BOTTOM OF SPECTRUM OF KAHLER MANIFOLDS WITH STRONGLY PSEUDOCONVEX BOUNDARY

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1. INTRODUCTION

The study of the spectrum of the Laplace operator has been a very active subject in Riemannian geometry. In the compact case, the spectrum consists of eigenvalues. On a noncompact complete Riemannian manifold (M^n, g), the L^2-spectrum is much more complicated. For many questions, it suffices to study the bottom of the spectrum which can be characterized

$$\lambda_0(M, g) = \inf_{u \in C_c^1(M)} \frac{\int_M |\nabla u|^2}{\int_M u^2}.$$

If $\text{Ric} \geq 0$, then $\lambda_0 = 0$ by Cheng's eigenvalue comparison theorem. If Ricci has a negative lower bound, Cheng [Ch] proved the following theorem.

Theorem 1. *Suppose (M^n, g) is a noncompact complete Riemannian manifold with $\text{Ric} \geq -(n-1)$, we have $\lambda_0 \leq (n-1)^2/4$.*

The estimate is sharp since the spectrum is the ray $[(n-1)^2/4, +\infty)$ for the hyperbolic space $\mathbb{H}^n$.

For Kähler manifolds, this estimate can be improved. On a Kähler manifold (M, g) of complex dimension n, where g is the Riemannian metric, let $\omega = g(J\cdot, \cdot)$ be the Kähler form. In local holomorphic coordinates $z_1, \cdots, z_n$, we have

$$\omega = \sqrt{-1} g_{i\bar{j}} dz_i \wedge d\bar{z}_j.$$

The Laplace operator on functions is given by the formula

$$\Delta f = 2g^{i\bar{j}} \frac{\partial^2 f}{\partial z_i \partial \bar{z}_j}.$$

It should be noted that we use different normalization here from ones appeared in [LW2, M, LT, Li3].

Munteanu [M] proved the following improved estimate for Kähler manifolds.

Theorem 2. *Suppose (M, g) is a noncompact complete Kähler manifold of complex dimension n. If $\text{Ric} \geq -(n+1)$, then $\lambda_0 \leq n^2/2$.*

We remark that prior to Munteanu's work, P. Li and J. Wang [LW2] established the same inequality under the stronger curvature assumption that the bisectional curvature $K_{\mathbb{C}} \geq -1$, i.e. for any vectors X, Y

$$R(X, Y, X, Y) + R(X, JY, X, JY) \geq -\left(|X|^2 |Y|^2 + \langle X, Y \rangle^2 + \langle X, JY \rangle^2\right)$$

where J is the complex structure. The Li-Wang [LW2] and Munteanu [M] estimates are sharp on the complex hyperbolic space $\mathbb{CH}^n$ normalized to have sectional curvature in $[-2, -1/2]$ so that $K_{\mathbb{C}} = -1$ and $\text{Ric} = -(n+1)$.

In general, it is more difficult to establish a positive lower bound for λ_0. In the Riemannian case, there is a nice theorem due to Lee [Lee1] for the so called conformally compact Einstein manifolds. A Riemannian manifold (M^n, g) is called conformally compact if M is the interior of a compact manifold $\overline{M}$ with boundary Σ and for any defining function r of the boundary (i.e. $r \in C^{\infty}(\overline{M})$ s.t. $r > 0$ on M, $r = 0$ on Σ and dr is nowhere zero along Σ), $\overline{g} = r^2 g$ extends to a C^3 metric on $\overline{M}$. The conformal class of the metric $\overline{g}|_{\Sigma}$ is invariantly defined and Σ with this conformal structure is called the conformal infinity of (M^n, g). We will also assume that $|dr|_{\overline{g}}^2 = 1$ on Σ. This condition is invariantly defined and such a metric can be termed AH (asymptotically hyperbolic) as one can check that the sectional curvature $K \to -1$ near the conformal infinity. If g is Einstein, i.e. $\text{Ric}(g) = -(n-1)g$, it must be asymptotically hyperbolic.

It is known that the continuous spectrum of an AH Riemannian manifold consists of the ray $[(n-1)^2/4, +\infty)$ with no embedded eigenvalues. In particular $\lambda_0 \leq (n-1)^2/4$. However, in general there may exist finitely many eigenvalues in the interval $\left(0, (n-1)^2/4\right)$ even if g is Einstein. The following theorem was proved by Lee in [Lee1].

Theorem 3. *Let (M^n, g) be a conformally compact Einstein manifold. If its conformal infinity has nonnegative Yamabe invariant, then $\lambda_0 = (n-1)^2/4$, i.e. the spectrum is $[(n-1)^2/4, +\infty)$.*

The main purpose of this paper is to investigate if there is a Kähler analogue of Lee's theorem. In Section 2, we consider a class of complete Kähler manifolds with a strictly pseudoconvex boundary at infinity. After studying its asymptotic geometry, we formulate a conjecture on its bottom of spectrum in the Kähler-Einstein case. In Section 3, we discuss a geometric approach to estimate the bottom of spectrum. Specifically, we prove a sharp lower estimate which illustrates the boundary effect. In the last section, we focus on the Kähler-Einstein metric constructed by Cheng-Yau [CY] on a strictly pseudoconvex domain in $\mathbb{C}^n$. We prove a theorem which yields the optimal lower bound for the bottom of spectrum under the condition that the induced pseudo-hermitian structure has nonnegative pseudo-hermitian scalar curvature.

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2. The formulation of the problem

Suppose $\Omega \subset\subset M$ is a smooth precompact domain in a complex manifold of complex dimension n. We assume the boundary $\Sigma = \partial\Omega$ is strongly pseudoconvex in the sense that there is a negative defining function ϕ for the boundary (i.e. $\phi \in C^{\infty}(\overline{\Omega})$ s.t. $\phi < 0$ on Ω, $\phi = 0$ on Σ and $d\phi$ is nowhere zero along Σ) s.t.

$\sqrt{-1}\partial\bar{\partial}\phi > 0$ near Σ. Then we can consider near Σ the following Kähler metric

$$\omega_0 = -\sqrt{-1}\partial\bar{\partial}\log(-\phi)$$

$$= \sqrt{-1}\left(-\frac{\phi_{i\bar{j}}}{\phi} + \frac{\phi_i\phi_{\bar{j}}}{\phi^2}\right) dz^i \wedge d\bar{z}^j.$$

By the assumption, the complex Hessian $H(\phi) = \left[\phi_{i\bar{j}}\right]$ is a positive definite matrix and its inverse will be denoted by $\left[\phi^{i\bar{j}}\right]$. In the following, we will also use the notation $\phi^i = \phi^{i\bar{j}}\phi_{\bar{j}}, \phi^{\bar{j}} = \phi^{i\bar{j}}\phi_i$. Let $\widehat{g}$ be the Riemannian metric defined by ω_0. We have

$$\widehat{g}_{i\bar{j}} = -\frac{\phi_{i\bar{j}}}{\phi} + \frac{\phi_i\phi_{\bar{j}}}{\phi^2},$$

$$\widehat{g}^{i\bar{j}} = -\phi\left(\phi^{i\bar{j}} + \frac{\phi^i\phi^{\bar{j}}}{\phi - |\partial\phi|_\phi^2}\right),$$

where $|\partial\phi|_\phi^2 = \phi^{i\bar{j}}\phi_i\phi_{\bar{j}}$. Notice

$$\widehat{g}^{i\bar{j}}\phi_{\bar{j}} = (-\phi)^2 \frac{\phi^i}{|\partial\phi|_\phi^2 - \phi}.$$

Moreover,

$$\widehat{G} = \det\left[\widehat{g}_{i\bar{j}}\right] = (-\phi)^{-(n+1)}\left(|\partial\phi|_\phi^2 - \phi\right)\det H(\phi),$$

Hence the Ricci tensor is given by

$$\widehat{R}_{i\bar{j}} = -(n+1)\widehat{g}_{i\bar{j}} - \frac{\partial^2}{\partial z^i \partial \bar{z}^j}\log\left[\left(|\partial\phi|_\phi^2 - \phi\right)\det H(\phi)\right].$$

Definition 1. *A Kähler metric ω on Ω is called ACH (asymptotically complex hyperbolic) if near $\partial\Omega$*

$$\omega = \omega_0 + \Theta,$$

where Θ extends to a C^3 form on $\overline{\Omega}$.

(Cf. [W3] where the complex hyperbolic space is normalized to have holomorphic sectional curvature -1 instead of -4.)

Proposition 1. *The curvature tensor is asymptotically constant. More precisely for any pair of $(1,0)$ vectors X, Y*

$$R\left(X, \overline{X}, Y, \overline{Y}\right) = (1 + o(1))\left(|X|^2 |Y^2| + \left\langle X, \overline{Y}\right\rangle^2\right)$$

near the boundary, i.e. $K_\mathbb{C} \to -1$ at infinity.

Proof. In local coordinates near the boundary, we write the metric as

$$g_{i\bar{j}} = -\frac{\phi_{i\bar{j}}}{\phi} + \frac{\phi_i\phi_{\bar{j}}}{\phi^2} + h_{i\bar{j}},$$

with $h_{i\bar{j}}$ C^3 up to the boundary. Direct calculation shows

$$\frac{\partial g_{k\bar{l}}}{\partial z_i} = \frac{\phi_i}{-\phi}g_{k\bar{l}} + \frac{\phi_k}{-\phi}g_{i\bar{l}} + \frac{\phi_{ik\bar{l}}}{-\phi} + \frac{\phi_{ik}\phi_{\bar{l}}}{\phi^2}$$

$$- \frac{\phi_i}{-\phi}h_{k\bar{l}} - \frac{\phi_k}{-\phi}h_{i\bar{l}} + \frac{\partial g_{k\bar{l}}}{\partial z_i},$$

$$\frac{\partial^2 g_{k\bar{l}}}{\partial z_i \partial z_{\bar{j}}} = g_{i\bar{j}} g_{k\bar{l}} + g_{k\bar{j}} g_{i\bar{l}} + \frac{\phi_i}{-\phi} \frac{\partial g_{k\bar{l}}}{\partial z_{\bar{j}}} + \frac{\phi_k}{-\phi} \frac{\partial g_{i\bar{l}}}{\partial z_{\bar{j}}}$$

$$+ \frac{\phi_{i\bar{j}k\bar{l}}}{-\phi} + \frac{\phi_{ik\bar{l}}\phi_{\bar{j}} + \phi_{ik\bar{j}}\phi_{\bar{l}} + \phi_{ik}\phi_{\bar{j}\bar{l}}}{\phi^2} - 2\frac{\phi_{ik}\phi_{\bar{j}}\phi_{\bar{l}}}{\phi^3}$$

$$- h_{i\bar{j}} g_{k\bar{l}} - h_{k\bar{j}} g_{i\bar{l}} - g_{i\bar{j}} h_{k\bar{l}} + g_{k\bar{j}} h_{i\bar{l}} + h_{i\bar{j}} h_{k\bar{l}} + h_{k\bar{j}} h_{i\bar{l}}$$

$$- \frac{\phi_i}{-\phi} \frac{\partial h_{k\bar{l}}}{\partial z_{\bar{j}}} - \frac{\phi_k}{-\phi} \frac{\partial h_{i\bar{l}}}{\partial z_{\bar{j}}} + \frac{\partial^2 h_{k\bar{l}}}{\partial z_i \partial z_{\bar{j}}}.$$

Then a straightforward but tedious calculation yields

$$R_{i\bar{j}k\bar{l}} = \frac{\partial^2 g_{k\bar{l}}}{\partial z_i \partial z_{\bar{j}}} - g^{p\bar{q}} \frac{\partial g_{p\bar{l}}}{\partial z_{\bar{j}}} \frac{\partial g_{k\bar{q}}}{\partial z_i}$$

$$= g_{i\bar{j}} g_{k\bar{l}} + g_{k\bar{j}} g_{i\bar{l}} + O\left(\frac{1}{-\phi}\right).$$

$\square$

Let $f = (-\phi)^{\alpha}$. Then

$$f_i = -\alpha (-\phi)^{\alpha-1} \phi_i,$$

$$f_{i\bar{j}} = -\alpha (-\phi)^{\alpha-1} \phi_{i\bar{j}} + \alpha (\alpha - 1) (-\phi)^{\alpha-2} \phi_i \phi_{\bar{j}}.$$

Thus, by direct calculation

$$\Delta f = 2g^{i\bar{j}} f_{i\bar{j}}$$

$$= 2\left[\hat{g}^{i\bar{j}} - \hat{g}^{i\bar{l}} h_{k\bar{l}} \hat{g}^{i\bar{j}} + O\left(\phi^3\right)\right] f_{i\bar{j}}$$

$$= 2\alpha (\alpha - n) (-\phi)^{\alpha} [1 + O(\phi)].$$

Proposition 2. *For an ACH Kähler manifold, we always have*

$$\lambda_0 \leq n^2/2.$$

Moreover, any number $\mu < n^2/2$ in the spectrum must be an L^2 eigenvalue.

Proof. We first prove $\lambda_0 \leq n^2/2$. For $f = (-\phi)^{\alpha}$ we have

$$\int f^2 \frac{\omega^n}{n!} = \int (-\phi)^{2\alpha-n-1} \det H(\phi) |\partial\phi|^2 (1 + O(\phi)).$$

Hence $f \in L^2$ as long as $\alpha > n/2$. On the other hand

$$|\nabla f|^2 = 2\alpha^2 (-\phi)^{2\alpha-2} g^{i\bar{j}} \phi_i \phi_{\bar{j}}$$

$$= 2\alpha^2 (-\phi)^{2\alpha} (1 + O(-\phi)),$$

$$\int |\nabla f|^2 \frac{\omega^n}{n!} = 2 \int g^{i\bar{j}} f_i f_{\bar{j}} \frac{\omega^n}{n!}$$

$$= 2\alpha^2 \int (-\phi)^{2\alpha-n-1} \det H(\phi) |\partial\phi|^2 (1 + O(\phi)).$$

It is then clear

$$\lim_{\alpha \searrow n/2} \frac{\int |\nabla f|^2 \frac{\omega^n}{n!}}{\int f^2 \frac{\omega^n}{n!}} = n^2/2.$$

Therefore $\lambda_0 \leq n^2/2$.

By the above calculation, for any $c \in \left(0, n^2/2\right)$ there exists $\varepsilon > 0$ such that the positive function $f = (-\phi)^{n/2}$ satisfies

$$-\Delta f \geq cf$$

outside the compact set $K_\varepsilon = \{-\phi \geq \varepsilon\}$. The second part of the theorem then follows from some general principle (see, e.g. [Lee3]). For any $\xi \in C_c^1 (M \backslash K_\varepsilon)$ we have

(2.1)
$$\int |\nabla \xi|^2 \geq c \int \xi^2.$$

Indeed, integrating by parts

$$\int |\nabla \xi|^2 - c\xi^2 \geq \int |\nabla \xi|^2 + \frac{\Delta f}{f} \xi^2$$

$$= \int |\nabla \xi|^2 - 2\frac{\xi}{f} \langle \nabla f, \nabla \xi \rangle + \frac{\xi^2}{f^2} |\nabla f|^2$$

$$= \int \left| \nabla \xi - \frac{\xi}{f} \nabla f \right|^2.$$

Let $\lambda < c$. Then from (2.1) we have for any $\xi \in C_c^2 (M \backslash K_\varepsilon)$

(2.2)
$$\|(\Delta + \lambda)\, \xi\|_{L^2(M)} \geq (c - \lambda) \|\xi\|_{L^2(M)}.$$

Let $\rho \in C_c^\infty (M)$ be such that $0 \leq \rho \leq 1$, $\rho \equiv 1$ in a neighborhood of K_ε and the support of ρ is contained in the larger $K_{\varepsilon/2}$. Then for any $\xi \in C_c^2 (M)$ applying (2.2) to $(1 - \rho)\, \xi$ yields

$$(c - \lambda) \|\xi\|_{L^2\left(M \backslash K_{\varepsilon/2}\right)} \leq (c - \lambda) \|(1 - \rho)\, \xi\|_{L^2(M)}$$

$$\leq \|(\Delta + \lambda) \left[(1 - \rho)\, \xi\right]\|_{L^2(M)}$$

$$\leq \|(1 - \rho)\, (\Delta + \lambda)\, \xi\|_{L^2(M)} + A_1 \|\xi\|_{H^1\left(K_{\varepsilon/2}\right)},$$

where $A_1 > 0$ is a constant depending on the C^2 norm of ρ. By elliptic estimate on the compact domain $K_{\varepsilon/2}$, there exists a constant $B > 0$ such that

$$\|\xi\|_{H^1\left(K_{\varepsilon/2}\right)} \leq A_2 \left(\|(\Delta + \lambda)\, \xi\|_{L^2\left(K_{\varepsilon/2}\right)} + \|\xi\|_{L^2\left(K_{\varepsilon/2}\right)} \right).$$

Combining the previous two inequalities, we conclude that there exists $A > 0$ such that for any $\xi \in D(\Delta)$

$$\|(\Delta + \lambda)\, \xi\|_{L^2(M)} + \|\xi\|_{L^2\left(K_{\varepsilon/2}\right)} \geq A \|\xi\|_{L^2(M)}.$$

From this inequality, it is easy to prove that $\Delta + \lambda$ is Fredholm on L^2. In particular, λ has to be an L^2 eigenvalue if it is in the spectrum. We emphasize that the argument works for any $\lambda < n^2/2$. $\qquad \square$

In summary, for an ACH Kähler manifold we have $\lambda_0 \leq n^2/2$ and in general there may exist L^2 eigenvalues below $n^2/2$. The interesting question is when $\lambda_0 = n^2/2$. We believe this is related to the CR geometry on the boundary when the metric is Kähler-Einstein.

Before we state a precise conjecture, it may be helpful to recall the rudiments of CR geometry (for details one can check the original sources [T, We] or the recent book [DT] among many other references). Let Σ be a smooth manifold of dimension $2m + 1$. A CR structure on Σ is a pair $(H(\Sigma), J)$, where $H(\Sigma)$ is a subbundle

of rank $2m$ of the tangent bundle $T(\Sigma)$ and J is an almost complex structure on $H(\Sigma)$ such that

$$\left[H^{1,0}(\Sigma), H^{1,0}(\Sigma)\right] \subset H^{1,0}(\Sigma).$$

where $H^{1,0}(\Sigma) = \{u - \sqrt{-1}Ju | u \in H(\Sigma)\} \subset T(\Sigma) \otimes \mathbb{C}$ and $H^{0,1}(\Sigma) = \overline{H^{1,0}(\Sigma)}$. We will assume that our CR manifold Σ is oriented. Then there is a contact 1-form θ on Σ which annihilates $H(\Sigma)$. Any such θ is called a pseudo-Hermitian structure on Σ. Let $\omega = d\theta$. Then $G_\theta(X, Y) = \omega(X, JY)$ defines a symmetric bilinear form on the vector bundle $H(\Sigma)$. We assume that G_θ is positive definite. Such a CR manifold is said to be strongly pseudoconvex. If $\widetilde{\theta} = f\theta$ with $f > 0$, then $\widetilde{\omega} = d\widetilde{\theta} = f d\theta + df \wedge \theta$ and $\widetilde{\omega}|_{H(\Sigma)} = f\omega|_{H(\Sigma)}$. Hence this definition is independent of the choice of θ.

There is a unique vector field T on Σ such that

$$\theta(T) = 1, T \rfloor d\theta = 0.$$

This gives rise to the decomposition

$$T(\Sigma) = H(\Sigma) \oplus \mathbb{R}T.$$

Using this decomposition we then extend J to an endomorphism on $T(\Sigma)$ by defining $J(T) = 0$. We can also define a Riemannian metric g_θ on Σ such that

$$g_\theta(X, Y) = G_\theta(X, Y), g_\theta(X, T) = 0, g_\theta(T, T) = 1,$$

$\forall X, Y \in H(\Sigma)$. Obviously $\theta = g_\theta(T, \cdot), \omega = d\theta = g_\theta(J \cdot, \cdot)$.

It is shown by Tanaka and Webster that there is a unique connection ∇ on $T(\Sigma)$ such that

(1) $H(\Sigma)$ is parallel, i.e. $\nabla_X Y \in \Gamma(H(\Sigma))$ for any $X \in T(\Sigma)$ and any $Y \in \Gamma(H(\Sigma))$.
(2) $\nabla J = 0, \nabla g_\theta = 0$.
(3) The torsion τ satisfies

$$\tau(Z, W) = 0,$$
$$\tau(Z, \overline{W}) = \omega(Z, \overline{W})T,$$
$$\tau(T, J\cdot) = -J\tau(T, \cdot)$$

for any $Z, W \in H^{1,0}(\Sigma)$.

Let $\{T_\alpha\}$ be a local frame for $H^{1,0}(\Sigma)$. Then $\{T_\alpha, T_{\overline{\alpha}} = \overline{T_\alpha}, T\}$ is a local frame for $T(\Sigma) \otimes \mathbb{C}$. The metric is determined by the positive Hermitian matrix $h_{\alpha\overline{\beta}} = g_\theta\left(T_\alpha, T_{\overline{\beta}}\right)$. We have the Webster curvature tensor

$$R_{\mu\overline{\nu}\alpha\overline{\beta}} = \left\langle -\nabla_\mu \nabla_{\overline{\nu}} T_\alpha + \nabla_{\overline{\nu}} \nabla_\mu T_\alpha + \nabla_{[T_\mu, T_{\overline{\nu}}]} T_\alpha, T_{\overline{\beta}} \right\rangle$$

The pseudo-hermitian Ricci tensor is defined to be $R_{\mu\overline{\nu}} = -h^{\alpha\overline{\beta}} R_{\mu\overline{\nu}\alpha\overline{\beta}}$ and the pseudo-hermitian scalar curvature $\mathcal{R} = h^{\mu\overline{\nu}} R_{\mu\overline{\nu}}$.

Given θ, all the pseudo-hermitian structure associated the CR manifold can be written as

$$[\theta] = \left\{ f^{2/m}\theta : f \in C^\infty(\Sigma), f > 0 \right\}.$$

If $\widetilde{\theta} = f^{2/m}\theta$ is another pseudo-Hermitian structure, then the scalar curvatures are related by the following transformation

$$-\frac{2(m+1)}{m}\Delta_b f + \mathcal{R} = \widetilde{\mathcal{R}}f^{(m+2)/m},$$

where $\Delta_b u = \operatorname{div}\left(\nabla^H u\right)$ and $\nabla^H u$ is the horizontal gradient.

This is similar to the formula that relates two conformal Riemannian metrics. Motivated by the Yamabe problem in Riemannian geometry, Jerison and Lee [JL1] initiated the CR Yamabe problem. Like the Riemannian case, for a compact strongly pseudoconvex CR manifold $(\Sigma, [\theta])$ one can define its CR Yamabe invariant

$$Y(\Sigma, [\theta]) = \inf \frac{\int \left(\left|\nabla^H f\right|^2 + \frac{m}{2(m+1)}\mathcal{R}f^2\right)\theta \wedge (d\theta)^m}{\left(\int f^{2(m+1)/m}\theta \wedge (d\theta)^m\right)^{m/(m+1)}}.$$

The CR Yamabe problem is whether the infimum is achieved. The CR Yamabe problem has been intensively studied. We do not need the solution of the CR Yamabe problem. If suffices to know the elementary fact that $(\Sigma, [\theta])$ admits a pseudo-Hermitian metric with positive, zero or negative scalar curvature iff the Yamabe invariant is positive, zero or negative, respectively.

We now go back to our domain $\Omega \subset\subset M$ with strongly pseudoconvex boundary. Clearly, $\partial\Omega$ is a strongly pseudoconvex with $\theta = \sqrt{-1}\partial\phi$, with ϕ any defining function. With all these definitions, we can formulate the following

Conjecture 1. *Suppose (Ω, g) is an ACH Kähler-Einstein manifold of complex dimension n. Then $\lambda_0 = n^2/2$ if $\partial\Omega$ has nonnegative CR Yamabe invariant.*

We give an example which has $\lambda_0 < n^2/2$ and the boundary has negative CR Yamabe invariant. Let $\pi : L \to \Sigma$ be a negative holomorphic line bundle over a compact complex manifold with $\dim_{\mathbb{C}}\Sigma = n - 1$. Suppose L is endowed with a Hermitian metric h such that its curvature form is negative, i.e. the $(1,1)$ form $\omega_0 = \sqrt{-1}\partial\bar{\partial}\log|\sigma|_h^2$ on $L\backslash\{0\}$ defines a Kähler metric on Σ. We further assume

$$\operatorname{Ric}(\omega_0) = -n\omega_0.$$

For example, we can take a smooth hypersurface of degree $2n + 1$ in $\mathbb{P}^n$ with L being the dual of the hyperplane line bundle. The existence of ω_0 is guaranteed by the well-known theorem of Aubin and Yau on Kähler-Einstein metrics (the Calabi conjecture in the negative C_1 case). Let $\Omega = \{\sigma \in L : |\sigma|_h < 1\}$ be the unit disc bundle. Then Calabi [C] constructed an ACH Kähler-Einstein ω metric on Ω explicitly

$$\omega = \frac{1}{1 - |\sigma|^2}\omega_0 + \frac{|\sigma|^2}{\left(1 - |\sigma|^2\right)^2}\sqrt{-1}\partial\log|\sigma|^2 \wedge \bar{\partial}\log|\sigma|^2$$

$$= -\sqrt{-1}\partial\bar{\partial}\log\left(1 - |\sigma|^2\right) + \omega_0.$$

Notice that the metric is smooth across $\sigma = 0$. Indeed, if we write $|\sigma|^2 = \rho|w|^2$ using a local holomorphic trivialization of L, then

$$\omega = \frac{1}{1 - |\sigma|^2}\omega_0 + \frac{\rho}{\left(1 - |\sigma|^2\right)^2}\sqrt{-1}\left(dw + w\partial\log\rho\right) \wedge \left(d\bar{w} + \bar{w}\bar{\partial}\log\rho\right).$$

Given the formula, it is easy to verify the Kähler-Einstein equation

$$\text{Ric}\left(\omega\right) = \text{Ric}\left(\omega_0\right) + \left(n+1\right)\sqrt{-1}\partial\bar{\partial}\log\left(1-\left|\sigma\right|^2\right) - \sqrt{-1}\partial\bar{\partial}\log\left|\sigma\right|^2$$

$$= \text{Ric}\left(\omega_0\right) + \left(n - \frac{n+1}{1-\left|\sigma\right|^2}\right)\sqrt{-1}\partial\bar{\partial}\log\left|\sigma\right|^2$$

$$- \left(n+1\right)\frac{\left|\sigma\right|^2}{\left(1-\left|\sigma\right|^2\right)^2}\sqrt{-1}\partial\log\left|\sigma\right|^2 \wedge \bar{\partial}\log\left|\sigma\right|^2$$

$$= -\left(n+1\right)\omega.$$

This example was studied in detail in [W3] using a different normalization. In particular it was found there that $\lambda_0 = 2\left(n-1\right)$. It is less than $n^2/2$ if $n > 2$.

On the boundary $\partial\Omega = \left\{\sigma \in L : \left|\sigma\right|_h = 1\right\}$ which is a circle bundle over Σ, the CR structure is determined by the pseudo-hermition metric

$$\theta = \sqrt{-1}\partial\log\left|\sigma\right|^2.$$

Suppose (U, z) is a local chart on Σ on which we choose a holomorphic trivialization. Then locally $\partial\Omega$ is given by

$$N = \left\{\left(z, w\right) \in U \times \mathbb{C} : \rho\left(z\right)\left|w\right|^2 = 1\right\}.$$

with $\theta = \sqrt{-1}\partial\log\left(\rho\left|w\right|^2\right)$ and $d\theta = \omega_0$. We have the local frame

$$T_a = \frac{\partial}{\partial z_a} - w\frac{\partial\log\rho}{\partial z_a}\frac{\partial}{\partial w},$$

$$T = \sqrt{-1}\left(w\frac{\partial}{\partial w} - \bar{w}\frac{\partial}{\partial\bar{w}}\right).$$

Simple calculation yields

$$\left[T_\alpha, T_{\bar{\beta}}\right] = -\sqrt{-1}g_{\alpha\bar{\beta}}T,$$

$$\left[T_\alpha, T\right] = 0.$$

Thus, $\nabla_T T_\beta = 0, \nabla_{T_{\bar{\alpha}}} T_\beta = 0$, while

$$\nabla_{T_\alpha} T_\beta = g^{\gamma\bar{\nu}}\frac{\partial g_{\beta\bar{\nu}}}{\partial z_a}T_\gamma.$$

In particular, the Tanaka-Webster connection is torsion-free. Further calculation yields that the Webster curvature tensor of the Tanaka-Webster connection agrees with the curvature tensor of the Kähler metric ω_0 on Σ. In particular, the pseudo-hermitian scalar curvature of θ equals $-n\left(n-1\right)$.

3. The boundary effect

To study a noncompact Riemannian manifold, we can often choose an exhaustion by certain compact domains with smooth boundary and then study these compact manifolds with boundary. This approach is illustrated by the second author's proof

of Lee's theorem [W1]. In fact this method proves a stronger result. Recall the isoperimetric constant I_1 is defined as

$$I_1 = \inf_{D \subset M} \frac{A(\partial D)}{V(D)},$$

where the infimum is taken over all compact domains D with smooth boundary. It is a well known fact that $\lambda_0 \geq I_1^2/4$. It is proved in [W1] that $I_1 \geq n - 1$.

The main idea is in fact the following theorem about a compact manifold with boundary.

Theorem 4. *Let (M^n, g) be a compact Riemannian manifold with nonempty boundary. We assume that*

- *$\mathrm{Ric}(g) \geq -(n-1)$,*
- *The boundary ∂M has mean curvature $H \geq n - 1$.*

Then the first Dirichlet eigenvalue $\lambda_0 \geq (n-1)^2/4$.

To complete the proof of Lee's theorem, we focus for simplicity on the case of positive Yamabe invariant. We choose a metric h on the boundary Σ s.t. it scalar curvature $s > 0$. Then there is a defining function r near the boundary s.t. near the boundary

$$g = r^{-2} \left(dr^2 + h_r \right),$$

where h_r is a family of metrics on Σ with $h_0 = h$. Moreover, under the Einstein condition, we have the following expansion

$$h_r = h - \frac{r^2}{n-3} \left(\mathrm{Ric}(h) - \frac{s}{2(n-2)} h \right) + o\left(r^2\right).$$

For $\varepsilon > 0$ small enough, we consider $M^\varepsilon = \{r \geq \varepsilon\}$ which is a compact manifold with boundary. A direct computation shows that the mean curvature of the boundary satisfies

$$H = n - 1 + \frac{s}{2(n-2)} \varepsilon^2 + o\left(\varepsilon^2\right).$$

Therefore we have $H \geq n-1$ for ε sufficiently small. Applying Theorem 4 for each M^ε we conclude $\lambda_0(M) \geq (n-1)^2/4$.

In the Kähler case, we have the following analogue of Theorem 4.

Theorem 5. *Let (M, g) be a compact Kähler manifold with nonempty boundary and $\dim_{\mathbb{C}} M = n$. We assume that*

- *$K_{\mathbb{C}} \geq -1$,*
- *The second fundamental form of boundary ∂M satisfies $\Pi(J\nu, J\nu) \geq \sqrt{2}$ and $\Pi(X, X) + \Pi(JX, JX) \geq \sqrt{2}$ for all unit X perpendicular to $J\nu$.*

Then the first Dirichlet eigenvalue $\lambda_0 \geq n^2/2$.

In the statement, ν is the outer unit normal along ∂M and the second fundamental form is defined by

$$\Pi(X, X) = \langle \nabla_X \nu, Y \rangle.$$

Let $r : M \to \mathbb{R}^+$ be the distance function to ∂M. For any geodesic $\gamma : [0, l] \to M$ with $p = \gamma(0) \in \partial M$ and $\gamma'(0) = \nu(p)$ and any piecewise C^1 vector field X along γ with $X(0) \in T_p \partial M$, we have the index form

$$I(X, X) = -\Pi(X(0), X(0)) + \int_0^l \left[\left| \dot{X}(t) \right|^2 - R\left(X(t), \dot{\gamma}(t), X(t), \dot{\gamma}(t) \right) \right] dt.$$

The proof is based on the following lemma.

Lemma 1. *Under the same assumptions, we have $\Delta r \leq -\sqrt{2}n$ in the support sense.*

It suffices to calculate Δr at a non-focal point q. Let $\gamma : [0, l] \to M$ be a minimizing geodesic from ∂M to q. We have

$$\Delta r = \sum_{i=0}^{2(n-1)} I(Z_i, Z_i),$$

where $\{Z_i\}$ are normal Jacobi fields s.t. $\{Z_i(l)\}$ are orthonormal. Let $\{e_i\}$ be an orthonormal set in $T_{\gamma(0)} \partial M$ with $e_0 = J\nu$ and $e_{2\alpha} = Je_{2\alpha-1}$ for $\alpha = 1, \cdots, n-1$. Let $\{E_i(t)\}$ be parallel vector fields along γ with $E_i(0) = e_i$. For $L > l$ we set

$$X_0(t) = \frac{\sinh \sqrt{2}(L-t)}{\sinh \sqrt{2}(L-l)} E_0(t), \quad X_\alpha(t) = \frac{\sinh(L-t)/\sqrt{2}}{\sinh(L-l)/\sqrt{2}} E_\alpha(t).$$

We calculate

$$
\begin{aligned}
I(X_0, X_0) &= \frac{1}{\sinh^2 \sqrt{2}(L-l)} \left(-\sinh^2 \sqrt{2}L \, \Pi(J\nu, J\nu) \right. \\
&\qquad \left. + \int_0^l \left[2\cosh^2 \sqrt{2}(L-t) - H\left(\dot{\gamma}(t) \right) \sinh^2 \sqrt{2}(L-t) \right] dt \right) \\
&\leq \frac{1}{\sinh^2 \sqrt{2}(L-l)} \left(-\sqrt{2} \sinh^2 \sqrt{2}L \right. \\
&\qquad \left. + 2 \int_0^l \left[\cosh^2 \sqrt{2}(L-t) + \sinh^2 \sqrt{2}(L-t) \right] dt \right) \\
&= \frac{-\sqrt{2} \sinh^2 \sqrt{2}L - \sqrt{2} \sinh \sqrt{2}(L-l) \cosh \sqrt{2}(L-l) + \sqrt{2} \sinh \sqrt{2}L \cosh \sqrt{2}L}{\sinh^2 \sqrt{2}(L-l)} \\
&= \frac{\sqrt{2} \sinh \sqrt{2}L \left(\cosh \sqrt{2}L - \sinh \sqrt{2}L \right)}{\sinh^2 \sqrt{2}(L-l)} - \sqrt{2} \frac{\cosh \sqrt{2}(L-l)}{\sinh \sqrt{2}(L-l)}.
\end{aligned}
$$

Similarly, for $\alpha = 1, \cdots, n-1$

$$I\left(X_{2\alpha-1}, X_{2\alpha-1}\right) + I\left(X_{2\alpha}, X_{2\alpha}\right)$$

$$= \frac{1}{\sinh^2\left(L-l\right)/\sqrt{2}}\left(-\sinh^2 L/\sqrt{2}\left[\Pi\left(e_{2\alpha-1}, e_{2\alpha-1}\right) + \Pi\left(e_{2\alpha}, e_{2\alpha}\right)\right]\right.$$

$$+ \int_0^l \left[\frac{1}{2}\cosh^2\left(L-t\right)/\sqrt{2} - R\left(\dot{E}_{2\alpha-1}\left(t\right), \dot{\gamma}\left(t\right), \dot{E}_{2\alpha-1}\left(t\right), \dot{\gamma}\left(t\right)\right)\sinh^2\left(L-t\right)/\sqrt{2}\right]dt$$

$$\left.+ \int_0^l \left[\frac{1}{2}\cosh^2\left(L-t\right)/\sqrt{2} - R\left(\dot{E}_{2\alpha}\left(t\right), \dot{\gamma}\left(t\right), \dot{E}_{2\alpha}\left(t\right), \dot{\gamma}\left(t\right)\right)\sinh^2\left(L-t\right)/\sqrt{2}\right]dt\right)$$

$$\leq \frac{1}{\sinh^2\left(L-l\right)/\sqrt{2}}\left(-\sqrt{2}\sinh^2 L/\sqrt{2} + \int_0^l \left[\cosh^2\left(L-t\right)/\sqrt{2} + \sinh^2\left(L-t\right)/\sqrt{2}\right]dt\right)$$

$$= \frac{-\sqrt{2}\sinh^2 L/\sqrt{2} - \sqrt{2}\sinh\left(L-l\right)/\sqrt{2}\cosh\left(L-l\right)/\sqrt{2} + \sqrt{2}\sinh L/\sqrt{2}\cosh L/\sqrt{2}}{\sinh^2\left(L-l\right)/\sqrt{2}}$$

$$= \frac{2\sinh L/\sqrt{2}\left(\cosh L/\sqrt{2} - \sinh L/\sqrt{2}\right)}{\sinh^2\left(L-l\right)/\sqrt{2}} - \sqrt{2}\frac{\cosh\left(L-l\right)/\sqrt{2}}{\sinh\left(L-l\right)/\sqrt{2}}.$$

By letting $L \to \infty$ we obtain

$$I\left(X_0, X_0\right) \leq -\sqrt{2},$$

$$I\left(X_{2\alpha-1}, X_{2\alpha-1}\right) + I\left(X_{2\alpha}, X_{2\alpha}\right) \leq -\sqrt{2} \text{ for } \alpha = 1, \cdots, n-1.$$

By the minimality of Jacobi fields, we have $\Delta r \leq \sum_{i=0}^{2(n-1)} I\left(X_i, X_i\right) \leq -n\sqrt{2}$.

Theorem 5 then follows from the previous lemma by an argument in [W1]. For completeness, we provide the detailed proof.

Proof of Theorem 5. We consider the first eigenfunction f on M

$$\begin{cases} -\Delta f = \lambda_0 f, & \text{on} \quad M, \\ \qquad f = 0 & \text{on} \quad \partial M. \end{cases}$$

We can assume that $f > 0$ in M. Suppose $fe^{-nr/\sqrt{2}}$ achieves its maximum at an interior point p. For any $\delta > 0$, let ϕ_δ be a C^2 lower support function for $-r$ at p, i.e.

$$\phi_\delta \leq -r \text{ in a neighborhood } U_\delta \text{ of } p;$$

$$\phi_\delta\left(p\right) \leq -r\left(p\right), \Delta\phi_\delta\left(p\right) \geq n\sqrt{2} - \delta.$$

As $-r$ is Lipschitz with Lipschitz constant ≤ 1, it is easy to prove $\left|\nabla\phi_\delta\right|\left(p\right) \leq 1$. The C^2 function $fe^{n\phi_\delta/\sqrt{2}}$ on U_δ achieves its maximum at p, so we have

$$\nabla f\left(p\right) = -\frac{n}{\sqrt{2}}\nabla\phi_\delta\left(p\right),$$

$$\Delta\left(fe^{n\phi_\delta/\sqrt{2}}\right)\left(p\right) \leq 0.$$

We calculate at p

$$\Delta\left(fe^{n\phi_\delta/\sqrt{2}}\right)(p) = e^{n\phi_\delta/\sqrt{2}}\left(\Delta f + n\sqrt{2}\left\langle\nabla f, \nabla\phi_\delta\right\rangle + \frac{n}{\sqrt{2}}f\Delta\phi_\delta + \frac{n^2}{2}f\left|\nabla\phi_\delta\right|^2\right)$$

$$= e^{n\phi_\delta/\sqrt{2}}\left(-\lambda_0 f + \frac{n}{\sqrt{2}}f\Delta\phi_\delta - \frac{n^2}{2}f\left|\nabla\phi_\delta\right|^2\right)$$

$$\geq fe^{n\phi_\delta/\sqrt{2}}\left(-\lambda_0 + n^2 - \frac{n\delta}{\sqrt{2}} - \frac{n^2}{2}\right)$$

$$= fe^{n\phi_\delta/\sqrt{2}}\left(-\lambda_0 + \frac{n^2}{2} - \frac{n\delta}{\sqrt{2}}\right).$$

Therefore $\lambda_0 \geq n^2/2 - n\delta/\sqrt{2}$. Let $\delta \to 0$ we get $\lambda_0 \geq n^2/2$. $\square$

4. Strictly pseudoconvex domain in $\mathbb{C}^n$

Let Ω be a smooth, bounded strictly pseudoconvex domain in $\mathbb{C}^n$. We want to study Kähler metrics $\omega_u = \sqrt{-1}\partial\bar{\partial}u$, where u is smooth Ω s.t. $\rho(z) = -e^{-u}$ is a defining function for $\partial\Omega$. The relation between u and ρ associated with Monge-Ampère operator and Fefferman operator is as follows

$$(4.1) \qquad\qquad \det H(u) = J(\rho)e^{(n+1)u},$$

where $H(u)$ is complex Hessian matrix and

$$J(\rho) = -\det\begin{bmatrix} \rho & \rho_{\bar{j}} \\ \rho_i & \rho_{i\bar{j}} \end{bmatrix}.$$

Notice that such a metric is ACH. Indeed, let ϕ be a strictly plurisubharmonic defining function, i.e. $\sqrt{-1}\partial\bar{\partial}\phi > 0$. Then we have $\rho = \phi f$, where f is smooth and positive on $\overline{\Omega}$. Therefore

$$\omega_u = -\sqrt{-1}\partial\bar{\partial}\log\left(-\rho\right)$$

$$= -\sqrt{-1}\partial\bar{\partial}\log\left(-\phi\right) - \sqrt{-1}\partial\bar{\partial}\log\left(-f\right),$$

clearly ACH. Then the Ricci form for ω_u is

$$\mathrm{Ric} = -\sqrt{-1}\partial\bar{\partial}\log\det H(u)$$

$$= -(n+1)\sqrt{-1}\partial\bar{\partial}u - \sqrt{-1}\partial\bar{\partial}\log J(\rho)$$

$$= -(n+1)\omega_u - \sqrt{-1}\partial\bar{\partial}\log J(\rho).$$

Cheng and Yau [CY] proved that there exists a unique Kähler-Einstein metric $\omega_u = \sqrt{-1}\partial\bar{\partial}u$, where the strictly plurisubharmonic u solves the following Monge-Ampère equation

$$\det H(u) = e^{(n+1)u} \quad \text{in} \quad \Omega,$$
$$u = \infty \qquad\quad \text{on} \quad \partial\Omega.$$

Or equivalent, $\rho(z) = -e^{-u}$ solves the Fefferman equation $J(\rho) = 1$. Moreover, they proved that $\rho \in C^{n+3/2}\left(\overline{\Omega}\right)$ and it is a defining function for $\partial\Omega$.

To study the spectrum of such metrics, our starting point is the following theorem proved in [LT].

Theorem 6. *Let $\omega_u = \sqrt{-1}\partial\bar{\partial}u$ be a Kähler metric on Ω with $\rho(z) = -e^{-u}$ a defining function for $\partial\Omega$. If ρ is plurisubharmonic, then $\lambda_0(\omega_u) = n^2/2$.*

Remark 1. *Notice that*

$$(4.2) \qquad \rho_{i\bar{j}} = e^{-u}\left(u_{i\bar{j}} - u_i u_{\bar{j}}\right).$$

Therefore ρ is plurisubharmonic iff $|\partial u|_g^2 \leq 1$.

From (4.2) we also obtain $\det H(\rho) = e^{-nu}(1 - |\partial u|_g^2)\det H(u)$. Combined with (4.1) it yields the following formula

$$\frac{\det H(\rho)}{J(\rho)} = e^u(1 - |\partial u|_g^2).$$

It follows that $|\partial u|_g \to 1$ as long as $\rho\frac{\det H(\rho)}{J(\rho)} \to 0$ as $z \to \partial D$.

For estimating the lower bound for $\lambda_1(\Delta_g)$, it was proved in [LT] that if ρ is plurisubharmonic in D then $\lambda_1(\Delta_g) \geq n^2/2$. Various version of the following lemma was proved and used in [Li1, Li2]. Here, we state and prove by using Ricci curvature.

Lemma 2. *Let $\omega_u = \sqrt{-1}\partial\bar{\partial}u$ be a Kähler metric on Ω with $\rho(z) = -e^{-u}$ a defining function for $\partial\Omega$. If $Ric \geq -(n+1)$, then*

$$\Delta\left[e^u\left(|\partial u|_g^2 - 1\right)\right] \geq 0.$$

Proof. We notice that

$$g^{i\bar{l}}g^{k\bar{j}}u_{i\bar{j}}u_{k\bar{l}} = m, \Box u = g^{i\bar{j}}u_{i\bar{j}} = m.$$

By the Bochner formula, we have

$$\Box|\partial u|_g^2 = g^{i\bar{l}}g^{k\bar{j}}u_{i\bar{j}}u_{k\bar{l}} + g^{i\bar{l}}g^{k\bar{j}}u_{i,k}u_{\bar{j},\bar{l}} + g^{i\bar{j}}\left(u_i(\Box u)_{\bar{j}} + (\Box u)_i u_{\bar{j}}\right) + R_{i\bar{j}}u^i u^{\bar{j}}$$

$$\geq m + g^{i\bar{l}}g^{k\bar{j}}u_{i,k}u_{\bar{j},\bar{l}} - (m+1)|\partial u|_g^2.$$

We calculate

$$\Box\left[e^u\left(|\partial u|_g^2 - 1\right)\right]$$

$$= e^u\left[\left(\Box u + |\partial u|_g^2\right)\left(|\partial u|_g^2 - 1\right) + \Box|\partial u|_g^2\right]$$

$$+ e^u\left[g^{i\bar{j}}\left(u_i\left(|\partial u|_g^2\right)_{\bar{j}} + \left(|\partial u|_g^2\right)_i u_{\bar{j}}\right)\right]$$

$$\geq e^u\left[\left(m + |\partial u|_g^2\right)\left(|\partial u|_g^2 - 1\right) + m + g^{i\bar{l}}g^{k\bar{j}}u_{i,k}u_{\bar{j},\bar{l}} - (m+1)|\partial u|_g^2\right]$$

$$+ e^u\left[g^{i\bar{j}}\left(u_i\left(|\partial u|_g^2\right)_{\bar{j}} + \left(|\partial u|_g^2\right)_i u_{\bar{j}}\right)\right]$$

$$= e^u\left[|\partial u|_g^4 - 2|\partial u|_g^2 + g^{i\bar{l}}g^{k\bar{j}}u_{i,k}u_{\bar{j},\bar{l}}\right] + e^u\left[g^{i\bar{j}}\left(u_i\left(|\partial u|_g^2\right)_{\bar{j}} + \left(|\partial u|_g^2\right)_i u_{\bar{j}}\right)\right]$$

On the other hand

$$\left(|\partial u|_g^2\right)_i = \left(g^{k\bar{l}}u_k u_{\bar{l}}\right)_i$$

$$= g^{k\bar{l}}\left(u_{k,i}u_{\bar{l}} + u_k u_{i\bar{l}}\right)$$

$$= g^{k\bar{l}}u_{k,i}u_{\bar{l}} + u_i.$$

Similarly

$$\left(|\partial u|_g^2\right)_{\overline{j}} = g^{k\overline{l}}u_k u_{\overline{j},\overline{l}} + u_{\overline{j}}.$$

Therefore

$$\square\left[e^u\left(|\partial u|_g^2 - 1\right)\right]$$
$$\geq e^u\left[|\partial u|_g^4 - 2|\partial u|_g^2 + g^{i\overline{l}}g^{k\overline{j}}u_{i,k}u_{\overline{j},\overline{l}}\right] + e^u\left[g^{i\overline{j}}g^{k\overline{l}}\left(u_{k,i}u_i u_{\overline{l}} + u_{\overline{j},\overline{l}}u_i u_k\right) + 2|\partial u|_g^2\right]$$
$$= e^u\left[|\partial u|_g^4 + g^{i\overline{l}}g^{k\overline{j}}u_{i,k}u_{\overline{j},\overline{l}} + g^{i\overline{j}}g^{k\overline{l}}\left(u_{k,i}u_i u_{\overline{l}} + u_{\overline{j},\overline{l}}u_i u_k\right)\right]$$
$$\geq 0,$$

by the Cauchy-Schwarz inequality. $\square$

Theorem 7. *Let Ω be a smoothly bounded and strictly pseudoconvex domain in $\mathbb{C}^n$ with defining function $\rho \in C^3(\overline{\Omega})$ so that $u = -\log(-\rho)$ is strictly plurisubharmonic in Ω. Let g be the Kähler metric with Kähler form $\omega_u = \sqrt{-1}\partial\overline{\partial}u$. Let $(\partial\Omega, \theta)$ be the pseudo-hermitian manifold with the contact form $\theta = \sqrt{-1}\partial\rho$ and let $\mathcal{R}_\theta$ be its Webster pseudo scalar curvature. Assume that g is Kähler-Einstein. If $\mathcal{R}_\theta \geq 0$ on $\partial\Omega$ then $\lambda_0 = n^2/2$.*

Proof. The proof is based on the following formula

$$Ric_\theta\left(X, \overline{Y}\right) = -D^2\left(\log J\left(\rho\right)\right)\left(X, \overline{Y}\right) + (m+1)\frac{\det H\left(\rho\right)}{J\left(\rho\right)}\left(X, \overline{Y}\right).$$

If $\sqrt{-1}\partial\overline{\partial}u$ is Kähler-Einstein, then $J\left(\rho\right) = 1$. Thus

$$Ric_\theta\left(X, \overline{Y}\right) = (m+1)\det H\left(\rho\right)\left(X, \overline{Y}\right),$$
$$\mathcal{R}_\theta = m\left(m+1\right)\det H\left(\rho\right).$$

If $\mathcal{R}_\theta \geq 0$, then $\det H\left(\rho\right) \geq 0$ on $\partial\Omega$. By the previous lemma and the maximum principle, we have $|\partial u|_g^2 \leq 1$. Therefore $\lambda_0 = n^2/2$ by Theorem 6. $\square$

Remark 2. *It is clear from the proof that the conclusion $\lambda_0 = n^2/2$ remains valid if the Kähler-Einstein condition is replaced by $\mathrm{Ric}\,(g) \geq -(n+1)g$ and $R_g + n(n+1) = O(\rho^2)$ in Ω.*

Remark 3. *When Ω is an ellipsoid with Kähler-Einstein metric, the above theorem was proved by the first author in [Li3].*

References

[C]   E. Calabi, Métriques kählériennes et fibrés holomorphes. Ann. Sci. École Norm. Sup. (4) 12 (1979), no. 2, 269–294.

[Ch]   S. Y. Cheng, Eigenvalue comparison theorems and its geometric application, Math. Z. 143 (1975), 289– 297.

[CY]   S. Cheng and S.T. Yau, On the existence of a complete Kähler metric on noncompact complex manifolds and the regularity of Fefferman's equation. Comm. Pure Appl. Math. 33 (1980), no. 4, 507–544.

[DT]   S. Dragomir and G. Tomassi, **Differential geometry and analysis on CR manifolds**. Birkhauser, Boston, 2006.

[JL1] D. Jerison and J. Lee, The Yamabe problem on CR manifolds. J. Diff. Geom. 25 (1987), 167-197.

[JL2] D. Jerison and J. Lee, Extremals for the Sobolev inequality on the Heisenberg group and the CR Yamabe problem. J. AMS 1 (1988), No. 1, 1-13.

[Lee1] J. Lee, The spectrum of an asymptotically hyperbolic Einstein manifold. Comm. Anal. Geom. 3 (1995), 253-271.

[Lee2] J. Lee, Pseudo-Einstein structures on CR manifolds. American J. Math. 110 (1988), 157-178.

[Lee3] J. Lee, Fredholm operators and Einstein metrics on conformally compact manifolds. (English summary) Mem. Amer. Math. Soc. 183 (2006), no. 864.

[Li1] S.-Y. Li, Characterization for balls by potential function of Kähler-Einstein metrics for domains in $\mathbb{C}^n$, Comm. Anal. Geom. 13 (2005), No. 2, 461–478.

[LW1] P. Li and J. Wang, Connectedness at infinity of complete Kähler manifolds. Amer. J. Math. 131 (2009), no. 3, 771–817.

[LW2] P. Li and J. Wang, Comparison theorem for Kähler manifolds and positivity of spectrum. J. Differential Geom. 69 (2005), no. 1, 43–74.

[Li2] S.-Y. Li, Characterization for a class of pseudoconvex domains whose boundaries having positive constant pseudo scalar curvature, Comm. Anal. Geom. 17(2009), 17–35.

[Li3] S.-Y. Li, On the Kähler manifolds with the largest the infimum of the spectrum of Laplace-Beltrami operators and sharp lower bound of Ricci or holomorphic bisectional curvatures, Preprint.

[LL] S.-Y. Li and H-S Luk, An explicit formula for the Webster pseudo Ricci curvature on real hypersurfaces and its application for characterizing balls in $\mathbb{C}^n$, Comm. Anal. Geom. 14 (2006), 673-701.

[LT] S-Y. Li and M-A. Tran, Infimum of the spectrum of Laplace-Beltrami operator on a bounded pseudoconvex domain with a Kähler metric of Bergman type, to appear in Comm. Anal. Geom.

[M] O. Munteanu, A sharp estimate for the bottom of the spectrum of the Laplacian on Kähler manifolds, to appear in Journal of Differential Geometry.

[S] T. Sakai, Riemannian geometry. Translations of Mathematical Monographs, 149. American Mathematical Society, Providence, RI, 1996.

[T] N. Tanaka, A differential geometric study on strongly pseudo-convex manifolds. Lectures in Mathematics, Department of Mathematics, Kyoto University, No. 9. Kinokuniya Book-Store Co., Ltd., Tokyo, 1975.

[W1] X. Wang, A new proof of Lee's theorem on the spectrum of conformally compact Einstein manifolds, Comm. Aanl. Geom. 10 (2002), 647-651.

[W2] X. Wang, On conformally compact Einstein manifolds, Math. Res. Letters 8 (2001), 671-688.

[W3] X. Wang, On L^2 cohomology of ACH Kähler manifolds, Proc. AMS 135 (2007), 2949-2960.

[We] S. Webster, Pseudo-Hermitian geometry on a real hypersurface, J. Diff. Geom.13 (1978), 25–41.

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