SOME RECENT RESULTS IN CR GEOMETRY

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1. Introduction

CR geometry originated from the study of real hypersurfaces in complex manifolds. In their foundational work Chern and Moser [CM] developed local invariants for CR manifolds. Shortly afterward Tanaka [T] and Webster [W2] introduced a canonical connection associated to a given pseudohermitian structure (i.e. a contact form), so pseudohermitian geometry was born. Today CR geometry or pseudohermitian geometry has become an independent subject with fascinating connections with complex analysis, Riemannian and sub-Riemannian geometry and other areas of mathematics.

We will only consider oriented strictly pseudoconvex CR manifolds with a chosen pseudohermitian structure. With the induced Tanaka-Webster connection and the adapted metric one may try to develop a whole theory parallel to Riemannian geometry. There are many different directions. On the geometric side one can study the induced Carnot-Caratheodory distance, its geodesics, the Hausdorff measures etc. We will not say anything on these fascinating topics except by giving a few references. On the Carnot-Caratheodory distance in the more general sub-Riemannian setting one can read Gromov's long paper [Gr] which contains a wealth of fascinating ideas. There have been a lot works on the isoperimetric problem in the Heisenberg space, cf. the book [CDPT]. Partly motivated by this problem, Paul Yang and his collaborators have developed a theory of p-mean curvature for surfaces in 3-D pseudohermitian manifolds. We refer to his survey [Y] and references therein.

On the more analytic side, there are also many natural problems. The CR Yamabe problem initiated by Jerison and Lee [JL1] has been quite well understood and a recent reference is the book [DT]. We will not discuss it here. It is also natural to study the fundamental operators, the sub-Laplacian and the Kohn Laplacian on functions, and their spectrum on pseudohermitian manifolds. One would hope that this study will be as fruitful as the study of the spectrum of the Laplacian in Riemannian geometry. These operators are not elliptic and therefore their analysis involves new analytic challenges. Another major new complication is that the Tanaka-Webster connection has nontrivial torsion. In this paper we discuss some recent results in this direction.

In Section 2 we give a quick summary of the basics in CR geometry. We take the opportunity to discuss a basic classification result which seems missing from the literature. In Section 3 we discuss some estimates on the eigenvalues of the fundamental operators. In Section 4 we discuss some Obata-type results in CR geometry and address the rigidity question in the sharp eigenvalue estimate. The last Section, in which we discuss a problem on the CR structure on circle bundles over compact Kahler manifolds, is in some sense independent of the previous sections. There are some new results and complete proofs are given.

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2. Basics in CR geometry

We recall the basic concepts in CR geometry. Let M be a smooth manifold of dimension $2m+1$. An almost CR structure on M is a pair $(H(M), J)$, where $H(M)$ is a subbundle of rank $2m$ of the tangent bundle $T(M)$ and J is an almost complex structure on $H(M)$. We then define

$$\begin{aligned}T^{1,0}(M) &= \{u - \sqrt{-1}Ju | u \in H(M)\} \subset T(M) \otimes \mathbb{C},\\ T^{0,1}(M) &= \overline{T^{1,0}(M)}.\end{aligned}$$

An almost CR structure is integrable if

$$\left[T^{1,0}(M), T^{1,0}(M)\right] \subset T^{1,0}(M).$$

Notice that this is always true when $m=1$. M with an integrable CR structure is called a CR manifold.

We will always assume that our CR manifold M is orientable. Thus there is a 1-form θ on M which annihilates exactly $H(M)$. Any such θ is called a pseudohermitian structure on M. Let $\omega = d\theta$. Then $G_\theta(X,Y) = \omega(X, JY)$ defines a symmetric bilinear form on the vector bundle $H(M)$. A CR manifold M is nondegenerate if ω is nondegenerate on $H(M)$. It is strictly pseudoconvex if G_θ is positive definite.

Let (M,θ) be a nondegenerate pseudo-Hermitian CR manifold. Then there is a unique vector field T on M such that

$$\theta(T) = 1, T \rfloor d\theta = 0.$$

This gives rise to the decomposition

$$T(M) = H(M) \oplus \mathbb{R}T.$$

Using this decomposition we then extend J to an endomorphism ϕ on $T(M)$ by defining $\phi(T) = 0$. We can also define a pseudo-Riemannian metric g_θ on M such that

$$g_\theta(X,Y) = G_\theta(X,Y), g_\theta(X,T) = 0, g_\theta(T,T) = 1,$$

$\forall X, Y \in H(M)$. Clearly, ϕ is skew-symmetric, i.e.

$$g_\theta(\phi X, Y) = -g_\theta(X, \phi Y).$$

It is a Riemannian metric is (M,θ) is strictly pseudoconvex. Obviously $\theta = \langle T, \cdot\rangle, \omega = d\theta = \langle J\cdot, \cdot\rangle$.

Let (M,θ) be a nondegenerate pseudohermitian CR manifold. By the fundamental work of Tanaka [T] and Webster [W1], there is a unique connection ∇ on $T(M)$ such that

(1) $H(M)$ is parallel, i.e. $\nabla_X Y \in \Gamma(H(M))$ for any $X \in \mathcal{T}(M)$ and any $Y \in \Gamma(H(M))$.
(2) $\nabla\phi = 0, \nabla g_\theta = 0$.
(3) The torsion τ satisfies

$$\tau(Z, W) = 0,$$
$$\tau\left(Z, \overline{W}\right) = \omega\left(Z, \overline{W}\right) T,$$
$$\tau(T, J\cdot) = -J\tau(T, \cdot)$$

for any $Z, W \in T^{1,0}(M)$.

Clearly, θ and ω are parallel as well.

We define $A : T(M) \to T(M)$ by $AX = \tau(T, X)$. It is obvious that $AT = 0, AH(M) \subset H(M)$ and $A\phi X = -\phi AX$. Moreover

$$\tau(X, Y) = \omega(X, Y) T + \theta(X)AY - \theta(Y)AX,$$
$$\langle AX, Y\rangle = \langle X, AY\rangle.$$

We will simply refer A as the torsion of (M, θ). It is a well known fact that A vanishes iff g_θ is Sasakain.

From now on we only consider strictly pseudoconvex CR manifolds. With the Tanaka-Webster connection ∇ we can consider its curvature tensor

$$R(X, Y) Z = -\nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z + \nabla_{[X,Y]} Z.$$

We often work with a local frame $\{T_\alpha\}$ for $T^{1,0}(M)$ and the dual frame $\{\theta^\alpha\}$. Thus

$$d\theta = \sqrt{-1} h_{\alpha\overline{\beta}} \theta^\alpha \wedge \overline{\theta}^\beta.$$

The connection ∇ is determined by 1-forms ω_α^β s.t.

$$\nabla T_\alpha = \omega_\alpha^\beta \otimes T_\beta$$

and the torsion matrix $A_{\alpha\beta} = \langle AT_\alpha, T_\beta\rangle$. Equivalently, we have

$$d\theta^\beta = \theta^\alpha \wedge \omega_\alpha^\beta + \theta \wedge \tau^\beta,$$

where $\tau^\beta = \theta^\beta(\tau(T, \cdot)) = A_{\overline{\nu}}^\beta \theta^{\overline{\nu}}$. By direct calculation, the curvature form is given by

$$\begin{aligned}\Omega_\alpha^\beta &= d\omega_\alpha^\beta - \omega_\alpha^\gamma \wedge \omega_\gamma^\beta \\ &= -R_{\mu\overline{\nu}\alpha}^\beta \theta^\mu \wedge \theta^{\overline{\nu}} + A_{\alpha\gamma,\overline{\nu}} h^{\beta\overline{\nu}} \theta^\gamma \wedge \theta - A_{\overline{\gamma},\alpha}^\beta \theta^{\overline{\gamma}} \wedge \theta \\ &\quad + \sqrt{-1}\left(h_{\alpha\overline{\gamma}} A_{\overline{\nu}}^\beta \theta^{\overline{\gamma}} \wedge \theta^{\overline{\nu}} - A_{\alpha\mu}\theta^\mu \wedge \theta^\beta\right).\end{aligned}$$

We call $R_{\mu\overline{\nu}\alpha}^\beta$ or equivalently

$$R_{\mu\overline{\nu}\alpha\overline{\beta}} = \left\langle -\nabla_\mu \nabla_{\overline{\nu}} T_\alpha + \nabla_{\overline{\nu}} \nabla_\mu T_\alpha + \nabla_{[T_\mu, T_{\overline{\nu}}]} T_\alpha, T_{\overline{\beta}}\right\rangle$$

the pseudohermitian curvature tensor and its trace $R_{\mu\overline{\nu}} = -R_{\mu\overline{\nu}\alpha}^\alpha$ the pseudohermitian Ricci tensor. The pseudohermitian scalar curvature $\mathcal{R}$ is defined to be $h^{\alpha\overline{\beta}} R_{\alpha\overline{\beta}}$.

The Chern tensor is defined by

$$\begin{aligned} C^{\beta}_{\mu\overline{\nu}\alpha} &:= -R^{\beta}_{\mu\overline{\nu}\alpha} - \frac{1}{m+2}\left[R^{\beta}_{\alpha}h_{\mu\overline{\nu}} + R^{\beta}_{\mu}h_{\alpha\overline{\nu}} + \delta^{\beta}_{\alpha}R_{\mu\overline{\nu}} + \delta^{\beta}_{\mu}R_{\alpha\overline{\nu}}\right] \\ &+ \frac{\mathcal{R}}{(m+1)(m+2)}\left[\delta^{\beta}_{\alpha}h_{\mu\overline{\nu}} + \delta^{\beta}_{\mu}h_{\alpha\overline{\nu}}\right]. \end{aligned}$$

The Cartan tensor is defined by

$$Q_{\alpha\beta} = \sqrt{-1}\left(A_{\alpha\beta,0} - 2\phi_{\alpha,\beta}\right) + 2P_{\alpha\overline{\nu}}A^{\overline{\nu}}_{\beta},$$

with

$$\begin{aligned} \phi_{\alpha} &= \frac{1}{n+2}\left(\frac{\mathcal{R}_{\alpha}}{2n+2} - \sqrt{-1}A_{\alpha\beta,\overline{\nu}}h^{\beta\overline{\nu}}\right), \\ P_{\alpha\overline{\beta}} &= \frac{1}{n+2}\left(R_{\alpha\overline{\beta}} - \frac{\mathcal{R}}{2n+2}h_{\alpha\overline{\beta}}\right). \end{aligned}$$

Here $A_{\alpha\beta,0}$ and $A_{\alpha\beta,\overline{\nu}}$ are covariant derivatives of the torsion A. These tensors are invariant under pseudoconformal deformations: if $\widetilde{\theta} = e^{2f}\theta$ is another pseudohermitian structure, for example, its Chern tensor is given by

$$\widetilde{C}^{\beta}_{\sigma\overline{\lambda}\alpha} = C^{\beta}_{\sigma\overline{\lambda}\alpha}.$$

Moreover, we have the following fundamental theorem in CR geometry.

Theorem 1. *(Cartan, Chern-Moser) Let (M,θ) be a strongly pseudoconvex pseudohermitian CR manifold of dimension $2m+1$.*

- *If dimension $2m+1 \geq 5$, then M is locally CR spherical iff the Chern tensor vanishes.*
- *If dimension $2m+1 = 3$, then M is locally CR spherical iff the Cartan tensor vanishes.*

The second part was proved by Cartan (cf. [J]) and the first part was proved by Chern-Moser [CM].

Let $\widetilde{\nabla}$ be the Levi-Civita connection of g_{θ}. We have the following formula relating ∇ and $\widetilde{\nabla}$

$$\begin{aligned} \widetilde{\nabla}_X Y &= \nabla_X Y + \theta(Y)AX + \frac{1}{2}\left(\theta(Y)\phi X + \theta(X)\phi Y\right) \\ &- \left[\langle AX, Y\rangle + \frac{1}{2}\omega(X,Y)\right]T. \end{aligned}$$

With this formula we can compare curvature tensors.

Proposition 1. *Suppose X, Y and Z are horizontal vector fields. Then*

$$\begin{aligned} \widetilde{R}(X,Y,X,Y) &= R(X,Y,X,Y) - \frac{3}{4}\langle JX, Y\rangle^2 + \langle AX, Y\rangle^2 - \langle AX, X\rangle\langle AY, Y\rangle, \\ \widetilde{R}(X,T,Y,T) &= -\langle \nabla_T AX, Y\rangle - \langle AX, AY\rangle + \langle AX, JY\rangle + \frac{1}{4}\langle X, Y\rangle, \\ \widetilde{R}(X,Y,Z,T) &= \langle \nabla_X AY, Z\rangle - \langle \nabla_Y AX, Z\rangle. \end{aligned}$$

Proposition 2. *Suppose $X = c_\alpha T_\alpha + \overline{c_\alpha} T_{\overline{\alpha}}$ is a horizontal vector field w.r.t. a unitary frame. Then*

$$\begin{aligned}
\widetilde{Ric}\,(X,X) &= 2R_{\alpha\overline{\beta}} c_\alpha \overline{c_\beta} + \sqrt{-1}\,(m-1)\left(A_{\alpha\beta} c_\alpha c_\beta - A_{\overline{\alpha}\overline{\beta}} \overline{c_\alpha c_\beta}\right) \\
&\quad - \frac{1}{2}\,|X|^2 - \langle \nabla_T AX, X\rangle + \langle AX, JX\rangle\,, \\
\widetilde{Ric}\,(X,T) &= 2\left\langle X, \operatorname{Re} A_{\alpha\beta,\overline{\alpha}} T_{\overline{\beta}}\right\rangle, \\
\widetilde{Ric}\,(T,T) &= \frac{m}{2} - |A|^2\,.
\end{aligned}$$

The simplest examples of pseudohermitian manifolds are those with constant curvature.

(1) The Heisenberg group $\mathbb{H}^{2m+1} = \mathbb{C}^m \times \mathbb{R}$ with

$$\theta_c = dt + \sqrt{-1}\sum z^j d\overline{z}^j - \overline{z}^j dz^j.$$

It is torsion-free and has zero curvature, i.e. $R_{\mu\overline{\nu}\alpha\overline{\beta}} = 0$.

(2) The unit sphere $\mathbb{S}^{2m+1} \subset \mathbb{C}^{m+1}$ with

$$\theta_c = \sqrt{-1}\sum_{i=1}^{m+1} z^j d\overline{z}^j - \overline{z}^j dz^j.$$

It is torsion-free and has constant pseudohermitian curvature

$$R_{\mu\overline{\nu}\alpha\overline{\beta}} = -\left(h_{\mu\overline{\nu}} h_{\alpha\overline{\beta}} + h_{\mu\overline{\beta}} h_{\alpha\overline{\nu}}\right).$$

(3) $\mathbb{Q}^{2m+1} = \left\{z \in \mathbb{C}^{m+1} : |z_{m+1}|^2 - \sum_{i=1}^m |z_i|^2 = 1\right\}$ with

$$\theta_c = \frac{\sqrt{-1}}{2}\left[\sum_{j=1}^m (z_j d\overline{z}_j - \overline{z}_j dz_j) - (z_{m+1} d\overline{z}_{m+1} - \overline{z}_{m+1} dz_{n+1})\right].$$

It is torsion-free and has constant pseudohermitian curvature

$$R_{\mu\overline{\nu}\alpha\overline{\beta}} = \left(h_{\mu\overline{\nu}} h_{\alpha\overline{\beta}} + h_{\mu\overline{\beta}} h_{\alpha\overline{\nu}}\right).$$

The 3rd example is not simply connected. Define $\Phi : \mathbb{Q}^{2m+1} \to \mathbb{B}^m$ by

$$\Phi\,(z) = \left(\frac{z_1}{z_{n+1}}, \cdots, \frac{z_n}{z_{n+1} - 1}\right).$$

This is circle fiberation. Let $w_i = z_i / z_{n+1}$ and $\rho = \sqrt{\sum_i |w_i|^2}$. Writing $z_{n+1} = e^{-\mathbf{i}\theta}/\sqrt{1-\rho^2}$ we have $z_i = e^{-\mathbf{i}\theta} w_i / \sqrt{1-\rho^2}$. In the new coordinates (w,θ)

$$\theta_c = d\theta - \frac{\sqrt{-1}}{2}\left(\overline{\partial} - \partial\right)\log\left(1 - |w|^2\right).$$

Therefore it is Sasakian and the transverse geometry is the complex hyperbolic space.

Therefore we consider the universal covering $\widetilde{\mathbb{Q}}^{2m+1}=\mathbb{R}\times\mathbb{B}^n$ with the contact 1-form

$$\begin{aligned}\theta_c &= dt-\frac{\sqrt{-1}}{2}\left(\overline{\partial}-\partial\right)\log\left(1-|z|^2\right)\\ &= dt-\frac{\sqrt{-1}}{2}\left(\frac{\overline{z}_\alpha}{1-|z|^2}dz_\alpha-\frac{z_\alpha}{1-|z|^2}d\overline{z}_\alpha\right).\end{aligned}$$

In Riemannian geometry the first global result is the classification of simply connected complete Riemannian manifolds with constant curvature. The following result is the CR analogue.

Theorem 2. *Let (M,θ) be a simply connected pseudohermitian manifold of dimension $2m+1$ with constant pseudohermitian curvature tensor $a\in\mathbb{R}$. Suppose the adapted Riemannian metric g_θ is complete. Then*

(1) *if $a=0$, (M,θ) is CR equivalent to $\left(\mathbb{H}^{2m+1},\theta_c\right)$;*
(2) *if $a>0$, (M,θ) is CR equivalent to $\left(\mathbb{S}^{2m+1},a^{-1}\theta_c\right)$;*
(3) *if $a<0$, (M,θ) is CR equivalent to $\left(\widetilde{\mathbb{Q}}^{2m+1},|a|^{-1}\theta_c\right)$.*

This should be known to the experts, but the author cannot find it in the literature. To prove this we first observe that geodesics of Tanaka-Webster connection are of constant speed. Since g_θ is complete, all such geodesics can be extended to all time. Then the theorem follows from Theorem 7.8 in Kobayashi and Nomizu [KN].

The 2nd case can be proved directly as follows. By scaling we can assume without loss of generality that $c=1/2$. Then by Proposition 1 it is straightforward to check that g_θ has constant sectional curvature $1/4$. Without loss of generality, we can take (M,g_θ) to be $\left(\mathbb{S}^{2m+1},4g_0\right)$. Then θ is a pseudohermitian structure on $\mathbb{S}^{2m+1}$ whose adapted metric is $4g_0$ and the associated Tanaka-Webster connection is torsion-free. It is a well known fact that the Reeb vector field T is then a Killing vector field for g_0 . Therefore there exists a skew-symmetric matrix A such that for all $X\in\mathbb{S}^{2m+1},T(X)=AX$, here we use the obvious identification between $z=(z_1,\ldots,z_{m+1})\in\mathbb{C}^{m+1}$ and $X=(x_1,y_1,\ldots,x_{m+1},y_{m+1})\in\mathbb{R}^{2m+2}$. Changing coordinates by an orthogonal transformation we can assume that A is of the following form

$$A=\begin{bmatrix} 0 & -a_1 & & & \\ a_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & -a_{m+1} \\ & & & a_{m+1} & 0 \end{bmatrix}$$

where $a_i\geq 0$. Therefore

$$T=\sum_i a_i\left(y_i\frac{\partial}{\partial x_i}-x_i\frac{\partial}{\partial y_i}\right)$$

Since T is of unit length we must have

$$4\sum_i a_i^2(x_i^2+y_i^2)=1$$

on $\mathbb{S}^{2m+1}$. Therefore all the a_i's are equal to $1/2$. It follows that

$$\theta = g_0(T, \cdot) = 2\sqrt{-1}\partial |z|^2.$$

Therefore we have produced a diffeomorphism $F : M \to \mathbb{S}^{2m+1}$ s.t. $F^*(2\theta_c) = \theta$ and moreover F is an isometry between g_θ and $4g_0$ (the adapted metric of $2\theta_c$ on $\mathbb{S}^{2m+1}$). It remains to show that F is CR. It is obvious that F_* maps $H(M)$ to $H\left(\mathbb{S}^{2m+1}\right)$. Since $F^*(2d\theta_c) = d\theta$, for any $X, Y \in H(M)$ we have $2d\theta_c(F_*X, F_*Y) = d\theta(X, Y)$ or $4g_0(JF_*X, F_*Y) = g_\theta(JX, Y)$. As F is an isometry we must have $JF_*X = F_*JX$, i.e. F is CR.

3. Spectrum of the sub-Laplacian and Kohn Laplacian

On a pseudohermitian manifold there is a natural second order differential operator which is subelliptic, namely the sub-Laplacian Δ_b. In terms of a unitary frame we have

$$\Delta_b u = u_{\overline{\alpha},\alpha} + u_{\alpha,\overline{\alpha}}.$$

On a closed pseudohermitian manifold the sub-Laplacian Δ_b satisfies the Hormander estimate: if $\Delta_b u \in W^s(M)$ (Sobolev space of order $s \geq 0$) for $u \in L^2(M)$, then $u \in W^{s+1}(M)$ and

$$\|u\|_{s+1}^2 \leq C_s \left(\|\Delta_b u\|_s^2 + \|u\|^2 \right).$$

It follows that Δ_b defines a selfadjoint operator with a discrete spectrum

$$\lambda_0 = 0 < \lambda_1 \leq \lambda_2 \leq \cdots$$

with $\lim_{k\to\infty} \lambda_k = +\infty$.

In [G] Greenleaf proved an analogue of the Lichnerowicz estimate for the sub-Laplacian.

Theorem 3. *Let M be a compact pseudohermitian manifold of dimension $2m+1 \geq 5$. Suppose for any $X \in H^{1,0}(M)$*

$$Ric(X, X) - \frac{m+1}{2} Tor(X, X) \geq \kappa |X|^2,$$

where κ is a positive constant. Then the first eigenvalue of $-\Delta_b$ satisfies

$$\lambda_1 \geq \frac{m}{m+1}\kappa.$$

The proof breaks down in dimension 3. Chang and Chiu [CC] established the eigenvalue estimate under the additional condition that the Panietz operator is nonnegative. The Panietz operator $P_0 : C^\infty(M) \to C^\infty(M)$ on a closed pseudohermitian manifold is defined by

$$P_0 f = (P_\alpha f)_{,\overline{\alpha}} = f_{\overline{\gamma},\gamma\alpha\overline{\alpha}} + m\sqrt{-1}\left(A_{\alpha\beta} f_{\overline{\beta}}\right)_{,\overline{\alpha}}.$$

We say that P_0 is nonnegative if for any u

$$\int_M u P_0 u \geq 0.$$

This is always the case in dimension $2m+1 \geq 5$ by [GL], but there are 3-dimensional CR manifolds whose Panietz operator is NOT nonnegative.

Theorem 4 ([CC]). *Let M^3 be a closed pseudohermitian manifold such that for any $X = cT_1$*

$$R_{1\overline{1}}\left|c\right|^2 - \sqrt{-1}\left(A_{11}c^2 - A_{\overline{11}}\overline{c}^2\right) \geq \kappa\left|c\right|^2,$$

where κ is a positive constant. If the Panietz operator is nonnegative, then the first eigenvalue of $-\Delta_b$ satisfies

$$\lambda_1 \geq \frac{1}{2}\kappa.$$

On a closed pseudohermitian manifold, there is another natural second order operator, the Kohn Laplacian $\Box_b$. For a complex function f

$$\Box_b f = \overline{\partial}_b^*\overline{\partial}_b f = -f_{\overline{\alpha},\alpha}.$$

It defines a nonnegative self-adjoint operator on the Hilbert space $L^2(M)$ of complex square integrable functions with the inner product

$$\langle f_1, f_2\rangle = \int_M f_1\overline{f}_2.$$

We have

$$-\Delta_b = \Box_b + \overline{\Box}_b = 2\Box_b + \sqrt{-1}mT = 2\overline{\Box}_b - \sqrt{-1}mT.$$

But unlike Δ_b, it is not hypo-elliptic. As a result, its resolvent is not compact. In fact, its kernel is the infinite dimensional space of CR holomorphic functions. However, it turns out that the spectral theory of $\Box_b$ is quite simple.

In dimension $2m+1 \geq 5$, one can use the fundamental work of Kohn [Ko] (see also [CS]) to prove the following

Theorem 5. *Let (M,θ) be closed pseudohermitian manifold with dimension $2m+1 \geq 5$. The* $\operatorname{spec}(\Box_b)$ *consists of countably many eigenvalues $\lambda_0 = 0 < \lambda_1 < \lambda_2 < \cdots$ with $\lambda_i \to \infty$ as $i \to \infty$. Moreover, for $i \geq 1$, each λ_i is an eigenvalue of finite multiplicity and all the eigenfunctions are smooth.*

In dimension 3 things are more complicated as the Hodge theory for $(0,1)$-forms is not valid. Nevertheless, based on the work of Beals and Greiner [BG], Burn and Epstein [BE] proved the following theorem.

Theorem 6. *Let M be a closed pseudohermitian manifold of dimension 3. The* $\operatorname{spec}(\Box_b)$ *in $(0,\infty)$ consists of point eigenvalues of finite multiplicity. Moreover all these eigenfunctions are smooth.*

In general there may exist a sequence of 'small' eigenvalues rapidly decreasing to zero. In fact zero is an isolated eigenvalue iff the range of $\Box_b$ is closed. Recently, Chanillo, Chiu and Yang [CCY] proved that there is no 'small' eigenvalue if the scalar curvature is positive and the Panietz operator is nonnegative.

Theorem 7. *Let M be a closed pseudohermitian manifold of dimension 3. Suppose the Panietz operator is nonnegative and the scalar curvature $R \geq \kappa > 0$. Then any nonzero eigenvalue of $-\Box_b$ satisfies*

$$\lambda \geq \kappa/2.$$

The above theorem can be generalized to higher dimension by the same argument.

Theorem 8. *Let (M,θ) be a closed pseudohermitian manifold of dimension $2m+1\geq 5$. Suppose for any $X\in H^{1,0}(M)$*

$$Ric\,(X,X)\geq \kappa\,|X|^2,$$

where κ is a positive constant. Then any nonzero eigenvalue of $-\Box_b$ satisfies

$$\lambda_1\geq \frac{m}{m+1}\kappa.$$

For details we refer to [LSW].

It is interesting to compare the eigenvalue estimate for the sub-Laplacian and that for the Kohn Laplacian. For the sub-Laplacian the assumption involves both the pseudohermitian Ricci tensor and the torsion while for the Kohn Laplacian we only need to assume a positive lower bound for the pseudohermitian Ricci tensor.

4. Rigidity results

The Greenleaf estimate is sharp as one can verify that equality holds on the CR sphere $\mathbb{S}^{2m+1}$. A natural question is whether the equality case characterizes the CR sphere. Motivated by this question, the following theorem is proved in [LW].

Theorem 9. *Let M be a closed pseudohermitian manifold of dimension $2m+1\geq 5$. Suppose there is a real nonzero function $u\in C^\infty(M)$ satisfying*

$$\begin{aligned} u_{\alpha,\beta} &= 0,\\ u_{\alpha,\overline{\beta}} &= \left(-\frac{\kappa}{2(m+1)}u+\frac{\sqrt{-1}}{2}u_0\right)\delta_{\alpha\beta}, \end{aligned}$$

for some constant $\kappa>0$. Then M is CR equivalent to the sphere $\mathbb{S}^{2m+1}$ with its standard pseudohermitian structure up to a scaling.

This can be viewed as the CR analogue of the following classic theorem in Riemannian geometry.

Proposition 3. [O] *Suppose (N^n,g) is a complete Riemannian manifold and u a smooth, nonzero function on N satisfying $D^2u=-c^2ug$ with $c>0$, then N is isometric to a sphere $\mathbb{S}^n(c)$ of radius $1/c$ in the Euclidean space $\mathbb{R}^{n+1}$.*

From Theorem 9, one can easily deduce that the equality holds in the Greenleaf estimate iff M is CR equivalent to the CR sphere. In dimension 3, the following result is proved in [LW].

Theorem 10. *Let M^3 be a closed pseudohermitian manifold. Suppose there exists a non-constant function u satisfying*

$$\begin{aligned} u_{1,1} &= 0,\\ u_{1,\overline{1}} &= -\frac{\kappa}{4}u+\frac{\sqrt{-1}}{2}u_0,\\ u_{0,1} &= 2A_{11}u_{\overline{1}}+\frac{\sqrt{-1}}{2}u_1. \end{aligned}$$

Then M is CR equivalent to $\mathbb{S}^3$ up to a scaling.

The main step in the proofs of the above theorems is to show that the torsion A must vanish. This involves a lot of integration by parts. After we have proved $A=0$, we can check by direct calculation that when u satisfies the following equation

$$D^2u=-\frac{1}{4}ug_\theta,$$

where D^2u is the Riemannian Hessian of u, here without loss of generality we take $\kappa=(m+1)/2$ by scaling. Then we can apply the classic Obata theorem to finish the proof. We refer to [LW] for details.

In [LSW], we prove the following variant of Theorem 9.

Theorem 11. *Let M be a closed pseudohermitian manifold of dimension $2m+1\geq 5$. Suppose that there exists a nonzero complex-valued function f on M satisfying*

$$\begin{aligned} f_{\alpha,\beta}&=0,\\ f_{\alpha,\overline{\beta}}&=-cf\delta_{\alpha\beta}, \end{aligned}$$

for some constant $c>0$. Then M is CR equivalent to the sphere $\mathbb{S}^{2m+1}$ with its standard pseudohermitian structure up to a scaling.

From this theorem it easily follows that equality holds for the estimate in Theorem 8 iff M is CR equivalent to the sphere $\mathbb{S}^{2m+1}$ up to a scaling. The proof of this theorem is quite different from that of Theorem 9. We refer to our paper [LSW] for details. The 3-dimensional version is still work in progress.

5. CR geometry on the boundary of a holomorphic disc bundle

Let $\pi:L\rightarrow M$ be a holomorphic line bundle over a compact complex manifold M of complex dimension m and h a Hermitian metric on L. We assume that the Chern form of (L,h) is negative, i.e. $\omega:=-c_1(L,h)$ is a Kahler form on M. If $\sigma:U\rightarrow L\backslash\{0\}$ is a local holomorphic trivializing section, then $\omega=\sqrt{-1}\partial\overline{\partial}\log|\sigma|_h^2$ on U. Consider the disc bundle

$$D=\left\{v\in L:\rho(v):=1-|v|_h^2<0\right\}$$

and its boundary the circle bundle $\Sigma=\partial D$. It is a well known fact that Σ is a strictly pseudoconvex CR manifold. It is an interesting problem to study the relationship between the CR geometry on ∂D and the Kahler geometry on (M,ω). Recently, several authors have studied the following question (see, e.g. Question 2 in [ALZ])

Problem 1. *Assume that Σ is locally spherical (i.e. locally CR equivalent to the sphere $\mathbb{S}^{2m+1}$). Is it true that M is biholomorphic to $\mathbb{CP}^m$?*

Engliš and Zhang [EZ] provided a positive answer to this question when L is a negative line bundle over an Hermitian symmetric space of compact type. Arezzo, Loi and Zuddas [ALZ] have positive results when M is a homogeneous Hodge manifold.

We can give a complete classification of (M,ω) when Σ is locally CR equivalent to the sphere.

Theorem 12. *Let $\pi : L \to M$ be a holomorphic line bundle over a compact complex manifold M of complex dimension m and h a Hermitian metric on L s.t. $\omega = -c_1(L,h)$ is a Kahler form on M. Consider the strictly pseudoconvex CR manifold*

$$\Sigma = \left\{ v \in L : |v|_h^2 = 1 \right\}$$

If $m = 1$, then Σ is locally spherical iff (M,ω) is a Riemannian surface with constant scalar curvature. If $m > 1$, then Σ is locally spherical iff (M,ω) is biholomorphically isometric to one of the following

1. *the complex projective space $\mathbb{CP}^m$,*
2. *a complex Euclidean space form $\mathbb{T}^m/F, F \subset U(m)$ a finite group,*
3. *a complex hyperbolic space form $\mathbb{B}^m/\Gamma, \Gamma \subset PU(m,1)$ a cocompact lattice,*
4. *the fibre space $\left(\mathbb{B}^l \times \mathbb{CP}^{m-l}\right)/\Gamma, \Gamma \subset PU(l,1) \times PU(m-l+1)$ a cocompact lattice ($l = 1, \cdots, m-1$).*

Let $\iota : \Sigma \to L \backslash \{0\}$ be the inclusion map. On Σ we consider the following pseudohermitian structure

$$\theta = \iota^* \sqrt{-1} \partial \log |v|^2 ,$$

In the following we compute the Tanaka-Webster connection and its curvature on (Σ,θ). The calculations we give are essentially due to Webster [W2] where it is formulated in a local setting. Suppose (U,z) is a local chart on Σ on which we have a local holomorphic trivializing section $\sigma : U \to L \backslash \{0\}$. Set $\rho = |\sigma|_h^2$. Then on U we have

$$\omega = \sqrt{-1}\partial\overline{\partial} \log \rho = \sqrt{-1} g_{\alpha\overline{\beta}} dz_\alpha \wedge dz_{\overline{\beta}},$$

with

$$g_{\alpha\overline{\beta}} = \frac{\partial^2 \log \rho}{\partial z_a \partial z_{\overline{\beta}}}.$$

Locally X is given by

$$N = \left\{ (z,w) \in U \times \mathbb{C} : \rho(z) |w|^2 = 1 \right\}.$$

with $\theta = \sqrt{-1}\partial \log \left(\rho |w|^2\right)$ and $d\theta = \omega$. We have the local frame

$$T_a = \frac{\partial}{\partial z_a} - w \frac{\partial \log \rho}{\partial z_a} \frac{\partial}{\partial w},$$
$$T = \sqrt{-1}\left(w \frac{\partial}{\partial w} - \overline{w} \frac{\partial}{\partial \overline{w}} \right).$$

Simple calculation yields

$$\left[T_\alpha, T_{\overline{\beta}} \right] = -\sqrt{-1} g_{\alpha\overline{\beta}} T,$$
$$[T_\alpha, T] = 0.$$

From the second identity it follows that ∇ is torsion free. Then we obtain

$$\nabla_T T_\beta = 0,$$
$$\nabla_{T_{\overline{\alpha}}} T_\beta = 0.$$
$$\nabla_{T_\alpha} T_\beta = \Gamma_{\alpha\beta}^\gamma T_\gamma.$$

with $\Gamma_{\alpha\beta}^{\gamma} = g^{\gamma\overline{\nu}}\frac{\partial g_{\beta\overline{\nu}}}{\partial z_a}$. Notice that this is the also the Christoffel symbol of the Levi-Civita connection of (M, ω). The curvature tensor of the Tanaka-Webster connection is then given by

$$\begin{aligned} R_{\mu\overline{\nu}\alpha\overline{\beta}} &= \left\langle -\nabla_\mu \nabla_{\overline{\nu}} T_\alpha + \nabla_{\overline{\nu}} \nabla_\mu T_\alpha + \nabla_{[T_\mu, T_{\overline{\nu}}]} T_\alpha, T_{\overline{\beta}} \right\rangle = \frac{\partial \Gamma_{\mu\alpha}^{\gamma}}{\partial z_{\overline{\nu}}} g_{\gamma\overline{\beta}} \\ &= R^M_{\mu\overline{\nu}\alpha\overline{\beta}}, \end{aligned}$$

where

$$R^M_{\mu\overline{\nu}\alpha\overline{\beta}} = R^M \left(\frac{\partial}{\partial z_\mu}, \frac{\partial}{\partial z_{\overline{\nu}}}, \frac{\partial}{\partial z_a}, \frac{\partial}{\partial z_{\overline{\beta}}} \right)$$

is the curvature tensor of (M, ω). Taking trace, we also obtain

$$R_{\mu\overline{\nu}} = R^M_{\mu\overline{\nu}}, \mathcal{R} = R^M,$$

where $R^M_{\mu\overline{\nu}} = -g^{\alpha\overline{\beta}} R^M_{\mu\overline{\nu}\alpha\overline{\beta}}, R^M = g^{\mu\overline{\nu}} R^M_{\mu\overline{\nu}}$ are the Ricci and scalar curvature of (M, ω). The Bochner tensor of (M, ω) is defined by

$$\begin{aligned} B_{\mu\overline{\nu}\alpha\overline{\beta}} &:= -R^M_{\mu\overline{\nu}\alpha\overline{\beta}} - \frac{1}{m+2} \left[R^M_{\alpha\overline{\beta}} g_{\mu\overline{\nu}} + R^M_{\mu\overline{\beta}} g_{\alpha\overline{\nu}} + g_{\alpha\overline{\beta}} R^M_{\mu\overline{\nu}} + g_{\mu\overline{\beta}} R^M_{\alpha\overline{\nu}} \right] \\ &\quad + \frac{R^M}{(m+1)(m+2)} \left[g_{\alpha\overline{\beta}} g_{\mu\overline{\nu}} + g_{\mu\overline{\beta}} g_{\alpha\overline{\nu}} \right]. \end{aligned}$$

Therefore the Chern tensor of (Σ, θ) equals the Bochner tensor B of the Kahler manifold (M, ω). Since the torsion A of the Tanaka-Webster connection is zero, we also have the following formula for the Cartan tensor

$$Q_{\alpha\beta} = -\frac{2\sqrt{-1}}{(n+2)(n+1)} R^M_{,\alpha\beta},$$

where $R^M_{,\alpha\beta}$ denoted the covariant derivative of the scalar curvature R^M.

For Bochner flat metric, we also have

$$R_{\alpha\overline{\beta},\mu} = \frac{1}{m+1} \left(R_\alpha \delta_{\mu\beta} + R_\mu \delta_{\alpha\beta} \right)$$

Proposition 4. *We have the following*

- *If dimension $2m+1 \geq 5$, then Σ is locally CR spherical iff the (M, ω) is Bochner-Kahler metric (i.e. the Bochner tensor vanishes).*
- *If dimension $2m+1 = 3$, then Σ is locally CR spherical iff the $\nabla^{(1,0)} R^M$ is a holomorphic vector field on (M, ω).*

Bochner-Kahler manifolds were first studied by Bochner [B]. There had been a lot of work on such manifolds. But the more recent paper of Bryant [Br] is the definitive work in which Bochner-Kahler metrics are classified even locally. For compact ones, we have the following classification.

Theorem 13. *(Kamishima* [K]*;Bryant* [Br] *) Let M be a compact Kahler manifold of complex dimension $n > 1$. Suppose the Bochner tensor vanishes. Then M is biholomorphically isometric to*

1. *the complex projective space $\mathbb{CP}^m$,*
2. *a complex Euclidean space form $\mathbb{T}^m/F, F \subset U(m)$ a finite group,*
3. *a complex hyperbolic space form $\mathbb{B}^m/\Gamma, \Gamma \subset PU(m,1)$ a cocompact lattice,*

(4) *the fibre space* $\left(\mathbb{B}^l \times \mathbb{CP}^{m-l}\right)/\Gamma, \Gamma \subset PU(l,1) \times PU(m-l+1)$ *a cocompact lattice* $(l = 1, \cdots, m-1)$.

Remark 1. *We should point out that the proof in* [K] *uses the connection between Bochner-Kahler manifolds and spherical CR structures. But his proof is not complete.*

We now prove Theorem 12. When $m \geq 2$ it follows from the above theorem. When $m = 1$, (M, ω) is a surface with $\nabla^{(1,0)} R^M$ is a holomorphic vector field. If the genus of M is bigger than 1, then M has no nonzero holomorphic vector field and hence R^M is constant. When the genus is 1, the same is true as $\nabla^{(1,0)} R^M$ is a holomorphic vector field with zeros. When the genus is zero, we are on the Riemann sphere and ω is a so called extremal metric. It is well known that ω must have constant scalar curvature.

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