HARMONIC FUNCTIONS, ENTROPY, AND A CHARACTERIZATION OF THE HYPERBOLIC SPACE

XIAODONG WANG

ABSTRACT. Let (M^n, g) be a compact Riemannian manifold with $Ric \geq -(n-1)$. It is well known that the bottom of spectrum λ_0 of its unverversal covering satisfies $\lambda_0 \leq (n-1)^2/4$. We prove that equality holds iff M is hyperbolic. This follows from a sharp estimate for the Kaimanovich entropy.

1. INTRODUCTION

Complete Riemannian manifolds with nonnegative Ricci curvature have been intensively studied by many people and there are various methods and many beautiful results (see e.g., the book [P]). One of the most important theorems on such manifolds is the following Cheeger-Gromoll splitting theorem:

Theorem 1. *(Cheeger-Gromoll) If (N, g) contains a line and has $Ric \geq 0$, then (N, g) is isometric to a product $\left(\mathbb{R} \times \Sigma, dt^2 + h\right)$.*

This theorem has the following important corollaries on the structure of manifolds with nonnegative Ricci curvature:

- A complete Riemannian (N, g) with $Ric \geq 0$ either has only one end or is isometric to a product $\left(\mathbb{R} \times \Sigma, dt^2 + g_\Sigma\right)$, with Σ compact.
- If (M^n, g) is compact with $Ric \geq 0$ then its universal covering $\widetilde{M}$ splits isometrically as a product $\mathbb{R}^k \times \Sigma^{n-k}$, where Σ is a simply connected compact manifold with $Ric \geq 0$. If furthermore $\widetilde{M}$ has Euclidean volume growth, then $\widetilde{M}$ is isometric to $\mathbb{R}^n$.

Riemannian manifolds with a negative lower bound for Ricci curvature are considerably more complicated and less understood. It is naive to expect such splitting results in general. Nevertheless there have been very interesting results due to Li and J. Wang recently. It has been discovered that the bottom of the L^2 spectrum plays an important role (see also the earlier work [W] in the conformally compact case). Let us assume that (N^n, g) is a complete Riemannian manifold with $Ric \geq -(n-1)$. The bottom of the L^2 spectrum of the Laplace operator on functions is denoted by $\lambda_0(N)$ and can be characterized as

$$\lambda_0(N) = \inf \frac{\int_N |\nabla u|^2}{\int_N u^2},$$

where the infimum is taken over all smooth functions with compact support. It is well known $\lambda_0(N) \leq (n-1)^2/4$. Li-Wang proved the following theorems for manifolds with positive λ_0.

Theorem 2. *(Li-Wang) Let (N^n, g) be a complete Riemannian manifold with $Ric \geq -(n-1)$ and $\lambda_0(N) \geq n-2$. Then either*
  (1) *N has only one end with infinite volume; or*
  (2) *$N = \mathbb{R} \times \Sigma$ with warped product metric $g = dt^2 + \cosh^2 tg_\Sigma$, where (Σ, g_Σ) is compact with $Ric \geq -(n-2)$.*

When $\lambda_0(N) = (n-1)^2/4$, they can also handle ends of finite volume.

Theorem 3. *(Li-Wang) Let (N^n, g) be a complete Riemannian manifold with $Ric \geq -(n-1)$ and $\lambda_0(N) = (n-1)^2/4$. Then either*
  (1) *N has only one end; or*
  (2) *$N = \mathbb{R} \times \Sigma$ with warped product metric $g = dt^2 + e^{2t}g_\Sigma$, where (Σ, h) is compact manifold with nonnegative Ricci curvature;*
  (3) *$n = 3$ and $N = \mathbb{R} \times \Sigma$ with warped product metric $g = dt^2 + \cosh^2 tg_\Sigma$, where (Σ, h) is compact surface with Gaussian curvature ≥ -1.*

The basic point is that $\lambda_0(N)$ is sensitive to the connectedness at infinity. In both cases if there are two ends they are able to prove that the manifold splits as a warped product. Since the splitting is only obtained under this restrictive situation, their theorems are not as powerful as the Cheeger-Gromoll splitting theorem is for manifolds with nonnegative Ricci curvature. An intriguing question is if there is a more general mechanism under which a complete (N^n, g) with $Ric \geq -(n-1)$ and $\lambda_0(N) = (n-1)^2/4$ splits as a warped product.

We will consider a special situation inspired by an aforementioned corollary of the Cheeger-Gromoll theorem. Suppose $\widetilde{M}$ is the universal covering of a compact Riemannian manifold (M^n, g) with $Ric \geq -(n-1)$. What happens if $\lambda_0\left(\widetilde{M}\right) = (n-1)^2/4$, the largest possible value? From Theorem 3 it seems the only information we can draw is that $\widetilde{M}$ has only one end. On the other hand it is reasonable to expect that $\widetilde{M}$ is isometric to $\mathbb{H}^n$, i.e. (M^n, g) is a hyperbolic manifold. In fact this has been conjectured by Jiaping Wang.

The main result of this paper is an affirmative answer to this conjecture, i.e. we prove

Theorem 4 (Main Theorem). *Let (M^n, g) be a compact Riemannian manifold with $Ric \geq -(n-1)$ and $\pi: \widetilde{M} \to M$ its universal covering. If $\lambda_0\left(\widetilde{M}\right) = (n-1)^2/4$, then $\widetilde{M}$ is isometric to the hyperbolic space $\mathbb{H}^n$.*

It is worth pointing out that complete Riemannian manifolds N^n with $Ric \geq -(n-1)$ and $\lambda_0 = (n-1)^2/4$ are abundant. There are many conformally compact examples by a theorem of Lee [Lee]. Therefore the more subtle assumption that $\widetilde{M}$ covers a compact manifold is essential.

If we further assume that g is negatively curved, then the result follows from the following well known theorem. Recall the volume entropy h is defined to be

$$h = \lim_{r \to +\infty} \frac{\log V(p, r)}{r},$$

where $V(p,r)$ is the volume of the geodesic ball with center p and radius r in $\widetilde{M}$.

Theorem 5. *Let (M^n, g) be a compact Riemannian manifold with negative curvature and $\pi : \widetilde{M} \to M$ its universal covering. If $\lambda_0\left(\widetilde{M}\right) = h^2/4$, then M is locally symmetric.*

This is a deep theorem whose proof is difficult and involved : first it is proved by Ledrappier [L1] that $\widetilde{M}$ is asymptotically harmonic in the sense that all the level sets of the Buseman functions have constant mean curvature; then by a theorem of Foulon and Labourie [FL] the geodesic flow of (M, g) is C^∞-conjugate to that of a locally symmetric space of rank one; finally by the work of Besson-Courtois-Gallot [BCG] one concludes that (M, g) is locally symmetric.

In contrast, we make no additional assumption on the sectional curvature and prove the Main Theorem in a direct way. A key ingredient in our proof is the entropy introduced by Kaimanovich [K], which we learned from [L1]. The paper is organized as follows. In Section 2, we discuss positive harmonic functions and the Martin boundary. In Section 3, we present the Kaimanovich entropy. We establish a sharp inequality which characterizes hyperbolic manifolds. The main theorem is then proved in Section 4.

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2. Harmonic functions and the Martin boundary

In this section we collect some fundamental facts on positive harmonic functions. There are two aspects: potential theory and geometric analysis. On potential theory (more specifically, the theory of Martin boundary) our primary reference is Ancona [A] ([AG] and [H] are also very useful) . Let $\widetilde{M}^n$ be a complete Riemannian manifold with a base point o. We assume that $\widetilde{M}$ is non-parabolic, that is, it has a positive Green's function. It is well known that $\widetilde{M}$ is non-parabolic if $\lambda_0\left(\widetilde{M}\right) > 0$ (see, e.g. [SY]). The vector space $\mathcal{H}\left(\widetilde{M}\right)$ of harmonic functions with seminorms

$$\|u\|_K = \sup_K |u(x)|, K \subset \widetilde{M} \text{ compact}$$

is a Frechet space. Let $\mathcal{K}_o = \{u \in \mathcal{H}(M) : u(o) = 1, u > 0\}$. This is a convex and compact set.

Definition 1. *A harmonic function $h > 0$ on $\widetilde{M}$ is called minimal if any nonnegative harmonic function $\leq h$ is proportional to h.*

Remark 1. *If $h(o) = 1$, then h is minimal iff h is an extremal point of $\mathcal{K}_o$.*

Definition 2. *The minimal Martin boundary of M is*

$$\partial^*\widetilde{M} = \{h \in \mathcal{K}_o : h \text{ is minimal}\}.$$

According to a theorem of Choquet ([A]), for any positive harmonic function h there is a unique Borel measure μ^h on $\partial^*\widetilde{M}$ such that

$$h(x) = \int_{\partial^*\widetilde{M}} \xi(x) \, d\mu^h(\xi).$$

Let ν be the measure corresponding to the harmonic function 1. Thus

(2.1) $$1 = \int_{\partial^* \widetilde{M}} \xi(x)\, d\nu(\xi).$$

For $f \in L^\infty\left(\partial^* \widetilde{M}\right)$ we get a bounded harmonic function

$$H_f(x) = \int_{\partial^* \widetilde{M}} f(\xi)\, \xi(x)\, d\nu(\xi).$$

When there is a lower Ricci bound, Yau's gradient estimate for positive harmonic functions (see e.g. [SY]) is a very powerful tool. The following sharp version is due to Li-Wang [LW2].

Lemma 1. *Let (N^n, g) be a complete Riemannian manifold with $Ric \geq -(n-1)$. For a positive harmonic function f on N we have*

$$|\nabla \log f| \leq n - 1$$

on N.

Let $\phi = \log f$. Then $\Delta \phi = -|\nabla \phi|^2$. Using the Bochner formula we have

(2.2) $$\frac{1}{2}\Delta |\nabla \phi|^2 = |D^2 \phi|^2 + \langle \nabla \phi, \nabla \Delta \phi \rangle + Ric(\nabla \phi, \nabla \phi)$$
$$\geq |D^2 \phi|^2 - \left\langle \nabla \phi, \nabla |\nabla \phi|^2 \right\rangle - (n-1)|\nabla \phi|^2.$$

Using the equation of ϕ and some algebra one can derive the following inequality at any point where $\nabla \phi \neq 0$

(2.3) $$|D^2 \phi|^2 \geq \frac{n}{4(n-1)} \frac{\left|\nabla |\nabla \phi|^2\right|^2}{|\nabla \phi|^2} + \frac{|\nabla \phi|^4 + \left\langle \nabla \phi, \nabla |\nabla \phi|^2 \right\rangle}{n-1}.$$

Moreover equality holds iff

(2.4) $$D^2 \phi = -\frac{|\nabla \phi|^2}{n-1}\left[g - \frac{1}{|\nabla \phi|^2} d\phi \otimes d\phi\right].$$

Combining (2.3) with (2.2) yields

(2.5) $$\frac{1}{2}\Delta |\nabla \phi|^2 \geq \frac{n}{4(n-1)} \frac{\left|\nabla |\nabla \phi|^2\right|^2}{|\nabla \phi|^2} - \frac{n-2}{n-1}\left\langle \nabla \phi, \nabla |\nabla \phi|^2 \right\rangle$$
$$- (n-1)|\nabla \phi|^2 + \frac{|\nabla \phi|^4}{n-1}.$$

The remaining part of the proof is to construct an appropriate cut-off function and apply the maximum principle to show $|\nabla \phi| \leq n - 1$.

If it happens that $|\nabla \phi| \equiv n - 1$, then (2.5) is an equality. Therefore by (2.4) we have

$$D^2 \phi = -(n-1)\left[g - \frac{1}{(n-1)^2} d\phi \otimes d\phi\right].$$

One can then prove that N splits as a warped product. More precisely we have the following lemma from [LW2].

Lemma 2. *Let (N^n, g) be a complete Riemannian manifold with $Ric \geq -(n-1)$. If there exists a positive harmonic function f on N such that $|\nabla \log f| \equiv n-1$ on N, then (N^n, g) is isometric to $(\mathbb{R} \times \Sigma^{n-1}, dt^2 + e^{2t} g_\Sigma)$, where (Σ^{n-1}, g_Σ) is complete and has nonnegative Ricci curvature. Moreover $\log f = -(n-1) t$.*

3. THE KAIMANOVICH ENTROPY

In [K] Kaimanovich studied Brownian motion on a regular covering $\widetilde{M}$ of a compact Riemannian manifold M. He introduced a remarkable entropy which will play an important role in the proof of our main theorem. We will present his theory using the minimal Martin boundary instead of the stationary boundary for Brownian motion. The equivalence of the two approaches can be seen from [A] (section 3 in particular). For more detailed discussions related to the Kaimanovich entropy we refer to several papers by Ledrappier [L1, L2, L3].

From then on we assume that $\widetilde{M}$ is the universal covering of a compact manifold M. We identify $\pi_1(M)$ with the group Γ of deck transformations on $\widetilde{M}$ and therefore $M = \widetilde{M}/\Gamma$. There is a natural Γ-action on $\partial^* \widetilde{M}$: for $\xi \in \partial^* \widetilde{M}$ and $\gamma \in \Gamma$

$$(\gamma \cdot \xi)(x) = \frac{\xi(\gamma^{-1}x)}{\xi(\gamma^{-1}o)}.$$

As a result for each $\gamma \in \Gamma$ we have the pushforward measure $\gamma_* \nu$ such that $\gamma_* \nu(E) = \nu(\gamma^{-1} \cdot E)$ for any Borel set $E \subset \partial^* \widetilde{M}$. By the definition of ν and a change of variables

$$
\begin{aligned}
1 &= \int_{\partial^* \widetilde{M}} \xi(\gamma^{-1}x) \, d\nu(\xi) \\
&= \int_{\partial^* \widetilde{M}} (\gamma \cdot \xi)(x) \, \xi(\gamma^{-1}o) \, d\nu(\xi). \\
&= \int_{\partial^* \widetilde{M}} \eta(x) \frac{1}{\eta(\gamma o)} \, d\gamma_* \nu(\eta).
\end{aligned}
$$

By the uniqueness of ν we have

$$d\gamma_* \nu(\xi) = \xi(\gamma o) \, d\nu(\xi).$$

We define

$$\phi(x, y) = -\int_{\partial^* \widetilde{M}} \xi(y) \log \frac{\xi(x)}{\xi(y)} \, d\nu(\xi).$$

It is easy to show that $\phi(\gamma x, \gamma y) = \phi(x, y)$ for any $\gamma \in \Gamma$ and

$$
\begin{aligned}
\phi(x, x) &= 0, \\
\Delta_y \phi(x, y) &= \int_{\partial^* \widetilde{M}} \xi(y) |\nabla \log \xi(y)|^2 \, d\nu(\xi).
\end{aligned}
$$

By Yau's gradient estimate we have

$$
\begin{aligned}
|\phi(x, y)| &\leq Cd(x, y), \\
|\nabla_y \phi(x, y)| &\leq Cd(x, y), \\
|\Delta_y \phi(x, y)| &\leq C.
\end{aligned}
$$

Let $p(t, x, y)$ be the heat kernel on $\widetilde{M}$. For any $\gamma \in \Gamma$

$$p(t, \gamma x, \gamma y) = p(t, x, y).$$

We define
$$u(\tau, x) = \frac{1}{\tau} \int_{\widetilde{M}} \phi(x, y) \, p(\tau, x, y) \, dv(y).$$

It is easy to see that u descends to M. Indeed,

$$
\begin{aligned}
u(\tau, \gamma x) &= \frac{1}{\tau} \int_{\widetilde{M}} \phi(\gamma x, y) \, p(\tau, \gamma x, y) \, dv(y) \\
&= \frac{1}{\tau} \int_{\widetilde{M}} \phi(x, \gamma^{-1} y) \, p(\tau, \gamma x, y) \, dv(y) \\
&= \frac{1}{\tau} \int_{\widetilde{M}} \phi(x, z) \, p(\tau, \gamma x, \gamma z) \, dv(z) \\
&= \frac{1}{\tau} \int_{\widetilde{M}} \phi(x, z) \, p(\tau, x, z) \, dv(z) \\
&= u(\tau, \gamma x).
\end{aligned}
$$

We can rewrite

$$
\begin{aligned}
u(\tau, x) &= \frac{1}{\tau} \int_{\widetilde{M}} \phi(x, y) \, p(\tau, x, y) \, dv(y) \\
&= \frac{1}{\tau} \int_{\widetilde{M}} \phi(x, y) \left(\int_0^\tau \frac{\partial}{\partial t} p(t, x, y) \, dt \right) dv(y) \\
&= \frac{1}{\tau} \int_0^\tau \left(\int_{\widetilde{M}} \phi(x, y) \, \Delta_y p(t, x, y) \, dv(y) \right) dt \\
&= \frac{1}{\tau} \int_0^\tau \left(\int_{\widetilde{M}} \Delta_y \phi(x, y) \, p(t, x, y) \, dv(y) \right) dt \\
&= \int_{\partial^* \widetilde{M}} \left[\frac{1}{\tau} \int_0^\tau \left(\int_{\widetilde{M}} \xi(y) \, |\nabla \log \xi(y)|^2 \, p(t, x, y) \, dv(y) \right) dt \right] d\nu(\xi) \\
&= \int_{\widetilde{M} \times \partial^* \widetilde{M} \times [0,1]} \xi(y) \, |\nabla \log \xi(y)|^2 \, p(\tau s, x, y) \, dv(y) \, d\nu(\xi) \, ds
\end{aligned}
$$

i.e.

(3.1) $$u(\tau, x) = \int_{\widetilde{M} \times \partial^* \widetilde{M} \times [0,1]} \xi(y) \, |\nabla \log \xi(y)|^2 \, p(\tau s, x, y) \, dv(y) \, d\nu(\xi) \, ds.$$

Moreover

$$
\begin{aligned}
\frac{\partial u}{\partial \tau}(\tau, x) &= \int_{\widetilde{M} \times \partial^* \widetilde{M} \times [0,1]} \xi(y) \, |\nabla \log \xi(y)|^2 \, s \frac{\partial p}{\partial t}(\tau s, x, y) \, dv(y) \, d\nu(\xi) \, ds \\
&= \int_{\widetilde{M} \times \partial^* \widetilde{M} \times [0,1]} \xi(y) \, |\nabla \log \xi(y)|^2 \, s \Delta_x p(\tau s, x, y) \, dv(y) \, d\nu(\xi) \, ds \\
&= \Delta \left(\int_{\widetilde{M} \times \partial^* \widetilde{M} \times [0,1]} \xi(y) \, |\nabla \log \xi(y)|^2 \, s p(\tau s, x, y) \, dv(y) \, d\nu(\xi) \, ds \right).
\end{aligned}
$$

As a result $\int_M u(\tau, x) \, dx$ is independent of τ. Let dm be the normalized volume form.

Definition 3. *The number*
$$\beta\left(\widetilde{M}\right) = \int_M u(\tau, x) \, dm(x)$$
is called the Kaimanovich entropy.

By (3.1) it is clear that

$$u\left(\tau, x\right) \rightarrow \int_{\partial^* \widetilde{M}} \xi\left(x\right) \left|\nabla \log \xi\left(x\right)\right|^2 d\nu\left(\xi\right)$$

as $\tau \rightarrow 0$. Hence we also obtain the following formula for β from [K].

Proposition 1.

$$\beta\left(\widetilde{M}\right) = \int_M \left(\int_{\partial^* \widetilde{M}} \xi\left(x\right) \left|\nabla \log \xi\left(x\right)\right|^2 d\nu\left(\xi\right)\right) dm\left(x\right).$$

Our main result of this section is the following sharp estimate for the entropy under a lower Ricci bound.

Theorem 6. *Let (M^n, g) be a compact Riemannian manifold with $Ric \geq -(n-1)$ and $\pi : \widetilde{M} \rightarrow M$ its universal covering. Then $\beta\left(\widetilde{M}\right) \leq (n-1)^2$ and equality holds iff $\widetilde{M}$ is isometric to the hyperbolic space $\mathbb{H}^n$.*

We first prove the following theorem on manifolds with $Ric \geq 0$ which may be of independent interest.

Theorem 7. *Let $\left(\Sigma^{n-1}, g\right)$ be a simply connected complete Riemannian manifold with $Ric \geq 0, n \geq 3$. Suppose there is a smooth positive, nonconstant function $u : \Sigma \rightarrow \mathbb{R}$ such that*

(3.2)
$$\begin{cases} D^2 u = u\left(1 + \lambda\right) g, \\ \frac{|\nabla u|^2}{u^2} = 1 - \lambda^2 \end{cases}$$

for some smooth function λ, then $\left(\Sigma^{n-1}, g\right)$ is isometric to $\mathbb{R}^{n-1}$ and on $\mathbb{R}^{n-1}$

$$u = c\left(1 + |x - x_0|^2\right)$$

for some constant $c > 0$ and $x_0 \in \mathbb{R}^{n-1}$.

Proof. We divide the proof into several steps. Set $\mu = u\left(1 + \lambda\right)$.

Step 1: Since u is nonconstant and satisfies an elliptic equation, the set $\{u \neq 0\}$ is open and dense. This is also true of the set $\{\lambda \neq 0\}$. Indeed, if $\lambda = 0$ on some open set U, then $|\nabla u|^2 = u^2, D^2 u = ug$ on U. Using the Bochner formula

$$\frac{1}{2}\Delta |\nabla u|^2 = \left|D^2 u\right|^2 + \langle \nabla u, \nabla \Delta u \rangle + Ric\left(\nabla u, \nabla u\right)$$

we then easily get $(n-2) |\nabla u|^2 + Ric\left(\nabla u, \nabla u\right) = 0$ by a simple computation. Since $Ric \geq 0$, we conclude that u is constant on U. A contradiction.

Step 2: For any vector filed X we have

(3.3)
$$\frac{1}{2} X |\nabla u|^2 = \langle \nabla_X \nabla u, \nabla u \rangle = \mu \langle X, \nabla u \rangle.$$

Taking X to be ∇u and using the second equation of (3.2) yields

$$\begin{aligned} \mu |\nabla u|^2 &= \frac{1}{2} \left\langle \nabla u, \nabla |\nabla u|^2 \right\rangle \\ &= \frac{1}{2} \left\langle \nabla u, \nabla \left(u^2 \left(1 - \lambda^2\right)\right) \right\rangle \\ &= \left(1 - \lambda^2\right) u |\nabla u|^2 - \lambda u^2 \langle \nabla u, \nabla \lambda \rangle. \end{aligned}$$

Hence
$$\lambda \langle \nabla u, \nabla \mu \rangle = 0.$$

On the other hand again from (3.3) for any X with $\langle X, \nabla u \rangle = 0$ we have $\left\langle X, \nabla |\nabla u|^2 \right\rangle = 0$ and hence $\langle X, \nabla \mu \rangle = 0$ as $\nabla \mu$ is a linear combination of ∇u and $\nabla |\nabla u|^2$. Therefore $\nabla \mu = 0$ on the set $\{\nabla u \neq 0, \lambda \neq 0\}$. Since this set is open and dense in Σ, we conclude that μ is a positive constant.

Step 3: Since $\nabla_X \nabla u = \mu X$ and μ is constant, it is easy to see

$$(3.4) \qquad\qquad R(X, Y, Z, \nabla u) = 0.$$

If $n = 3$, then Σ is flat and hence isometric to $\mathbb{R}^2$. In the remaining steps we assume $n > 3$.

Step 4: We show that each regular level set of u is compact. Let $S = u^{-1}(c)$ with c a regular value. The unit normal of S is $\nu = \nabla u / |\nabla u|$ and its second fundamental form is given by

$$\begin{aligned}
\Pi(X, Y) &= \langle \nabla_X \nu, Y \rangle \\
&= \frac{D^2 u(X, Y)}{|\nabla u|} \\
&= \frac{\mu}{|\nabla u|} \langle X, Y \rangle
\end{aligned}$$

for X, Y tangent to S. Similarly

$$X |\nabla u| = \frac{D^2 u(X, \nabla u)}{|\nabla u|} = \frac{\mu X u}{|\nabla u|} = 0$$

for X tangent to S. As a result $|\nabla u|$ and $a = \frac{\mu}{|\nabla u|}$ are positive constants along S. We compute the intrinsic Ricci curvature of S

$$\begin{aligned}
Ric_S(X, X) &= Ric(X, X) - R(X, \nu, X, \nu) + (n-3) a^2 |X|^2 \\
&\geq (n-3) a^2 |X|^2.
\end{aligned}$$

It follows that S is compact by Bonnet-Myers theorem. Since Σ is simply connected, each connected component of S separates Σ into two components.

Step 5: The first equation of (3.2) implies $D^2 u > 0$, i.e. u is convex. Then it is easy to see that S is connected and $\{u \leq c\}$ is the inner component of $\Sigma - S$ and hence compact. In other words u is proper. Let p be a point where u achieves its minimum $\mu/2$. For $X \in \mathcal{S}_p M$ let $\gamma(t)$ be the geodesic with $\dot{\gamma}(0) = X$. Then $f(t) = u \circ \gamma(t)$ satisfies

$$f''(t) = \mu, f(0) = \mu/2, f'(0) = 0.$$

Hence $f(t) = \mu(1 + t^2)$. In other words

$$u(\exp_p rX) = \frac{\mu}{2}(1 + r^2).$$

In geodesic polar coordinates r is the distance function to p. The first equation of (3.2) then simply means

$$D^2 r^2 = 2g$$

at least within the cut locus. It is then easy to show that $\left(\Sigma^{n-1}, g\right)$ is flat and hence isometric to $\mathbb{R}^{n-1}$. $\qquad\square$

Proof of Theorem 6. By Lemma 1 we have for any $\xi \in \partial^* \widetilde{M}$

$$|\nabla \log \xi(x)| \leq n - 1.$$

Hence, using Proposition 1

$$\begin{aligned}
\beta\left(\widetilde{M}\right) &= \int_M \left(\int_{\partial^* \widetilde{M}} \xi(x) |\nabla \log \xi(x)|^2 \, d\nu(\xi) \right) dm(x) \\
&\leq (n-1)^2 \int_M \left(\int_{\partial^* \widetilde{M}} \xi(x) \, d\nu(\xi) \right) dm(x) \\
&= (n-1)^2.
\end{aligned}$$

If $\beta\left(\widetilde{M}\right) = (n-1)^2$, then there exists $A \subset \partial^* \widetilde{M}$ with $\nu\left(\partial^* \widetilde{M} \backslash A\right) = 0$ such that for any $\xi \in A$

(3.5) $$|\nabla \log \xi(x)| \equiv n - 1.$$

Let $\xi \in A$ be such a point. By Lemma we have $\widetilde{M} = \mathbb{R} \times \Sigma^{n-1}$ with $g = dt^2 + e^{2t} g_\Sigma, \xi = \exp[-(n-1)t]$, where (Σ, g_Σ) is a complete Riemannian manifold with $Ric \geq 0$. Notice that $o \in \{0\} \times \Sigma$. Moreover Σ is simply connected as $\widetilde{M}$ is. If $n = 2$, we are done. From then on we assume $n \geq 3$. It is clear that $A \backslash \{\xi\}$ is not empty by (2.1). Let $\eta \in A \backslash \{\xi\}$. We know that $\phi = \log \eta$ satisfies

$$\begin{aligned}
|\nabla \phi| &= (n-1), \Delta \phi = -(n-1)^2, \\
D^2 \phi &= -(n-1)\left[g - \frac{1}{(n-1)^2} d\phi \otimes d\phi\right].
\end{aligned}$$

Let $\psi = \exp\left(-\frac{\phi}{n-1}\right)$. Then a simple computation shows

$$\begin{aligned}
\frac{|\nabla \psi|^2}{\psi^2} &= 1, \\
D^2 \psi &= \psi g.
\end{aligned}$$

Notice that Σ, view as the $t = 0$ slice in $\widetilde{M} = \mathbb{R} \times \Sigma^{n-1}$ is umbilic in the sense that the second fundamental form w.r.t. the unit normal $\frac{\partial}{\partial t}$ equals the metric h. As a result $u = \psi|_\Sigma$, the restriction of ψ on Σ satisfies the following equations

$$\begin{aligned}
\frac{|\nabla u|^2}{u^2} &= 1 - \lambda^2, \\
D^2 u &= u(1 + \lambda) g_0,
\end{aligned}$$

where $\lambda = \frac{\partial \log \psi}{\partial t}$ along Σ. We claim that u is not constant on Σ. If this is NOT true, then $\psi(0, x) \equiv 1$ as $o \in \{0\} \times \Sigma$. Then either $\nabla \psi(0, x) = \frac{\partial}{\partial t}$ or $\nabla \psi(0, x) = -\frac{\partial}{\partial t}$ along Σ. In the first case $\psi(t, x)$ satisfies

$$\frac{\partial^2 \psi}{\partial t^2}(t, x) = \psi(t, x), \psi(0, x) \equiv 1, \frac{\partial \psi}{\partial t}(0, x) = 1$$

and hence $\psi(t, x) = e^t$. This then implies that $\eta = \exp[-(n-1)t] = \xi$, a contradiction. In the second case we get $\psi(t, x) = e^{-t}$ and $\eta = \exp[(n-1)t] = \frac{1}{\xi}$. But this is not harmonic as it is easy to check: $\Delta \eta = 2\frac{|\nabla \xi|^2}{\xi^3} = \frac{2(n-1)^2}{\xi}$.

Since u is not constant on Σ, by Theorem 7 (Σ, g_Σ) is isometric to $\mathbb{R}^{n-1}$. Therefore $\widetilde{M}$ is isometric to $\mathbb{H}^n$. $\qquad\square$

Remark 2. *For the hyperbolic space $\mathbb{H}^n$ the Martin boundary is the same as the ideal boundary. In the ball model $\mathbb{H}^n$ is simply the unit ball B^n in $\mathbb{R}^n$ with the metric $g = \frac{4}{(1-|x|^2)}dx^2$. We take the base point to be the origin. The ideal boundary is the unit sphere S^{n-1}. For any $\xi \in S^{n-1}$ there corresponds to a normalized minimal positive function*

$$h_\xi(x) = \left(\frac{1-|x|^2}{|x-\xi|^2} \right)^{n-1}.$$

It is easy to verify that they all satisfy $|\nabla \log h_\xi(x)| \equiv (n-1)$.

4. Proof of the main theorem

First we need another remarkable formula from [K].

Theorem 8.

$$\beta\left(\widetilde{M}\right) = -\lim_{t\to\infty} \frac{1}{t} \int_{\widetilde{M}} p(t,x,y) \log p(t,x,y)\, dv(y).$$

We also need the following lemma from [L1].

Lemma 3. *Let (M,g) be a compact Riemannian manifold and $\pi : \widetilde{M} \to M$ its universal covering. Then $\beta\left(\widetilde{M}\right) \geq 4\lambda_0\left(\widetilde{M}\right)$.*

Proof. Since the proof is short and instructive, we present it for the convenience of the reader. For $\varepsilon > 0$

$$-\frac{1}{t}\int_{\widetilde{M}} p(t,x,y)\log p(t,x,y)\,dv(y) + \frac{1}{t}\int_{\widetilde{M}} p(\varepsilon,x,y)\log p(\varepsilon,x,y)\,dv(y)$$

$$= -\frac{1}{t}\int_{\widetilde{M}}\int_{\varepsilon}^{t} (\log p(s,x,y)+1)\frac{\partial p}{\partial s}(s,x,y)\,dsdv(y)$$

$$= -\frac{1}{t}\int_{\widetilde{M}}\int_{\varepsilon}^{t} (\log p(s,x,y)+1)\Delta_y p(s,x,y)\,dsdv(y)$$

$$= -\frac{1}{t}\int_{\widetilde{M}}\int_{\varepsilon}^{t} \Delta_y(\log p(s,x,y)+1)\,p(s,x,y)\,dsdv(y)$$

$$= \frac{1}{t}\int_{\varepsilon}^{t}\int_{\widetilde{M}} \frac{|\nabla_y p(s,x,y)|^2}{p(s,x,y)}\,dsdv(y)\,ds$$

$$= \frac{4}{t}\int_{\varepsilon}^{t}\int_{\widetilde{M}} \left|\nabla_y\sqrt{p(s,x,y)}\right|\,dv(y)\,ds$$

$$\geq \frac{4}{t}\lambda_0\left(\widetilde{M}\right)\int_{\varepsilon}^{t}\int_{\widetilde{M}} p(s,x,y)\,dv(y)\,ds$$

$$= \frac{4}{t}\lambda_0\left(\widetilde{M}\right)(t-\varepsilon).$$

In the last step we use the fact that $\widetilde{M}$ is stochastically complete, i.e.

$$\int_{\widetilde{M}} p(s,x,y)\,dv(y) = 1.$$

Letting $t \to \infty$ yields $\beta\left(\widetilde{M}\right) \geq 4\lambda_0\left(\widetilde{M}\right)$. $\square$

We now prove our main theorem

Theorem 9. *Let (M^n, g) be a compact Riemannian manifold with $Ric \geq -(n-1)$ and $\pi : \widetilde{M} \to M$ its universal covering. Then*

(1) $\lambda_0\left(\widetilde{M}\right) \leq (n-1)^2/4$,

(2) *If equality holds, then $\widetilde{M}$ is isometric to the hyperbolic space $\mathbb{H}^n$.*

Proof. The first part is well known and follows from a theorem of Cheng [C] or the inequality $\lambda_0\left(\widetilde{M}\right) \leq h^2/4$, where h is the volume entropy. The second part clearly follows from Lemma 3 and Theorem 6. $\square$

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Department of Mathematics, Michigan State University, East Lansing, MI 48824

E-mail address: xwang@math.msu.edu