COMPACTIFICATIONS OF COMPLETE RIEMANNIAN MANIFOLDS AND THEIR APPLICATIONS

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Dedicated to Professor Richard Schoen in honor of his 60th birthday

1. Introduction

To study a noncompact Riemannian manifold, it is often useful to find a compactification or attach a boundary. For example, in hyperbolic geometry a lot of investigation is carried out on the sphere at infinity. An eminent illustration is Mostow's proof of his rigidity theorem for hyperbolic manifolds [Mo]. More generally, if $\widetilde{M}$ is simply connected and nonpositively curved, one can compactify it by equivalent geodesic rays and the boundary is a topological sphere, called the geometric boundary. This compactification was first introduced in [EO] and has been indispensable in the study of negatively curved manifolds. If $\widetilde{M}$ is not nonpositively curved, then the geometric compactification does not work in general. But there are other compactifications which are useful for various studies. In this short survey, we will discuss some of these compactifcations and the relationships among them. Our discussion will focus on general Riemannian manifolds and therefore we ignore the large literature on compactifications of symmetric spaces (see the book [GJL]).

We first discuss the geometric compactification for Cartan-Hadamard manifolds and Gromov hyperbolic spaces in Section 2. In Section 3 we discuss the Martin compactification. In Section 4 we discuss the Busemann compactification. In the last section, we discuss how these compactifications are used. In particular, we consider certain invariants defined on the Martin boundary and prove a comparison inequality using a method of Besson, Courtois and Gallot. It should be noted that when the author showed this inequality to François Ledrappier he was informed that it had been known to Besson, Courtois and Gallot (unpublished).

Acknowledgement: It is my great pleasure to dedicate this article to Professor Rick Schoen on the occasion of his 60th birthday. Professor Schoen's many fundamental contributions to geometry are well-known and greatly admired. I had the good fortune to be one of his students and his influence on my mathematical career, through his teaching, his work and his example, has been tremendous. I wish him good health and more great theorems in the years to come.

In preparing this article, I have benefited from several stimulating discussions with François Ledrappier. I want to thank him warmly for teaching me a lot of things. I would also like to thank Jianguo Cao for helpful discussion on the work of Ancona [A] and for bringing to my attention his paper [Cao].

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2. The geometric compactification

The most familiar compactification is the geometric compactification first introduced by Eberlein and O'Neill [EO] for Cartan-Hadamard manifolds. Let $\widetilde{M}^n$ be a Cartan-Hadamard manifold. We can compactify $\widetilde{M}$ using geodesic rays. More precisely, two geodesic rays γ_1 and γ_2 are said to be equivalent, if $d\left(\gamma_1\left(t\right),\gamma_2\left(t\right)\right)$ is bounded for $t\in[0,\infty)$. The set of equivalence classes, denoted by $\widetilde{M}\left(\infty\right)$, is called the geometric boundary and can be naturally identified with the unit sphere $\mathbb{S}^{n-1}$ if we fix a base point. We then obtain a compactification $\widetilde{M}^*=\widetilde{M}\sqcup\widetilde{M}\left(\infty\right)$ that is homeomorphic to the closed unit n-ball with the natural "cone" topology. If the sectional curvature satisfies $-b^2\leq K_{\widetilde{M}}\leq -a^2$, where $a,b>0$, it is proved by Anderson and Schoen that $\widetilde{M}^*$ has a C^α-structure, where $\alpha=a/b$. For details, cf. [E, SY].

The same compactification works for the so called Gromov hyperbolic spaces. We first recall one of several equivalent definitions of Gromov hyperbolic spaces.

Definition 1. *A complete geodesic metric space (X,d) is called Gromov hyperbolic if for some $\delta>0$ s.t. for all points $o,x,y,z\in X$*

$$\left(x\cdot y\right)_o\geq\min\left\{\left(x\cdot z\right)_o,\left(y\cdot z\right)_o\right\}-\delta.$$

where we use Gromov products, e.g. $\left(x\cdot y\right)_o=\frac{1}{2}\left(d\left(o,x\right)+d\left(o,y\right)-d\left(x,y\right)\right)$.

We make the following remarks

Remark 1. *In the definition one can take o to be fixed.*

Remark 2. *A Cartan-Hadamard manifold $\widetilde{M}$ is Gromov hyperbolic if the sectional curvature has a negative upper bound.*

This concept was introduced by Gromov [G]. For detailed study of Gromov hyperbolic spaces, see the excellent books [BH, GH, O]. It suffices to say that the definition captures the global features of the geometry of a complete simply connected manifold of negative curvature. It is very robust as illustrated by the following remarkable fact.

Theorem 1. *Let X and Y be geodesic spaces. Suppose $f:X\to Y$ is a quasi-isometry, i.e. there are $L,\varepsilon>0$ s.t. for any $x_1,x_2\in X$*

$$L^{-1}d\left(x_1,x_2\right)-\varepsilon\leq d\left(f\left(x_1\right),f\left(x_1\right)\right)\leq Ld\left(x_1,x_2\right).$$

If Y is Gromov hyperbolic, then so is X.

We will further assume that X is proper. Then we can define the geometric boundary $X\left(\infty\right)$ for a Gromov hyperbolic spaces in the same way s.t. $\overline{X}=X\sqcup X\left(\infty\right)$ with a natural topology is a compact metrizable space. Moreover $X\left(\infty\right)$ has a canonical quasi-conformal structure.

Theorem 2. *Let X and Y be proper Gromov hyperbolic spaces. If $f:X\to Y$ is a quasi-isometry, then f extends to a homeomorphism $\overline{f}:X\left(\infty\right)\to Y\left(\infty\right)$.*

In fact, the boundary map is furthermore a quasi-conformal map. The boundary map can be described as follows: given the equivalence class $\xi\in\partial X$ of a geodesic ray $\gamma:[0,\infty)\to X$, $f\circ\gamma$ is a quasi-geodesic in Y and hence has a well defined end point $f\circ\gamma\left(\infty\right)\in Y\left(\infty\right)$ which is defined to be $\overline{f}\left(\xi\right)$.

3. The Martin compactification

We can also compactify $\widetilde{M}$ using all positive harmonic functions. The vector space $\mathcal{H}\left(\widetilde{M}\right)$ of harmonic functions with seminorms

$$\|u\|_K = \sup_K |u(x)|, K \subset \widetilde{M} \text{ compact}$$

is a Frechet space. Let $\mathcal{K}_o = \{u \in \mathcal{H}_+\left(\widetilde{M}\right) : u(o) = 1\}$. It is a convex and compact set in $\mathcal{H}\left(\widetilde{M}\right)$. We assume that $\widetilde{M}$ is nonparabolic and $G(x,y)$ is the minimal positive Green's function. Define the Martin kernel

$$k(x,y) = \frac{G(x,y)}{G(o,y)}.$$

A sequence $y_i \to \infty$ is called a Martin sequence if $\lim_{i\to\infty} k(x,y_i)$ converges to a harmonic function. By Harnack inequality and the elliptic theory every sequence $y_i \to \infty$ has a Martin subsequence. Two Martin sequences are called equivalent if they have the same harmonic function as limit. The collection of all such equivalence classes is called the Martin boundary and will be denoted by $\partial_\Delta \widetilde{M}$. It is easy to see that $\partial_\Delta \widetilde{M} \subset \mathcal{K}_o$ is a compact set. The Martin compactification is defined to be

$$\widehat{M} = \widetilde{M} \sqcup \partial\widetilde{M}$$

with a natural topology that makes it a compact metrizable space. An excellent reference on Martin compactification is Ancona [A].

Definition 2. *A harmonic function $h > 0$ on $\widetilde{M}$ is called minimal if any nonnegative harmonic function $\leq h$ is proportional to h.*

Remark 3. *If $h(o) = 1$, then h is minimal iff h is an extremal point of $\mathcal{K}_o$.*

It is proved that all minimal harmonic function h with $h(o) = 1$ belong to $\partial_\Delta \widetilde{M}$. Therefore we can introduce the following

Definition 3. *The minimal Martin boundary of M is*

$$\partial^* \widetilde{M} = \{h \in \mathcal{K}_o : h \textit{ is minimal}\}.$$

Moreover $\partial^* \widetilde{M} \subset \partial\widetilde{M}$ is at least a Borel subset (cf. [A]). According to a theorem of Choquet ([A]), for any positive harmonic function h there is a unique Borel measure μ^h on $\partial^* \widetilde{M}$ such that

$$h(x) = \int_{\partial^* \widetilde{M}} \xi(x)\, d\mu^h(\xi).$$

Let ν be the measure corresponding to the harmonic function 1. Thus

$$1 = \int_{\partial^* \widetilde{M}} \xi(x)\, d\nu(\xi). \tag{3.1}$$

The family of probability measures $\left\{\nu^x : x \in \widetilde{M}\right\}$ with $\nu^x = \xi(x)\nu$ are called the harmonic measures. For $f \in L^\infty\left(\partial^* \widetilde{M}\right)$ we get a bounded harmonic function

$$H_f(x) = \int_{\partial^* \widetilde{M}} f(\xi)\, \xi(x)\, d\nu(\xi).$$

This defines an isomorphism between $L^{\infty}\left(\partial^{*}\widetilde{M}\right)$ and the space of bounded harmonic functions on $\widetilde{M}$.

The study of the Martin compactification is closely related to the study of Brownian motion on $\widetilde{M}$. For simplicity we further add a mild condition that the Ricci curvature is bounded from below to ensure stochastic completeness. Therefore we have the sample space $\Omega\left(\widetilde{M}\right)=C\left(\mathbb{R}^{+},\widetilde{M}\right)$ with a family of probability measure $\left\{\mathbb{P}^{x}:x\in\widetilde{M}\right\}$ s.t. for any $0<t_1<\cdots<t_k$ and open sets $U_1,\cdots,U_k$

$$\begin{aligned}&\mathbb{P}^{x}\left\{\omega\in\Omega\left(M\right):\omega\left(t_1\right)\in U_1,\cdots\omega\left(t_k\right)\in U_k\right\}\\ =&\int_{U_1\times\cdots\times U_k}p_{t_1}\left(x,y_1\right)p_{t_2-t_1}\left(y_1,y_2\right)\cdots p_{t_k-t_{k-1}}\left(y_{k-1},y_k\right)dy_1\times\cdots dy_k.\end{aligned}$$

Here $p_t\left(x,y\right)$ is the heat kernel on $\widetilde{M}$. For each $t\geq 0$ we have a random variable $X_t:\Omega\left(\widetilde{M}\right)\rightarrow\widetilde{M}$ which is simply the position at t, i.e. $X_t\left(\omega\right)=\omega\left(t\right)$. It is an intriguing and important problem to understand the asymptotic behavior for $\omega\left(t\right)$ as $t\rightarrow\infty$. The answer is closely related to the Martin boundary.

Theorem 3. (1) *For any $x\in\widetilde{M}$ and for $\mathbb{P}^{x}$-a.e. $\omega\in\Omega\left(M\right)$, $X_t\left(\omega\right)$ admits a limit $X_{\infty}\left(\omega\right)\in\partial^{*}\widetilde{M}$ as $t\rightarrow\infty$, i.e.*

$$\lim_{t\rightarrow\infty}k\left(x,\omega\left(t\right)\right)$$

exists and is a minimal harmonic function.

(2) *Under $\mathbb{P}^{x}$, the distribution of X_{∞} is ν^{x}, i.e. $\left(X_{\infty}\right)_{*}\mathbb{P}^{x}=\nu^{x}$.*

For detailed discussion, see Ancona [A].

For a Cartan-Hadamard manifold $\widetilde{M}$ with sectional curvature bounded between two negative constants, Anderson and Schoen [AS] proved that the Martin boundary is homeomorphic to the geometric boundary.

Theorem 4. *Suppose that $\widetilde{M}$ is a Cartan-Hadamard manifold with whose sectional curvature satisfies $-b^{2}\leq K\leq -a^{2}<0$. Then there exists a natural homeomorphism $\Phi:\partial\widetilde{M}\rightarrow\widetilde{M}\left(\infty\right)$ between the Martin boundary and the geometric boundary. Moreover, Φ^{-1} is Hölder continuous.*

From the proof, it is also clear that $\partial\widetilde{M}=\partial^{*}\widetilde{M}$ in this case.

This theorem was generalized by Ancona who proved

Theorem 5. *(Ancona* [A, Theorem 6.2]*) Suppose that $\widetilde{M}$ is Gromov hyperbolic and $\lambda_0\left(\widetilde{M}\right)>0$. Then the Martin boundary is homeomorphic to the geometric boundary. Moreover $\partial\widetilde{M}=\partial^{*}\widetilde{M}$.*

In the statement, $\lambda_0\left(\widetilde{M}\right)$ is the bottom of the L^{2} spectrum of $\widetilde{M}$, i.e.

$$\lambda_0\left(\widetilde{M}\right)=\inf\frac{\int_{\widetilde{M}}\left|\nabla u\right|^{2}}{\int_{\widetilde{M}}u^{2}},$$

where the infimum is taken over all smooth functions with compact support. It is easy to see that for a Cartan-Hadamard manifold $\widetilde{M}^{n}$ with $K\leq -a^{2}$, we have $\lambda_0\left(\widetilde{M}\right)\geq\left(n-1\right)^{2}a^{2}/4$.

4. The Busemann boundary

Instead of harmonic functions, one can use distance functions to compactify $\widetilde{M}$. This leads to the Busemann compactification, first introduced by Gromov in [BGS]. Fix a point $o \in \widetilde{M}$ and define, for $x \in \widetilde{M}$ the function $\xi_x(z)$ on $\widetilde{M}$ by:

$$\xi_x(z) \; = \; d(x,z) - d(x,o).$$

The assignment $x \mapsto \xi_x$ is continuous, one-to-one and takes values in a relatively compact set of functions for the topology of uniform convergence on compact subsets of $\widetilde{M}$. The Busemann compactification $\widehat{M}$ of $\widetilde{M}$ is the closure of $\widetilde{M}$ for that topology. The space $\widehat{M}$ is a compact separable space. The *Busemann boundary* $\partial\widehat{M} := \widehat{M} \backslash \widetilde{M}$ is made of 1-Lipschitz continuous functions ξ on $\widetilde{M}$ such that $\xi(o) = 0$ and there exists a sequence $\{a_k\} \subset \widetilde{M}$ s.t. $d\,(o.a_k) \to \infty$ and

$$\xi\,(x) = \lim_{k\to\infty} d(a_k, x) - d(a_k, o),$$

where the convergence is uniform over compact sets. Elements of $\partial\widehat{M}$ are called *horofunctions*. We note that this compactification works for any proper metric space X (cf. [KL]) But in general, X may fail to be open in its Busemann compactification. This pathology does not happen for Riemannian manifolds, i.e. we have

Proposition 1. *$\widetilde{M}$ is open in its Busemann compactification $\widehat{M}$. Hence the Busemann boundary $\partial\widehat{M}$ is compact.*

For proof see [LW1]. For a Cartan-Hadamard manifold, the Busemann compactification coincides with the geometric compactification. More precisely,

Proposition 2. *Let $\widetilde{M}$ be a Cartan-Hadamard manifold and $\{a_k\}$ a sequence in $\widetilde{M}$ s.t. $d\,(o.a_k) \to \infty$. Let σ_k be the unique geodesic ray from o to a_k. Then ξ_{a_k} converges to a horofunction ξ iff σ_k converges to a ray σ. Furthermore, we have*

$$\xi\,(x) = \lim_{t\to\infty} d\,(\sigma\,(t)\,, x) - t.$$

For proof see Ballmann [B] (p30).

Recently, the Busemann compactification has found to be very useful in various questions, cf. [KL, L1, LW1]. We first describe the application in [LW1]. Suppose M is a compact Riemannian manifold and $\widetilde{M}$ its universal covering (noncompact). Let G be the fundamental group of M acting on $\widetilde{M}$ isometrically. Observe that we may extend by continuity the action of G from $\widetilde{M}$ to $\widehat{M}$, in such a way that for ξ in $\widehat{M}$ and g in G,

$$g.\xi(z) \; = \; \xi(g^{-1}z) - \xi(g^{-1}o).$$

The volume entropy of M is defined to be the limit

$$v\,(g) = \lim_{r\to\infty} \frac{\ln \mathrm{vol} B_{\widetilde{M}}\,(x,r)}{r},$$

where $B_{\widetilde{M}}\,(x,r)$ is the ball of radius r centered at x in the universal covering space $\widetilde{M}$. This important invariant was introduced by Manning [Ma] who proved

(1) the limit exists and is independent of the center $x \in \widetilde{M}$,

(2) $v \leq H$, the topological entropy of the geodesic flow on the unit tangent bundle of M,
(3) $v = H$ if M is nonpositively curved.

In [LW1], we extended the classical theory of Patterson-Sullivan measure by constructing a family of measures on the *Busemann boundary* $\partial \widehat{M}$.

Theorem 6. *There exists a family* $\left\{ \nu_x : x \in \widetilde{M} \right\}$ *of finite measures on the Busemann boundary* $\partial \widehat{M}$ *s.t.*

(1) *for any pair* $x, y \in \widetilde{M}$ *the two measures* ν_x *and* ν_y *are equivalent with*

$$\frac{d\nu_x}{d\nu_y}(\xi) = e^{-v(\xi(x) - \xi(y))},$$

(2) *for any* $g \in G$ *and* $x \in \widetilde{M}$

$$g_* \nu_x = \nu_{gx}.$$

This family of measures plays a crucial role in the proofs of the following rigidity results involving the volume entropy.

Theorem 7. *Let* M^n *be a compact Riemannian manifold with* $Ric \geq -(n-1)$*. Then the volume entropy satisfies* $v \leq n-1$ *and equality holds iff* M *is hyperbolic.*

Remark 4. *This result was proved by Knieper* [Kn] *under the additional assumption that* M *is negatively curved.*

As a corollary, in view of the well-known inequality $\lambda_0 \left(\widetilde{M} \right) \leq v^2/4$, we deduce the following result which was previously proved in [W] by a different method.

Theorem 8. *Let* M^n *be a compact Riemannian manifold with* $Ric \geq -(n-1)$*. Then* $\lambda_0 \left(\widetilde{M} \right) \leq (n-1)^2/4$ *and equality holds iff* M *is hyperbolic.*

Theorem 9. *Let* M *be a compact Kähler manifold with* $\dim_{\mathbb{C}} M = m$*. If the bisectional curvature* $K_{\mathbb{C}} \geq -2$*, then the volume entropy satisfies* $v \leq 2m$*. Moreover equality holds iff* M *is complex hyperbolic (normalized to have constant holomorphic sectional curvature* -4*).*

Theorem 10. *Let* M *be a compact quaternionic Kähler manifold of* $\dim = 4m$ *with* $m \geq 2$ *and scalar curvature* $-16m(m+2)$*. Then the volume entropy satisfies* $v \leq 2(2m+1)$*. Moreover equality holds iff* M *is quaternionic hyperbolic.*

We refer to the original paper [LW1] for details. More recently, we can prove some pinching theorems using our method. The first step is the following rigidity result for $C^{1,\alpha}$ metrics.

Theorem 11. *Let* M^n *be a (smooth) compact smooth manifold and* g *a* $C^{1,\alpha}$ *Riemannian metric. Suppose that* g_i *is a sequence of smooth Riemannian metrics on* M *s.t.*

(1) $\mathrm{Ric}(g_i) \geq -(n-1)$ *for each* i,
(2) $g_i \to g$ *in* $C^{1,\alpha}$ *norm as* $i \to \infty$,
(3) *the volume entropy* $v(g_i) \to n-1$ *as* $i \to \infty$.

Then g *is hyperbolic.*

From this result, we then deduce the following pinching theorem.

Theorem 12. *There exists a positive constant $\varepsilon = \varepsilon(n, D)$ s.t. if (M^n, g) is a compact Riemannian manifold of dimension n satisfying the following conditions*

- *g has **negative** sectional curvature,*
- $\mathrm{Ric}(g) \geq -(n-1)$,
- $\mathrm{diam}(M, g) \leq D$,
- *the volume entropy $v(g) \geq n - 1 - \varepsilon$,*

then M is diffeomorphic to a hyperbolic manifold (X, g_0). Moreover, the Gromov-Hausdorff distance $d_{GH}(M, X) \leq \alpha(\varepsilon) \to 0$ as $\varepsilon \to 0$.

We note that this theorem was established by Courtois [C] (unpublished) in 2000 using the Cheeger-Colding theory. Our proof is different and simpler. The details will appear in [LW2].

In [L1] Ledrappier studied another fundamental invariant: the linear drift (introduced by Guivarc'h [Gu]) which is the following limit for almost every path ω of the Brownian motion on $\widetilde{M}$

$$l = \lim_{t \to \infty} \frac{1}{t} d(\omega(0), \omega(t)).$$

If M is negatively curved, Kaimanovich [K1] established a remarkable integral formula for l. Let $\partial\widetilde{M}$ be the geometric boundary of $\widetilde{M}$. As usual we fix a base point $o \in \widetilde{M}$. Recall that there is a homeomorphism Φ from $\partial\widetilde{M}$ to the Martin boundary $\partial_\Delta\widetilde{M}$ by the theorem of Anderson-Schoen. For each $\xi \in \partial\widetilde{M}$, $h_\xi = \Phi(\xi)$ is the unique harmonic function on $\widetilde{M}$ s.t. $h_\xi(o) = 1$ and $h_\xi \in C\left(\widetilde{M}^* \backslash \{\xi\}\right)$ with boundary value zero. With these notations, the Kaimanovich formula can be written as

$$l = -\int_M \left(\int_{\partial\widetilde{M}} \langle \nabla B_\xi, \nabla \ln h_\xi \rangle (x)\, h_\xi(x)\, d\nu(\xi) \right) dm(x),$$

where ν is the harmonic measure on $\partial\widetilde{M}$ defined by (3.1) and m is the normalized Lebesgue measure on M. The main result in [L1] is a similar integral formula for l in the general case. The key step is to construct certain measures on the *Busemann boundary* $\partial\widehat{M}$.

5. A comparison theorem

In this section we discuss why compactifications are useful. The basic principle is that often times a geometric object is much simpler near infinity. When we look at it on the boundary, we capture its essential features while all the background noise dies off. The first illustration of this principle is perhaps Mostow's rigidity theorem [Mo]: If $f : M \to N$ is a (smooth) homotopy equivalence between two compact hyperbolic manifolds then f is homotopic to an isometry. In the proof Mostow considers the lifting $\widetilde{f} : \widetilde{M} \to \widetilde{N}$ between the universal coverings (which are both $\mathbb{H}^n$ in this case) which is a quasi-isometry. Then $\widetilde{f}$ extends to a homeomorphism $\overline{f} : \partial\widetilde{M} \to \partial\widetilde{N}$ between the boundaries. Using the theory of quasi-conformal maps and the fact that the fundamental group acts on $\partial\widetilde{M}$ ergodically, Mostow shows that $\overline{f}$ is in fact a Mobius transformation.

More recently, in a seminal paper [BCG1], Besson, Courtois and Gallot proved the following theorem which implies the Mostow rigidity theorem in the rank one cases.

Theorem 13. *Let (N^n, g_0) be a compact locally symmetric space of negative curvature. Let M^n be another compact manifold and $f : M \to N$ is a continuous map of nonzero degree. Then for any metric g on M*

1. $v(g)^n \operatorname{vol}(M, g) \geq |\deg f| \, v(g_0)^n \operatorname{vol}(N, g_0)$*;*
2. *the equality holds iff f is homotopic to an covering map.*

The proof involves embedding $\widetilde{M}$ into a Hilbert space and a calibration argument. In a later paper [BCG2], the same authors gave a very elegant and simpler proof of their theorem under the additional assumption that g is also negatively curved. In this second approach, the Patterson-Sullivan measure on the geometric boundary $\partial\widetilde{M}$ plays a fundamental role. Their method is geometric and flexible and we will apply it in a slight different situation.

Let (M^n, g) be a compact Riemannian manifold and $\pi : \widetilde{M} \to M$ its universal covering. We pick a base point $o \in \widetilde{M}$. Let $\partial\widetilde{M}$ be the Martin boundary.

Definition 4. *For any $p > 0$ let*

$$\beta_p(g) = \int_M \left(\int_{\partial\widetilde{M}} |\nabla \log \xi(x)|^2 \xi(x) \, d\nu(\xi) \right)^p dm(x).$$

Similarly we can consider

$$\widetilde{\beta}_p(g) = \int_M \int_{\partial\widetilde{M}} |\nabla \log \xi(x)|^{2p} \xi(x) \, d\nu(\xi) \, dm(x).$$

We have $\beta_p(g) \leq \widetilde{\beta}_p(g)$ by the Hölder inequality. When $p = 1$, $\beta_p(g) = \widetilde{\beta}_p(g)$ is the Kaimanovich entropy, an invariant of fundamental importance. This was introduced by Kaimanovich [K1]. We summarize its main properties:

1. [K1] $\beta_1 = \lim_{t\to\infty} -\frac{1}{t} \int_{\widetilde{M}} p_t(x, y) \log p_t(x, y) \, dy$ for any $x \in \widetilde{M}$;
2. [K1] $\beta_1 > 0$ iff $\widetilde{M}$ has nonconstant bounded harmonic functions.
3. [L1, L2] $4\lambda_0 \leq \beta_1 \leq v^2$, where v is the volume entropy and λ_0 is the bottom of the L^2 spectrum of $\widetilde{M}$.

Let (N^n, g_0) be a compact locally symmetric space of negative curvature. Theorem 13 says that among all metrics g on N with $\operatorname{vol}(N, g) = \operatorname{vol}(N, g_0)$ the metric g_0 has the smallest volume entropy. A natural question is whether the same is true for the Kaimonovich entropy.

Problem 1. *Let (N^n, g_0) be a compact locally symmetric space of negative curvature. Is it true that for any metric g*

$$\beta_1(g)^{n/2} \operatorname{vol}(N, g) \geq \beta_1(g_0)^{n/2} \operatorname{vol}(N, g_0)?$$

We do not know the answer to this question. What we can prove is the following result which gives an affirmative answer to the same question for $\beta_{n/2}$.

Theorem 14. *Let (N^n, g_0) be a compact locally symmetric space of negative curvature. Let M^n be another compact manifold and $f : M \to N$ a (smooth) homotopy equivalence. Then for any metric g on M*

1. $\beta_{n/2}(g) \operatorname{vol}(M, g) \geq \beta_{n/2}(g_0) \operatorname{vol}(N, g_0)$*;*
2. *the equality holds iff f is homotopic to an isometry.*

As a consequence we have the following result which is also an easy corollary of Theorem 13.

Corollary 1. *If g_0 is real hyperbolic and g satisfies $Ric \geq -(n-1)$, then* $\operatorname{vol}(M,g) \geq \operatorname{vol}(N,g_0)$. *Moreover, equality holds iff g is also hyperbolic.*

This follows from the sharp gradient estimate.

Proposition 3. *(Li-J. Wang* [LiW, Lemma 2.1]*) Let N^n be a complete manifold with $Ric \geq -(n-1)$. If u is a positive harmonic function on N, then $|\nabla \log u| \leq n-1$.*

We now prove Theorem 14. The homotopy equivalence $f:(M,g)\to(N,g_0)$ induces an isomorphism $\rho:\Gamma := \pi_1(M)\to\pi_1(N)$. We view the fundamental groups as groups of deck transformations acting on the universal covering manifolds. Lifting f we obtain a smooth map $\widetilde{f}:\widetilde{M}\to\widetilde{N}$ which is Γ-equivariant, i.e. for any $\gamma\in\Gamma$

$$\widetilde{f}(\gamma\cdot x)=\rho(\gamma)\cdot\widetilde{f}(x).$$

This is a quasi-isometry and hence $\widetilde{M}$ is Gromov hyperbolic as $\widetilde{N}$ is. Hence $\widetilde{f}$ extends to a homeomorphism $\overline{f}:\partial\widetilde{M}\to\partial\widetilde{N}$ between the boundaries. By a theorem of Brooks [Br], $\lambda_0\left(\widetilde{M}\right)>0$ since $\lambda_0\left(\widetilde{N}\right)>0$ and the two fundamental groups are isomorphic. Therefore, by Theorem 5 $\partial\widetilde{M}$ is also the Martin boundary of $\widetilde{M}$ and let $\left\{\nu^x : x\in\widetilde{M}\right\}$ be the harmonic measures. We now define a new map $\widetilde{F}:\widetilde{M}\to\widetilde{N}$ applying the construction in [BCG2]: $\widetilde{F}(x)$ is the barcenter of the measure $\overline{f}_*\nu^x$ on $\partial\widetilde{N}$, i.e. $\widetilde{F}(x)$ is the unique minimum point of the following function on $\widetilde{N}$

$$y\to\int_{\partial\widetilde{N}} B_\theta(y)\,d\left(\overline{f}_*\nu^x\right)(\theta),$$

where B_θ is the Busemann function on $\widetilde{N}$ associated to $\theta\in\partial\widetilde{N}$. For detailed discussion of the barcenter see [BCG2]. Note that this map is well defined as the support ν^x always has more that two points. By the implicit function theorem, it is easy to show that $\widetilde{F}$ is smooth. Moreover it is Γ-equivariant and hence yields a smooth map $F:M\to N$. What remains is to estimate the Jacobian of this map.

By the definition of F we have

$$\begin{aligned}&\int_{\partial\widetilde{N}} dB_\theta\left(\widetilde{F}(x)\right)(\cdot)\,d\left(\overline{f}_*\nu_x\right)(\theta)\\ =\;&\int_{\partial\widetilde{M}} dB_{\overline{f}(\xi)}\left(\widetilde{F}(x)\right)(\cdot)\,\xi(x)\,d\nu(\xi)\\ =\;&0.\end{aligned}$$

Differentiating in x we get

$$\begin{aligned}&\int_{\partial\widetilde{M}} D^2B_{\overline{f}(\xi)}\left(\widetilde{F}(x)\right)\left(\widetilde{F}_*(x)(\cdot),\cdot\right)\xi(x)\,d\nu(\xi)\\ =\;&-\int_{\partial\widetilde{M}} dB_{\overline{f}(\xi)}\left(\widetilde{F}(x)\right)(\cdot)\,d\xi(\cdot)\,d\nu(\xi).\end{aligned}$$

We define the following quadratic form K and H on $T_{F(x)}\widetilde{N}$

$$\begin{aligned} g_0\left(K_{\widetilde{F}(x)}(u), u\right) &= \int_{\partial\widetilde{M}} D^2 B_\theta\left(\widetilde{F}(x)\right)(u,u)\,\xi(x)\,d\nu(\xi), \\ g_0\left(H_{\widetilde{F}(x)}(u), u\right) &= \int_{\partial\widetilde{M}} dB_{f(\xi)}\left(\widetilde{F}(x)\right)(u)^2\,\xi(x)\,d\nu(\xi). \end{aligned}$$

Then for any $v \in T_x\widetilde{M}, u \in T_{\widetilde{F}(x)}\widetilde{N}$

$$\begin{aligned} &\left| g_0\left(K_{\widetilde{F}(x)}(F_*(x)(v)), u\right)\right| \\ \leq\ & g_0\left(H_{\widetilde{F}(x)}(u), u\right)^{1/2}\left(\int_{\partial\widetilde{M}} \frac{|\langle\nabla\xi(x), v\rangle|^2}{\xi(x)} d\nu(\xi)\right)^{1/2} \\ =\ & g_0\left(H_{\widetilde{F}(x)}(u), u\right)^{1/2}\left(\int_{\partial\widetilde{M}} |\langle\nabla\log\xi(x), v\rangle|^2\,\xi(x)\,d\nu(\xi)\right)^{1/2}. \end{aligned}$$

Therefore

$$|\det K|\,\mathrm{Jac}\widetilde{F}(x) \leq \frac{1}{n^{n/2}}|\det H|^{1/2}\left(\int_{\partial\widetilde{M}} |\nabla\log\xi(x)|^2\,\xi(x)\,d\nu(\xi)\right)^{n/2}.$$

By [BCG1, Appendix B] we obtain

$$\mathrm{Jac}\widetilde{F}(x) \leq \frac{1}{(n+d-2)^n}\left(\int_{\partial\widetilde{M}} |\nabla\log\xi(x)|^2\,\xi(x)\,d\nu(\xi)\right)^{n/2},$$

where $d = 1, 2, 4$ or 8 when $\left(\widetilde{N}, g_0\right)$ is the real, complex, quaternionic hyperbolic space or the Cayley hyperbolic plane, respectively. Integrating over M yields

$$\mathrm{vol}(N, g_0) \leq \frac{\mathrm{vol}(M,g)}{(n+d-2)^n}\int_M\left(\int_{\partial\widetilde{M}} |\nabla\log\xi(x)|^2\,\xi(x)\,d\nu(\xi)\right)^{n/2} dm(x).$$

This proves the inequality as $\beta_{n/2}(g_0) = (n+d-2)^n$. If equality holds, it is easy to see that F has to be an isometry up to a scaling.

References

[A] A. Ancona, Théorie du potentiel sur les graphes et les variétés. École d'été de Probabilités de Saint-Flour XVIII—1988, 1–112, Lecture Notes in Math., 1427, Springer, Berlin, 1990.

[AS] M. Anderson and R. Schoen, Positive harmonic functions on complete manifolds of negative curvature. Ann. of Math. (2) 121 (1985), no. 3, 429–461.

[B] W. Ballmann, **Lectures on spaces of Nonpositive Curvature**. Birkhäuser, Boston, 1985.

[BGS] W. Ballmann, M. Gromov and V. Schroeder, **Manifolds of Nonpositive Curvature**. With an appendix by Misha Brin. Birkhäuser Verlag, Basel, 1995.

[Br] R. Brooks, The fundamental group and the spectrum of the Laplacian. Comment. Math. Helv. 56 (1981), no. 4, 581–598.

[BH] M. Bridson and A. Haefliger, **Metric spaces of non-positive curvature**. Grundlehren der Mathematischen Wissenschaften, 319. Springer-Verlag, Berlin, 1999.

[BCG1] G. Besson, G. Courtois and S. Gallot, Entropies et rigidités des espaces localement symétriques de courbure strictement négative. Geom. Funct. Anal. 5 (1995), no. 5, 731–799.

[BCG2] G. Besson, G. Courtois and S. Gallot, Minimal entropy and Mostow's rigidity theorems. Ergodic Theory Dynam. Systems 16 (1996), no. 4, 623–649.

[Cao] J. Cao, Cheeger isoperimetric constants of Gromov-hyperbolic spaces with quasi-poles. Commun. Contemp. Math. 2 (2000), no. 4, 511–533.

[C] G. Courtois, Pincement spectral des variétés hyperboliques. Preprint, 2000.

[E] P. Eberlein, **Geometry of nonpostively curved manifolds**. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1996.

[EO] P. Eberlein and B. O'Neill, Visibility manifolds. Pacific J. Math. 46 (1973), 45–109.

[GH] É. Ghys and P. de la Harpe (ed), **Sur les groupes hyperboliques d'après Mikhael Gromov**. Progress in Mathematics, 83. Birkhäuser, Boston, MA, 1990.

[G] M. Gromov, Hyperbolic groups. **Essays in group theory**, 75–263, Math. Sci. Res. Inst. Publ., 8, Springer, New York, 1987.

[Gu] Y. Guivarc'h, Sur la loi des grands nombres et le rayon spectral d'une marche aléatoire. Astérisque, 74 (1980) 47–98.

[GJL] Y. Guivarc'h, L. Ji and J. C. Taylor, **Compactifications of Symmetric Spaces**. Birkhauser, Bostorn, MA, 1998.

[K1] V. Kaimanovich, Brownian motion and harmonic functions on covering manifolds: an entropy approach. Soviet Math. Dokl. 33 (1986) 812-816.

[K2] V. Kaimanovich, The Poisson formula for groups with hyperbolic properties. Ann. of Math. 152 (1990), 659-692.

[KL] A. Karlsson and F. Ledrappier, On laws of large numbers for random walks. Ann. Probab. 34 (2006), no. 5, 1693–1706.

[Kn] G. Knieper, Spherical means on compact Riemannian manifolds of negative curvature. Differential Geom. Appl. 4 (1994), no. 4, 361–390.

[L1] F. Ledrappier, Linear drift and entropy for regular covers. arXiv math.ds 0910.1425, to appear in Geom. Func. Anal.

[L2] F. Ledrappier, Harmonic measures and Bowen-Margulis measures. Israel J. Math. 71 (1990) 275-287.

[LW1] F. Ledrappier and X. Wang, An integral formula for the volume entropy with applications to rigidity. arXiv math.dg 0911.0370, to appear in J. Diff. Geom.

[LW2] F. Ledrappier and X. Wang, Pinching theorems for the volume entropy. Preprint.

[LiW] P. Li and J. Wang, Complete manifolds with positive spectrum. II. J. Differential Geom. 62 (2002), no. 1, 143–162.

[Ma] A. Manning, Topological entropy for geodesic flows. Ann. of Math. (2) 110 (1979), no. 3, 567–573.

[Mo] G. D. Mostow, Quasi-conformal mappings in n-space and the rigidity of hyperbolic space forms. Inst. Hautes Études Sci. Publ. Math. No. 34 1968 53–104.

[O] K. Ohshika, **Discrete groups**, Translations of Mathematical Monographs, 207. American Mathematical Society, Providence, RI, 2002.

[SY] R. Scheon and S.-T. Yau, **Lectures on Differential Geometry**. International Press, 1994.

[V] N. Varopoulos, Information theory and harmonic functions. Bull. Sci. Math. (2) 110 (1986), no. 4, 347–389.

[W] X. Wang, Harmonic functions, entropy, and a characterization of the hyperbolic space. J. Geom. Anal. 18 (2008), no. 1, 272–284.

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