|\nabla v|^2-v^{-2}\left\langle\nabla v,\nabla|\nabla v|^2\right\rangle-\frac{1}{2}v^{-2}\Delta v|\nabla v|^2+v^{-3}|\nabla v|^4\\&\geq v^{-3}|\nabla v|^4+v^{-1}\left\langle\nabla v,\nabla\left(v^{-1}|\nabla v|^2\right)\right\rangle-v^{-2}\left\langle\nabla v,\nabla|\nabla v|^2\right\rangle\\&=0.\end{aligned}$$

□

In the following we denote $v|_{\partial\Sigma}$ by f and $\chi=\frac{\partial v}{\partial\nu}=\lambda f-1$. We use the arclength s to parametrize the boundary. Then

$$\phi(s):=\phi|_{\partial\Sigma}=f(s)^{-1}\left(f'(s)^2+\chi(s)^2\right)=f^{-1}\left[(f')^2+(\lambda f-1)^2\right].$$

Lemma 2. *We have*

$$\frac{\partial\phi}{\partial\nu}\leq 2f^{-1}\left[\left((f')^2+\chi^2\right)\left(\frac{1}{2}f^{-1}\chi-1\right)+\lambda(f')^2-f''\chi\right].\tag{2.1}$$

Proof. We compute

$$\begin{aligned}\frac{\partial\phi}{\partial\nu}&=2f^{-1}D^2v(\nabla v,\nu)-f^{-2}\chi\left((f')^2+\chi^2\right)\\&=2f^{-1}\left[\chi D^2v(\nu,\nu)+f'D^2v\left(\frac{\partial}{\partial s},\nu\right)\right]-f^{-2}\chi\left((f')^2+\chi^2\right).\end{aligned}$$

On one hand

$$\begin{aligned}D^2v\left(\frac{\partial}{\partial s},\nu\right)&=\left\langle\nabla_{\frac{\partial}{\partial s}}\nabla v,\nu\right\rangle\\&=\chi'-\left\langle\nabla v,\nabla_{\frac{\partial}{\partial s}}\nu\right\rangle\\&=\lambda f'-f'\left\langle\frac{\partial}{\partial s},\nabla_{\frac{\partial}{\partial s}}\nu\right\rangle\\&=\lambda f'-\kappa f'.\end{aligned}$$

On the other hand from the equation of v we have on $\partial\Sigma$

$$D^2 v(\nu,\nu) + \kappa\chi + f'' = f^{-1}\left((f')^2 + \chi^2\right).$$

Plugging the above two identities into the formula for $\frac{\partial\phi}{\partial\nu}$ yields

$$\begin{aligned}
\frac{\partial\phi}{\partial\nu} &= 2f^{-1}\left[\left(f^{-1}\chi - \kappa\right)\left((f')^2 + \chi^2\right) + \lambda (f')^2 - f''\chi\right] - f^{-2}\chi\left((f')^2 + \chi^2\right) \\
&= 2f^{-1}\left[\left(\frac{1}{2}f^{-1}\chi - \kappa\right)\left((f')^2 + \chi^2\right) + \lambda (f')^2 - f''\chi\right] \\
&\leq 2f^{-1}\left[\left(\frac{1}{2}f^{-1}\chi - 1\right)\left((f')^2 + \chi^2\right) + \lambda (f')^2 - f''\chi\right],
\end{aligned}$$

where in the last step we use the assumption $\kappa \geq 1$. □

Remark 2. *From the proof it is clear that if equality holds in (2.1) at a point where ϕ is not zero, then $\kappa = 1$ there.*

Differentiating ϕ on the boundary we have on $\partial\Sigma$

$$\begin{aligned}
\phi'(s) &= f^{-1}\left(2f'f'' + 2\lambda\chi f'\right) - f^{-2}f'\left((f')^2 + \chi^2\right) \\
&= f^{-1}f'\left[2f'' + 2\lambda\chi - f^{-1}\left((f')^2 + \chi^2\right)\right].
\end{aligned}$$

We now prove Theorem 6. By the maximum principle ϕ achieves its maximum somewhere on the boundary, say at $p_0 \in \partial\Sigma$ whose local parameter is s_0. Then we have

$$\phi'(s_0) = 0, \phi''(s_0) \leq 0, \frac{\partial\phi}{\partial\nu}(s_0) \geq 0.$$

Moreover by the Hopf lemma, the 3rd inequality is strict unless ϕ is constant.

Case 1. $f'(s_0) \neq 0$.

Then we must have

$$2f'' + 2\lambda\chi - f^{-1}\left((f')^2 + \chi^2\right) = 0$$

or

$$f'' = \frac{1}{2}f^{-1}\left((f')^2 + \chi^2\right) - \lambda\chi$$

at s_0. Plugging into the inequality for $\frac{\partial\phi}{\partial\nu}$ at s_0 we have

$$\begin{aligned}
\frac{\partial\phi}{\partial\nu}(s_0) &\leq 2f^{-1}(\lambda - 1)\left((f')^2 + \chi^2\right) \\
&\leq 0, \text{ as } \lambda \leq 1.
\end{aligned}$$

Therefore ϕ is constant. Moreover if $\lambda < 1$, then ϕ is identically zero and hence v is constant.

Case 2. $f'(s_0) = 0$.

Then

$$\begin{aligned}
\phi''(s_0) &= f^{-1}f''\left[2f'' + 2\lambda\chi - f^{-1}\chi^2\right] \\
&= f^{-2}f''\left[2ff'' + (\lambda f)^2 - 1\right].
\end{aligned}$$

As $\phi''(s_0) \le 0$ and $\frac{\partial \phi}{\partial \nu}(s_0) \ge 0$ we have at s_0

$$\left[(\lambda f)^2 - 1\right] f'' + 2f\,(f'')^2 \le 0,$$
$$\chi^2\left(\frac{1}{2} f^{-1}\chi - 1\right) - f''\chi \ge 0$$

In particular the 1st inequality implies that $\left[(\lambda f)^2 - 1\right] f'' \le 0$. Therefore there are 2 possibilities.

Case 2a. $f'' \ge 0$ and $\chi = \lambda f - 1 \le 0$.

In this case the two inequalities simplify as

$$\left[(\lambda f)^2 - 1\right] + 2ff'' \le 0,$$
$$\chi\left(\frac{1}{2} f^{-1}\chi - 1\right) - f'' \le 0.$$

Thus canceling f'' yields

$$2\chi(\lambda - 1) f = \left[(\lambda f)^2 - 1\right] + \chi(\chi - 2f) \le 0.$$

As $\lambda \le 1$ we must conclude all the above inequalities are equalities. In particular $\frac{\partial \phi}{\partial \nu}(s_0) = 0$ and hence ϕ is constant. Moreover if $\lambda < 1$, then $\chi(s_0) = 0$ and hence $\phi(s_0) = 0$. It follows that ϕ is identically zero and hence v is constant.

Case 2b. $f'' \le 0$ and $\chi = \lambda f - 1 \ge 0$.

In this case the two inequalities simplify as

$$\left[(\lambda f)^2 - 1\right] + 2ff'' \ge 0,$$
$$\chi\left(\frac{1}{2} f^{-1}\chi - 1\right) - f'' \ge 0.$$

Thus canceling f'' yields

$$2\chi(\lambda - 1) f = \left[(\lambda f)^2 - 1\right] + \chi(\chi - 2f) \ge 0.$$

Clearly we can draw the same conclusion.

It remains to consider the case when $\lambda = 1$ and $\phi = 2a$ is a positive constant. Then from the proof of Lemma we conclude that $K = 0$ and

$$\begin{aligned} D^2 v &= ag \quad \text{on} \quad \Sigma, \\ \frac{\partial v}{\partial \nu} &= f - 1 \quad \text{on} \quad \partial\Sigma. \end{aligned}$$

From the proof of Lemma and Remark we also have $\kappa = 1$. By the Riemann mapping theorem we can take Σ to be $\overline{\mathbb{B}^2}$ with $g = e^{2w} g_0$, here g_0 is the Euclidean metric. As $K = 0$ and $\kappa = 1$ we have

$$\begin{aligned} \Delta w &= 0 \quad \text{on} \quad \overline{\mathbb{B}^2}, \\ \frac{\partial w}{\partial r} &= e^w - 1 \quad \text{on} \quad \mathbb{S}^1. \end{aligned}$$

Applying our argument to w we conclude that the Hessian of e^{-w} is $2cI$ for some constant $c \ge 0$. It follows that $e^{-w} = c\,|x - \xi|^2 + c'$. From the above equation we can easily show that w must be of the form

$$w = \log \frac{1 - |a|^2}{1 + |a|^2 |z|^2 - 2\,\mathrm{Re}\,(z\overline{a})}$$

for some $a \in \mathbb{B}^2$ and hence $g = F^* g_0$ where F is the linear fractional transformation

$$F(z) = \frac{z - a}{1 - \overline{a} z}.$$

Therefore (Σ, g) is isometric to $\left(\overline{\mathbb{B}^2}, g_0\right)$. The same argument can now be applied to v to finish the proof.

3. The supercritical case $\lambda > 1$ on $\overline{\mathbb{B}^2}$

A natural question is whether one can say something about the equation (1.1) in the supercritical case $\lambda > 1$. In general we do not have anything. But on $\overline{\mathbb{B}^2}$ we have a complete answer.

Theorem 7. *Suppose u is a smooth function on $\overline{\mathbb{B}^2}$ satisfying the following equation*

$$\begin{aligned} \Delta u &= 0 \quad \text{on} \quad \overline{\mathbb{B}^2}, \\ \tfrac{\partial u}{\partial \nu} &= \lambda (e^u - 1) \quad \text{on} \quad \mathbb{S}^1, \end{aligned}$$

where $\lambda > 0$ is a constant (the case $\lambda \leq 0$ is trivial by the maximal principle). Then

1. *If $\lambda \notin \mathbb{N}$ then u is constant.*
2. *If $\lambda = N \in \mathbb{N}$ then either $u \equiv 1$ or*

$$u(z) = \log \frac{|\xi|^2 - 1}{|\xi - z^N|^2},$$

for some $\xi \in \mathbb{C}$ with $|\xi| > 1$.

The 1st part was proved in [OPS, Lemma 2.3]. We first review their method and then deduce the 2nd part. Let $f = u|_{\mathbb{S}^1}$ which can be expressed in terms of its Fourier series

$$f = \sum a_n e^{in\theta} = \sum \widehat{f}(n) e^{in\theta}.$$

Then

$$\begin{aligned} u &= \sum a_n r^{|n|} e^{in\theta}, \\ \frac{\partial u}{\partial \nu} &= \sum |n| a_n e^{in\theta} = H\left(\frac{df}{d\theta}\right), \end{aligned}$$

where H is the Hilbert transform. Thus the boundary condition can be written as

$$H\left(\frac{df}{d\theta}\right) = \lambda \left(e^f - 1\right).$$

Differentiating the above equation yields

$$\begin{aligned} H\left(\frac{d^2 f}{d\theta^2}\right) &= \lambda e^f \frac{df}{d\theta} \\ &= \left(H\left(\frac{df}{d\theta}\right) + \lambda\right) \frac{df}{d\theta}. \end{aligned}$$

Setting $v = df/d\theta$, we have $\int_{\mathbb{S}^1} v d\theta = 0$ and

$$H\left(\frac{dv}{d\theta}\right) = vHv + \lambda v. \tag{3.1}$$

Since

$$v = \sum \widehat{v}(n) e^{in\theta}, Hv = -i \sum \operatorname{sgn}(n) \widehat{v}(n) e^{in\theta}$$

we have

$$vHv = -i\sum\left(\sum_{k+l=n}\operatorname{sgn}(k)\,\widehat{v}(k)\,\widehat{v}(l)\right)e^{in\theta}$$

By the equation (3.1) we must have

$$(|n|-\lambda)\,\widehat{v}(n) = -i\sum_{k+l=n}\operatorname{sgn}(k)\,\widehat{v}(k)\,\widehat{v}(l)\,.$$

We know $\widehat{v}(0)=0$. We observe that in the above summation each pair (k,l) with k,l having different sign will be canceled by the pair (l,k). Therefore for $n>0$

$$(n-\lambda)\,\widehat{v}(n) = -i\sum_{k=1}^{n-1}\widehat{v}(k)\,\widehat{v}(n-k)\,. \tag{3.2}$$

From the above equation Osgood, Phillips and Sarnak [OPS] proved by induction that the Fourier coefficients $\widehat{v}(n)=0$ for all $n\in\mathbb{Z}$, provided that $\lambda\notin\mathbb{Z}$.

Suppose now $\lambda = N$ is a positive integer. We prove by induction for all $n>0$

$$\widehat{v}(n) = \begin{cases} 0, & \text{if } N\nmid n \\ i\frac{(-i\widehat{v}(N))^m}{N^{m-1}} & \text{if } n = mN. \end{cases} \tag{3.3}$$

By conjugation we also get $\widehat{v}(-n)$. Suppose that it has been proved for all positive integers $<n$. If n is not a multiple of N, then in the summation on the RHS of (3.2) either k or l is not a multiple of N and hence $\widehat{v}(k)\,\widehat{v}(l)=0$ by the induction hypothesis. Therefore $\widehat{v}(n)=0$. If $n=(m+1)\,N$ we have from (3.2) and the induction hypothesis

$$\begin{aligned} mN\widehat{v}((m+1)\,N) &= -i\sum_{k=1}^{m}\widehat{v}(kN)\,\widehat{v}((m+1-k)\,N) \\ &= i\sum_{k=1}^{m}\frac{(-i\widehat{v}(N))^{m+1}}{N^{m-1}} \\ &= im\frac{(-i\widehat{v}(N))^{m+1}}{N^{m-1}}. \end{aligned}$$

Therefore

$$\widehat{v}((m+1)\,N) = i\frac{(-i\widehat{v}(N))^{m+1}}{N^{m}}.$$

This proves (3.3).

If $\widehat{v}(N)=0$ clearly $v=0$ and f is constant and as a result u must be identically 1. From now on we assume that $\widehat{v}(N)\neq 0$. Setting $\xi = N/(-i\widehat{v}(N))$, we have

$$\begin{aligned} v &= 2\operatorname{Re}\sum_{m=1}^{\infty} i\frac{(-i\widehat{v}(N))^m}{N^{m-1}}e^{imN\theta} \\ &= 2N\operatorname{Re} i\frac{e^{iN\theta}/\xi}{1-e^{iN\theta}/\xi} \\ &= \frac{2N(b\cos N\theta - a\sin N\theta}{|\xi - e^{iN\theta}|^2}, \end{aligned}$$

with $a = \operatorname{Re}\xi, b = \operatorname{Im}\xi$. For v to be defined on the entire $\mathbb{S}^1$ we must have $|\xi| \neq 1$. Without loss of generality we take $|\xi| > 1$. Integrating yields that for some $C > 0$

$$f = \log \frac{C}{|\xi - e^{iN\theta}|^2}.$$

Taking the harmonic extension we obtain

$$u(z) = \log \frac{C}{|\xi - z^N|^2}.$$

From the boundary condition $\frac{\partial u}{\partial \nu} = N(e^u - 1)$ one can easily determine C to be $|\xi|^2 - 1$. This finishes the proof.

4. An integral inequality

As before (Σ, g) denotes a compact surface with boundary with Gaussian curvature $K \geq 0$ and geodesic curvature $\kappa \geq 1$ on the boundary. By the Gauss-Bonnet formula

$$\int_\Sigma K + \int_{\partial\Sigma} \kappa = 2\pi,$$

we easily see that the perimeter of the boundary $L \leq 2\pi$. We remark that one can also prove the area $A \leq \pi$ without much difficulty, though this fact is not needed here.

Theorem 8. *For any smooth function u on Σ*

$$\frac{1}{2L}\int_\Sigma |\nabla u|^2 + \frac{1}{L}\int_{\partial\Sigma} u - \log\left(\frac{1}{L}\int_{\partial\Sigma} e^u\right) \geq 0. \tag{4.1}$$

Moreover if there exists a nonconstant extremal function then Σ is isometric to $\overline{\mathbb{B}^2}$ and all extremal functions are of the following form

$$u(x) = \log \frac{1 - |a|^2}{1 + |a|^2 |x|^2 - 2x \cdot a} + c,$$

for some $a \in \mathbb{B}^2$ and $c \in \mathbb{R}$.

On $\overline{\mathbb{B}^2}$ the above inequality becomes

$$\frac{1}{4\pi}\int_{\mathbb{B}^2} |\nabla u|^2 + \frac{1}{2\pi}\int_{\mathbb{S}^1} u d\theta - \log\left(\frac{1}{2\pi}\int_{\mathbb{S}^1} e^u d\theta\right) \geq 0,$$

a result proved in Osgood, Phillips and Sarnak [OPS].

We consider the following functional

$$\mathcal{F}_t(u) = \frac{1}{2t}\int_\Sigma |\nabla u|^2 + \frac{1}{L}\int_{\partial\Sigma} u - \log\left(\frac{1}{L}\int_{\partial\Sigma} e^u\right).$$

Notice that $\mathcal{F}_t(u + c) = \mathcal{F}_t(u)$ and $\mathcal{F}_t(0) = 0$.

We first prove that $\mathcal{F}_t$ is coercive if $t < 2\pi$. By the Riemann mapping theorem we can take Σ to be $\overline{\mathbb{B}^2}$ with $g = e^{2\phi} g_0$, here g_0 is the Euclidean metric. Then

$$\mathcal{F}_t(u) = \frac{1}{2t}\int_{\mathbb{B}^2} |\nabla u|^2 \, dxdy + \frac{1}{L}\int_{\mathbb{S}^1} u e^\phi d\theta - \log\left(\frac{1}{L}\int_{\mathbb{S}^1} e^{u+\phi} d\theta\right).$$

Without loss of generality we assume $\int_0^{2\pi} ud\theta = 0$. By Theorem

$$\begin{aligned}
\log\left(\frac{1}{L}\int_0^{2\pi} e^{u+\phi}d\theta\right) &\leq \frac{1}{4\pi}\int_{\overline{\mathbb{B}^2}} \left|\nabla\left(u+\phi\right)\right|^2 dxdy + \frac{1}{2\pi}\int_{\mathbb{S}^1}\left(u+\phi\right)d\theta + \log\frac{2\pi}{L} \\
&= \frac{1}{4\pi}\int_{\overline{\mathbb{B}^2}} \left|\nabla u\right|^2 + 2\left\langle \nabla u, \nabla\phi\right\rangle + \left|\nabla \phi\right|^2 + \frac{1}{2\pi}\int_{\mathbb{S}^1}\phi d\theta + \log\frac{2\pi}{L} \\
&\leq \frac{1+\varepsilon}{4\pi}\int_{\overline{\mathbb{B}^2}} \left|\nabla u\right|^2 + \frac{1+1/\varepsilon}{4\pi}\int_{\overline{\mathbb{B}^2}}\left|\nabla\phi\right|^2 + \frac{1}{2\pi}\int_{\mathbb{S}^1}\phi d\theta + \log\frac{2\pi}{L}.
\end{aligned}$$

Let U be the harmonic extension of $u|_{\mathbb{S}^1}$ on $\overline{\mathbb{B}^2}$. Using Fourier series it is easy to see

$$\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi} u^2 d\theta &\leq \frac{1}{2\pi}\int_{\overline{\mathbb{B}^2}}\left|\nabla U\right|^2 dxdy \\
&\leq \frac{1}{2\pi}\int_{\overline{\mathbb{B}^2}}\left|\nabla u\right|^2 dxdy.
\end{aligned}$$

Thus

$$\begin{aligned}
\frac{1}{L}\left|\int_0^{2\pi} ue^{\phi}d\theta\right| &\leq \frac{2\pi}{L}\left(\frac{1}{2\pi}\int_0^{2\pi} ud\theta\right)^{1/2}\left(\frac{1}{2\pi}\int_0^{2\pi} e^{2\phi}d\theta\right)^{1/2} \\
&\leq \frac{2\pi}{L}\left(\frac{1}{2\pi}\int_{\overline{\mathbb{B}^2}}\left|\nabla u\right|^2 dxdy\right)^{1/2}\left(\frac{1}{2\pi}\int_0^{2\pi} e^{2\phi}d\theta\right)^{1/2} \\
&\leq \frac{\varepsilon}{4\pi}\int_{\overline{\mathbb{B}^2}}\left|\nabla u\right|^2 + C/\varepsilon,
\end{aligned}$$

where C depends on $\int_0^{2\pi} e^{2\phi}d\theta$. Combining these inequalities we obtain

$$\mathcal{F}_t\left(u\right) \geq \left(\frac{1}{2t} - \frac{1+2\varepsilon}{4\pi}\right)\int_{\overline{\mathbb{B}^2}}\left|\nabla u\right|^2 dxdy + C_\varepsilon.$$

As $t < 2\pi$, we can choose $\varepsilon > 0$ small enough such that $\frac{1}{2t} - \frac{1+2\varepsilon}{4\pi} > 0$. Therefore $\mathcal{F}_t$ is coercive if $t < 2\pi$.

It then follows that the infimum of $\mathcal{F}_t$ is achieved by some u_0 which must satisfy the following Euler-Lagrange equation

$$\begin{aligned}
\Delta u = 0 \quad &\text{on} \quad \Sigma, \\
\tfrac{\partial u}{\partial \nu} + \lambda = ae^u \quad &\text{on} \quad \partial\Sigma,
\end{aligned}$$

with $\lambda = \frac{t}{L}$ and $a = t/\int_{\partial\Sigma} e^u$. If $t < L \leq 2\pi$, by Theorem 2 u_0 must be constant and hence the infimum of $\mathcal{F}_t$ is zero. Therefore if $t < L$

$$\mathcal{F}_t\left(u\right) = \frac{1}{2t}\int_\Sigma \left|\nabla u\right|^2 + \frac{1}{L}\int_{\partial\Sigma} u - \log\left(\frac{1}{L}\int_{\partial\Sigma} e^u\right) \geq 0.$$

Letting $t \nearrow L$ yields (4.1). The 2nd part about extremal functions obviously follows from Theorem 2.

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