A RIGIDITY THEOREM FOR THE HEMI-SPHERE

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1. Introduction

In this paper we prove the following rigidity theorem.

Theorem 1. *Let* (M^n, g) $(n \geq 2)$ *be a compact Riemannian manifold with non-empty boundary* $\Sigma = \partial M$*. Suppose*

- *Ric*$\geq (n-1)\, g$,
- $(\Sigma, g|_\Sigma)$ *is isometric to the standard sphere* $\mathbb{S}^{n-1} \subset \mathbb{R}^n$,
- Σ *is convex in* M *in the sense that its second fundamental form is nonnegative.*

Then (M^n, g) *is isometric to the hemisphere* $\mathbb{S}^n_+ \subset \mathbb{R}^{n+1}$.

It may be necessary to make precise certain definitions involved here as there are different conventions for the second fundamental form and the mean curvature in the literature. Let ν be the outer unit normal field of Σ in M. For any $p \in \Sigma$, for any $X, Y \in T_p\Sigma$ the second fundamental form is defined as

$$\Pi\,(X, Y) = \langle \nabla_X \nu, Y \rangle\,.$$

The mean curvature is the trace of the second fundamental form.

Put in another way, the theorem says that for a compact manifold with boundary, if we know that the boundary is $\mathbb{S}^{n-1}$(intrinsic geometry on the boundary) and convex (some extrinsic geometry) then we recognize the manifold as the hemisphere $\mathbb{S}^n_+$, provided Ric $\geq (n-1)\, g$. To put this result in a context, we first recall the following

Theorem 2. *Let* (M^n, g) *be a compact Riemannian manifold with boundary and scalar curvature* $R \geq 0$*. If the boundary is isometric to* $\mathbb{S}^{n-1}$ *and has mean curvature* $n-1$*, then* (M^n, g) *is isometric to the unit ball* $\overline{\mathbb{B}^n} \subset \mathbb{R}^n$*. (If* $n > 7$ *we need to assume that* M *is spin.)*

This remarkable result is a simple corollary of the positive mass theorem: indeed one may glue M with $\mathbb{R}^n \backslash \mathbb{B}^n$ along the boundary $\mathbb{S}^{n-1}$ to obtain an asymptotically flat manifold N with nonnegative scalar curvature. Since it is actually flat near infinity the positive mass theorem implies that N is isometric to $\mathbb{R}^n$ and hence M is isometric to $\overline{\mathbb{B}^n}$ (see [M, ST] for details). There are similar rigidity results for geodesic balls in the hyperbolic space assuming $R \geq -n\,(n-1)$ by applying the positive mass theorem for asymptotically hyperbolic manifolds.

It is a natural question to consider the hemisphere. The following conjecture was proposed by Min-Oo in 1995.

Conjecture 1. *(Min-Oo) Let (M^n, g) be a compact Riemannian manifold with boundary and scalar curvature $R \geq n(n-1)$. If the boundary is isometric to $\mathbb{S}^{n-1}$ and totally geodesic, then (M^n, g) is isometric to the hemisphere $\mathbb{S}^n_+$.*

The proof of Theorem 2 does not seem to work any more: there is no positive mass theorem providing a miraculous passage from the compact manifold in question to a noncompact manifold. As it stands this conjecture seems difficult. There have only been some partial results in [HW] and some recent progress in dimension three in [E]. Theorem 1 can be viewed as the Ricci version of Min-Oo's conjecture. It is a strong evidence that Min-Oo's conjecture should be true.

In dimension 2 it turns out that Theorem 1 is essentially equivalent to a result of Toponogov on the length of simple closed geodesics on a strictly convex surface. This connection is discussed in Section 2 in which we also present a different proof working only in dimension 2. This proof may have some independent interest. It is also interesting to compare this two dimensional argument, which is partly geometric and partly analytic, with the unified proof of purely analytic nature presented in Section 3.

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2. The two dimensional case

When $n = 2$ we consider a compact surface (M^2, g) with boundary. The boundary then consists of closed curves and there is no intrinsic geometry except the lengths of these curves. The extrinsic geometry of the boundary is given by the geodesic curvature. Therefore Theorem 1 follows from the following slightly stronger result.

Theorem 3. *Let (M^2, g) be compact surface with boundary and the Gaussian curvature $K \geq 1$. Suppose the geodesic curvature k of the boundary γ satisfies $k \geq c \geq 0$. Then $L(\gamma) \leq 2\pi/\sqrt{1+c^2}$. Moreover equality holds iff (M, g) is isometric to a disc of radius $\cot^{-1}(c)$ in $\mathbb{S}^2$.*

Proof. By Gauss-Bonnet formula

$$2\pi\chi(M) = \int_M K d\sigma + \int_\gamma k ds > 0,$$

where $\chi(M)$ is the Euler number of M. Therefore M is simply connected and in particular γ has only one component. By the Riemann mapping theorem, (M, g) is conformally equivalent to the unit disc $\overline{\mathbb{B}} = \{z \in \mathbb{C} : |z| \leq 1\}$. Without loss of generality, we take (M, g) to be $(\overline{\mathbb{B}}, g = e^{2u}|dz|^2)$ with $u \in C^\infty(\overline{\mathbb{B}}, \mathbb{R})$. By our assumptions we have

$$\begin{cases} -\Delta u \geq e^{2u} \text{ on } \overline{\mathbb{B}}, \\ \frac{\partial u}{\partial r} + 1 \geq c e^u \quad \text{on } \mathbb{S}^1 \end{cases}$$

Let $\underline{u} \in C^\infty(\overline{\mathbb{B}}, \mathbb{R})$ such that

$$\begin{cases} -\Delta \underline{u} = 0 \text{ on } B, \\ \underline{u}|_{\mathbb{S}^1} = u|_{\mathbb{S}^1}. \end{cases}$$

Then $\underline{u} \leq u$ as u is superharmonic. It follows from sub-sup solution method (see, e.g., [SY, page 187-189]) that we may find a $v \in C^{\infty}\left(\overline{\mathbb{B}}, \mathbb{R}\right)$ with

$$\begin{cases} -\Delta v = e^{2v} \text{ on } \overline{\mathbb{B}}, \\ \underline{u} \leq v \leq u. \end{cases}$$

Since $v \leq u$ and $v|_{\mathbb{S}^1} = u|_{\mathbb{S}^1}$ we have $\frac{\partial v}{\partial \nu} \geq \frac{\partial u}{\partial \nu}$ and hence $\left.\frac{\partial v}{\partial \nu}\right|_{S^1} + 1 \geq ce^u$, i.e. the boundary circle has has geodesic curvature $\geq c$. As the metric $\left(\overline{\mathbb{B}}, e^{2v}|dz|^2\right)$ has curvature 1 and the boundary circle is convex, it can be isometrically embedded as a domain in $\mathbb{S}^2$, say Ω. Denote $\sigma = \partial\Omega$ parametrized by arclength. Notice $L(\sigma) = L(\gamma)$ as $v = u$ on the boundary $\mathbb{S}^1$. Because the boundary has geodesic curvature $\geq c \geq 0$, it is known that the smallest geodesic disc D containing Ω has radius at most $\cot^{-1}(c)$. Hence $L(\gamma) = L(\sigma) \leq 2\pi/\sqrt{1+c^2} = L(\partial D)$. The equality case follows directly from the argument. $\square$

As a corollary we have the following theorem due to Toponogov.

Corollary 1. *(Toponogov* [T]*) Let (M^2, g) be a closed surface with Gaussian curvature $K \geq 1$. Then any simple closed geodesic in M has length at most 2π. Moreover if there is one with length 2π, then M is isometric to the standard sphere $\mathbb{S}^2$.*

Proof. Suppose γ is a simple close geodesic. We cut M along γ to obtain two compact surfaces with the geodesic γ as their common boundary. The result follows from applying the previous theorem to either of these two compact surfaces with boundary. $\square$

Toponogov's original proof, as presented in Klingenberg [K, page 297] uses his triangle comparison theorem. In applying the triangle comparison theorem, which requires at least two minimizing geodesics, the difficulty is to know how long a geodesic segment is minimizing without assuming an upper bound for curvature. As the proof presented above, this difficulty is overcome by using special features of two dimensional topology.

3. The proof of the main theorem

We now present a proof of Theorem 1 which works in any dimension $n \geq 2$. We first recall the following result due to Reilly.

Theorem 4. *(Reilly* [R]*) Let (M^n, g) be a compact Riemannian manifold with nonempty boundary $\Sigma = \partial M$. Assume that* $\mathrm{Ric} \geq (n-1)\, g$ *and the mean curvature of Σ in M is nonnegative. Then the first (Dirichlet) eigenvalue λ_1 of $-\Delta$ satisfies the inequality $\lambda_1 \geq n$. Moreover $\lambda_1 = n$ iff M is isometric to the standard hemisphere $\mathbb{S}^n_+ \subset \mathbb{R}^{n+1}$.*

Therefore to prove Theorem 1, it suffices to show $\lambda_1(M) = n$. If this were not the case, then $\lambda_1(M) > n$. Therefore for every $f \in C^{\infty}(\Sigma)$ there is a unique $u \in C^{\infty}(M)$ solving

$$\begin{cases} -\Delta u = nu & \text{on} \quad M, \\ u = f & \text{on} \quad \Sigma. \end{cases} \tag{3.1}$$

Define

$$\phi = |\nabla u|^2 + u^2.$$

Lemma 1. *ϕ is subharmonic, i.e. $\Delta \phi \geq 0$.*

Proof. Using the Bochner formula, the equation (3.1) and the assumption $Ric \geq (n-1) g$,

$$\begin{aligned} \frac{1}{2}\Delta\phi &= \left|D^2 u\right|^2 + \langle \nabla u, \nabla \Delta u\rangle + \mathrm{Ric}(\nabla u, \nabla u) + |\nabla u|^2 + u\Delta u \\ &\geq \left|D^2 u\right|^2 - nu^2 \\ &\geq \frac{(\Delta u)^2}{n} - nu^2 \\ &= 0. \end{aligned}$$

□

Denote $\chi = \frac{\partial u}{\partial \nu}$, the derivative on the boundary in the outer unit normal ν. By the assumption of Theorem 1 there is an isometry $F : (\Sigma, g|_\Sigma) \to \mathbb{S}^{n-1} \subset \mathbb{R}^n$. In the following let $f = \sum_{i=1}^n \alpha_i x_i \circ F$, where $x_1, \cdots, x_n$ are the standard coordinate functions on $\mathbb{S}^{n-1}$ and $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{S}^{n-1}$. We have

$$-\Delta_\Sigma f = (n-1) f, \quad |\nabla_\Sigma f|^2 + f^2 = 1.$$

Hence

$$\phi|_\Sigma = |\nabla_\Sigma f|^2 + \chi^2 + f^2 = 1 + \chi^2. \tag{3.2}$$

On the boundary Σ

$$-nf = \Delta u|_\Sigma = \Delta_\Sigma f + H\chi + D^2 u(\nu, \nu) = -(n-1) f + H\chi + D^2 u(\nu, \nu),$$

whence

$$D^2 u(\nu, \nu) + f = -H\chi. \tag{3.3}$$

Lemma 2. *On Σ*

$$\frac{1}{2}\frac{\partial \phi}{\partial \nu} = \langle \nabla_\Sigma f, \nabla_\Sigma \chi\rangle - H\chi^2 - \Pi(\nabla_\Sigma f, \nabla_\Sigma f).$$

Proof. Indeed

$$\begin{aligned} \frac{1}{2}\frac{\partial \phi}{\partial \nu} &= D^2 u(\nabla u, \nu) + f\chi \\ &= D^2 u(\nabla_\Sigma u, \nu) + \chi\left(D^2 u(\nu, \nu) + f\right) \\ &= D^2 u(\nabla_\Sigma f, \nu) - H\chi^2, \end{aligned}$$

here we have used (3.3) in the last step. On the other hand

$$\begin{aligned} D^2 u(\nabla_\Sigma f, \nu) &= \langle \nabla_{\nabla_\Sigma f} \nabla u, \nu\rangle \\ &= \nabla_\Sigma f \langle \nabla u, \nu\rangle - \langle \nabla u, \nabla_{\nabla_\Sigma f} \nu\rangle \\ &= \langle \nabla_\Sigma f, \nabla_\Sigma \chi\rangle - \Pi(\nabla_\Sigma f, \nabla_\Sigma f). \end{aligned}$$

The lemma follows. □

Lemma 3. *The function* $\phi = |\nabla u|^2 + u^2$ *is constant and*

$$D^2 u = -ug.$$

Moreover $\chi = \frac{\partial u}{\partial \nu}$ *is also constant and* $\Pi\left(\nabla_\Sigma f, \nabla_\Sigma f\right) \equiv 0$.

Proof. Since ϕ is subharmonic, by the maximum principle ϕ achieves its maximum on Σ, say at $p \in \Sigma$. Obviously we have

$$\nabla_\Sigma \phi\left(p\right) = 0, \quad \frac{\partial \phi}{\partial \nu}\left(p\right) \geq 0.$$

If $\frac{\partial \phi}{\partial \nu}\left(p\right) = 0$, then ϕ must be constant by the strong maximum principle and Hopf lemma (see [GT, page 34-35]). Then the proof of Lemma 1 implies $D^2 u = -ug$. By (3.2) χ is constant. It then follows from Lemma 2 that $\Pi\left(\nabla_\Sigma f, \nabla_\Sigma f\right) \equiv 0$.

Suppose $\frac{\partial \phi}{\partial \nu}\left(p\right) > 0$. Then $\chi\left(p\right) \neq 0$, for otherwise it follows from (3.2) that $\chi \equiv 0$ and hence $\frac{\partial \phi}{\partial \nu}\left(p\right) \leq 0$ by Lemma 2, a contradiction. From (3.2) we conclude $\nabla_\Sigma \chi\left(p\right) = 0$. By Lemma 2

$$\frac{1}{2}\frac{\partial \phi}{\partial \nu}\left(p\right) = \left\langle \nabla_\Sigma f, \nabla_\Sigma \chi\right\rangle\left(p\right) - H\chi^2 - \Pi\left(\nabla_\Sigma f, \nabla_\Sigma f\right) \leq 0,$$

here we have used the assumption that Σ is convex, i.e. $\Pi \geq 0$. This contradicts with $\frac{\partial \phi}{\partial \nu}\left(p\right) > 0$ again. □

Recall f depends on a unit vector $\alpha \in \mathbb{S}^{n-1}$. To indicate the dependence on α we will add subscript α to all the quantities. Since $\Pi\left(\nabla_\Sigma f_\alpha, \nabla_\Sigma f_\alpha\right) \equiv 0$ on Σ for any $\alpha \in \mathbb{S}^{n-1}$ and $\left\{\nabla_\Sigma f_\alpha : \alpha \in \mathbb{S}^{n-1}\right\}$ span the tangent bundle $T\Sigma$ we conclude that Σ is totally geodesic, i.e. $\Pi = 0$.

We now claim that we can choose α such that $\chi_\alpha \equiv 0$. Indeed, $\alpha \to \chi_\alpha$ is a continuous function on $\mathbb{S}^{n-1}$. Clearly $u_{-\alpha} = -u_\alpha$ and hence $\chi_{-\alpha} = -\chi_\alpha$. Therefore by the intermediate value theorem there exists some $\beta \in \mathbb{S}^{n-1}$ such that $\chi_\beta \equiv 0$. With this particular choice $f = f_\beta, u = u_\beta$ we have

$$\begin{cases} D^2 u = -ug, \\ \frac{\partial u}{\partial \nu} \equiv 0. \end{cases}$$

There is $q \in \Sigma$ such that $f\left(q\right) = \max f = 1$. Then $\nabla_\Sigma f\left(q\right) = 0$ and hence $\nabla u\left(q\right) = 0$ as $\frac{\partial u}{\partial \nu}\left(q\right) = 0$. For $X \in T_q M$ such that $\left\langle X, \nu\left(q\right)\right\rangle \leq 0$ let γ_X be the geodesic with $\dot{\gamma}_X\left(0\right) = X$. Note that γ_X lies in Σ if X is tangential to Σ since Σ is totally geodesic. The function $U\left(t\right) = u \circ \gamma_X\left(t\right)$ then satisfies the following

$$\begin{cases} \ddot{U}\left(t\right) = -U, \\ U\left(0\right) = 1, \\ \dot{U}\left(0\right) = 0. \end{cases}$$

Hence $U\left(t\right) = \cos t$. Because Σ is totally geodesic, every point may be connected to q by a minimizing geodesic. Using the geodesic polar coordinates $\left(r, \xi\right) \in \mathbb{R}^+ \times \mathbb{S}^{n-1}_+$at q we can write

$$g = dr^2 + h_r$$

where r is the distance function to q and h_r is r-family of metrics on $\mathbb{S}^{n-1}_+$ with

$$\lim_{r \to 0} r^{-2} h_r = h_0,$$

here h_0 is the standard metric on $\mathbb{S}^{n-1}_+$. Then $u = \cos r$. The equation $D^2 u = -ug$ implies

$$\frac{\partial h_r}{\partial r} = 2\frac{\cos r}{\sin r} h_r$$

which can be solved to give $h_r = \sin^2 r h_0$. It follows that (M, g) is isometric to $\mathbb{S}^n_+$. This implies $\lambda_1(M) = n$ and contradicts with the assumption $\lambda_1(M) > n$. Theorem 1 follows.

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