A NOTE ON A CONJECTURE OF SCHROEDER AND STRAKE

FENGBO HANG AND XIAODONG WANG

Abstract. We prove some rigidity results for compact manifolds with boundary. In particular for a compact Riemannian manifold with nonnegative Ricci curvature and simply connected mean convex boundary, it is shown that if the sectional curvature vanishes on the boundary, then the metric must be flat.

In [Schroeder and Strake 1989, Theorem 1], Schroeder and Strake proved the following rigidity theorem.

> Let (M, g) be a compact Riemannian manifold with convex boundary and nonnegative Ricci curvature. Assume that the sectional curvature is identically zero in some neighborhood U of ∂M and that one of the following conditions holds:
>
> - ∂M is simply connected
> - $\dim \partial M$ is even and ∂M is strictly convex at some point $p \in \partial M$.
>
> Then M is flat.

As remarked in [Schroeder and Strake 1989], the condition that the metric is flat in a whole neighborhood of ∂M is very strong. They conjectured that it suffices to only assume that the sectional curvature vanishes on ∂M and proved this in the special case of a convex metric ball. The problem was studied by Xia in [Xia 1997, Xia 2002] who confirmed the conjecture under various additional conditions: like the boundary has constant mean curvature or constant scalar curvature, or the second fundamental form satisfies some pinching condition etc. We refer to [Xia 1997, Xia 2002] for the precise statements. Here we present some results related to the conjecture.

Theorem 1. *Let M be a smooth compact connected Riemannian manifold with boundary and nonnegative Ricci curvature. If every component of ∂M is simply connected and has nonnegative mean curvature and the sectional curvature of M vanishes on ∂M, then M is flat and ∂M has only one component.*

Therefore when ∂M is simply connected the conjecture of Schroeder and Strake is true. Moreover one only needs ∂M to be mean convex instead of convex. We remark that the conclusion that ∂M has only one component follows from theorems in [Ichida 1981, Kasue 1983]. Below we will present a different argument for it based on the Reilly's formula ([Reilly 1977]).

To continue the discussion we need to fix some notations. We will often write $\langle\ ,\ \rangle$ for the metric on M and denote the connection as D. For convenience we write

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$\Sigma = \partial M$ and denote the Levi-Civita connection and curvature tensor etc. of the induced metric on Σ as standard notations with a subscript Σ. Let ν be the unit outer normal vector. The shape operator is given by $A(X) = D_X \nu$ and the second fundamental form is given by $h(X,Y) = \langle A(X), Y \rangle = \langle D_X \nu, Y \rangle$, here $X, Y \in T\Sigma$. The mean curvature $H = \operatorname{tr} A$. Recall Reilly's formula ([Reilly 1977, formula (14)]) for a smooth function u on M

$$\begin{aligned} \frac{1}{2}\int_M \left((\Delta u)^2 - \left|D^2 u\right|^2 \right) d\mu &= \frac{1}{2}\int_M Rc(\nabla u, \nabla u)\, d\mu + \int_\Sigma \Delta_\Sigma u \cdot \frac{\partial u}{\partial \nu} dS \\ &+ \frac{1}{2}\int_\Sigma H \left(\frac{\partial u}{\partial \nu} \right)^2 dS + \frac{1}{2}\int_\Sigma \langle A(\nabla_\Sigma u), \nabla_\Sigma u \rangle\, dS. \end{aligned}$$

A special case of theorems in [Ichida 1981, Kasue 1983] claims that if M^n is a compact connected Riemannian manifold with mean convex boundary Σ and nonnegative Ricci curvature, then Σ has at most two components; moreover if Σ has two components, then M is isometric to $\Gamma \times [0, a]$ for some connected compact Riemannian manifold Γ with nonnegative Ricci curvature and $a > 0$. For Theorem 1, it is clear M can not have the product metric, hence Σ has one component. It is interesting that one may give an argument for the above special case based on Reilly's formula. Indeed, assume Σ is not connected, fix a component Σ_0 of Σ, then we may solve the Dirichlet problem

$$\begin{cases} \Delta u = 0 \text{ on } M, \\ u|_{\Sigma_0} = 0, \\ u|_{\Sigma \backslash \Sigma_0} = 1. \end{cases}$$

Applying the Reilly's formula to u, we get

$$-\int_M \left|D^2 u\right|^2 d\mu = \int_M Rc(\nabla u, \nabla u)\, d\mu + \int_\Sigma H \left(\frac{\partial u}{\partial \nu} \right)^2 dS.$$

Hence $D^2 u = 0$. This implies $|\nabla u| \equiv c > 0$. Since $\nabla u = -c\nu$ on Σ_0 and $\nabla u = c\nu$ on $\Sigma \backslash \Sigma_0$, we see $D_X \nu = 0$ for $X \in T\Sigma$ i.e. Σ is totally geodesic. If we look at the flow generated by $\frac{\nabla u}{c}$, then it sends Σ_0 to $\Sigma \backslash \Sigma_0$ at time $\frac{1}{c}$ and hence Σ has exactly two components. Note that the flow lines are just geodesics. If we fix a coordinate on Σ_0, namely $\theta^1, \cdots, \theta^{n-1}$, let $r = \frac{u}{c}$, then we have $g = dr \otimes dr + g_{ij}(r, \theta)\, d\theta^i \otimes d\theta^j$. Using $D^2 r = 0$, we see $\partial_r g(r, \theta) = 0$. Hence M is isometric to $\Sigma_0 \times \left[0, \frac{1}{c}\right]$.

Under the assumption of Theorem 1 that the sectional curvature of M vanishes on Σ, it follows from Gauss and Codazzi equations that

$$\begin{aligned} R_\Sigma(X, Y, Z, W) &= h(X, Z)\, h(Y, W) - h(X, W)\, h(Y, Z), \\ (D_\Sigma)_X h(Y, Z) &= (D_\Sigma)_Y h(X, Z), \end{aligned}$$

where X, Y, Z and W belong to $T\Sigma$. By the fundamental theorem for hypersurfaces [Spivak 1999, part (2) of Theorem 21 on p63] and the fact Σ is simply connected, we may find a smooth isometric immersion $\phi : \Sigma \to \mathbb{R}^n$ such that the second fundamental form of the immersion $h_\phi = h$. If Σ is convex then ϕ is an embedding by a Hadamard type theorem of Sacksteder [Sacksteder 1960]. With this immersion ϕ at hands, Theorem 1 follows from the following proposition.

Proposition 1. *Assume M^n is a smooth compact connected Riemannian manifold with connected boundary $\Sigma = \partial M$ and $Rc \geq 0$. If $\phi : \Sigma \to \mathbb{R}^l$ is an isometric immersion with $|H_\phi| \leq H$ on Σ, here H_ϕ is the mean curvature vector of the* *immersion ϕ, then M is flat. If moreover ϕ is an imbedding, then M is isometric to a domain in $\mathbb{R}^n$.*

This is a generalization of a theorem of Ros [Ros 1988, Theorem 2], who derived a congruence theorem for hypersurface in Euclidean space. Following the argument of Ros, we will show by Reilly's formula that the harmonic extension of the map ϕ is in fact an isometric immersion.

Proof. We may find a smooth function $F : M \to \mathbb{R}^l$ such that

$$\begin{cases} \Delta F = 0 \text{ in } M; \\ F|_\Sigma = \phi. \end{cases}$$

Applying the Reilly's formula to each component of F and sum up we get

$$\begin{aligned} & -\frac{1}{2}\sum_\alpha \int_M \left|D^2 F^\alpha\right|^2 d\mu \\ = \ & \frac{1}{2}\int_M \sum_\alpha Rc\left(\nabla F^\alpha, \nabla F^\alpha\right) d\mu + \int_\Sigma \sum_\alpha \Delta_\Sigma \phi^\alpha \cdot \frac{\partial F^\alpha}{\partial \nu} dS \\ & + \frac{1}{2}\int_\Sigma \sum_\alpha H \left(\frac{\partial F^\alpha}{\partial \nu}\right)^2 dS + \frac{1}{2}\int_\Sigma \sum_\alpha \left\langle A\left(\nabla_\Sigma \phi^\alpha\right), \nabla_\Sigma \phi^\alpha \right\rangle dS. \end{aligned}$$

Note

$$\begin{aligned} \sum_\alpha \left\langle A\left(\nabla_\Sigma \phi^\alpha\right), \nabla_\Sigma \phi^\alpha \right\rangle &= \left\langle Ae_i, e_j \right\rangle e_i \phi^\alpha \cdot e_j \phi^\alpha = \left\langle Ae_i, e_j \right\rangle \phi_* e_i \cdot \phi_* e_j \\ &= \left\langle Ae_i, e_j \right\rangle \delta_{ij} = \operatorname{tr} A = H, \end{aligned}$$

here $e_1, \cdots, e_{n-1}$ is a local orthonormal frame on Σ, hence

$$\begin{aligned} 0 &= \frac{1}{2}\sum_\alpha \int_M \left|D^2 F^\alpha\right|^2 d\mu + \frac{1}{2}\int_M \sum_\alpha Rc\left(\nabla F^\alpha, \nabla F^\alpha\right) d\mu + \int_\Sigma H_\phi \cdot F_* \nu dS \\ & \quad + \frac{1}{2}\int_\Sigma H \left|F_* \nu\right|^2 dS + \frac{1}{2}\int_\Sigma H dS \\ &\geq \frac{1}{2}\sum_\alpha \int_M \left|D^2 F^\alpha\right|^2 d\mu + \frac{1}{2}\int_\Sigma H \left(\left|F_*\nu\right|^2 - 2\left|F_*\nu\right| + 1\right) dS. \end{aligned}$$

Hence $D^2 F^\alpha = 0$ for all α. It follows that $F^* g_{\mathbb{R}^l}$ is parallel on M. We may find some $p \in \Sigma$ such that $|H_\phi| > 0$ at p, hence $H(p) > 0$. From the argument above this implies $|F_*\nu| = 1$ at p and $F_*\nu$ is perpendicular to $\phi_* \Sigma_p$, hence $F^* g_{\mathbb{R}^l} = g_M$ at p. It follows that $F^* g_{\mathbb{R}^l} = g_M$ on M, that is, F is an isometric immersion and M is flat. Now assume ϕ is an imbedding. Let $\overline{D}$ be the connection on $\mathbb{R}^l$, then $\overline{D}_X F_* Y - F_* D_X Y = XYF - (D_X Y) F = 0$, it follows that $F : M \to \mathbb{R}^l$ is a totally geodesic submanifold, hence the image lies in a n dimensional affine subspace. Without losing of generality we may assume $l = n$ and Σ is a compact hypersurface in $\mathbb{R}^n$, then there exists a bounded open domain Ω such that $\partial\Omega = \Sigma$. Since F is an immersion, we see $F(M) \backslash \overline{\Omega}$ is both open and closed in $\mathbb{R}^n \backslash \overline{\Omega}$, hence it must be empty. Based on this we may show $F : M \to \overline{\Omega}$ is a covering map and hence it must be a diffeomorphism. □

If we assume that ∂M is convex, then it is clear from the above discussion that M is isometric to a convex domain in $\mathbb{R}^n$. In fact in this case one may replace the nonnegativity of the Ricci curvature by the much weaker nonnegativity of the scalar curvature, at least when M is spin.

Theorem 2. *Let M be a smooth compact connected Riemannian manifold with boundary and nonnegative scalar curvature. If M is spin, each component of ∂M is convex and simply connected and the sectional curvature of M vanishes on ∂M, then M is isometric to a convex domain in $\mathbb{R}^n$.*

Proof. For every component Γ of ∂M, we have an isometric embedding $\phi : \Gamma \to \mathbb{R}^n$ which has h as the second fundamental form. Let Ω be the convex domain enclosed by $\phi(\Gamma)$. We glue M and $\mathbb{R}^n \backslash \Omega$ along Γ via the diffeomorphism ϕ for all the Γ's and obtain a complete Riemannian manifold N which has nonnegative scalar curvature and is flat outside a compact set. Notice that the metric is C^1 along the gluing hypersurface. Since M is spin, we conclude by the generalized positive mass theorem proved in [Shi and Tam 2002, Theorem 3.1] that N is isometric to $\mathbb{R}^n$. It follows that M is isometric to a convex domain in $\mathbb{R}^n$. $\square$

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