RIGIDITY THEOREMS FOR COMPACT MANIFOLDS WITH BOUNDARY AND POSITIVE RICCI CURVATURE

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1. Introduction

The positive mass theorem, first proved by Schoen-Yau [SY1, SY2] and later by Witten [W] using spinors, is one of the profound results in differential geometry. In the recent work of Shi-Tam, it is used in a novel way to yield beautiful results on the boundary effect on compact Riemannian manifolds with nonnegative scalar curvature. The following theorem is only a special case of their main result.

Theorem 1. *Let (M^n, g) be a compact Riemannian manifold with boundary and scalar curvature $R \geq 0$. If the boundary is isometric to $\mathbb{S}^{n-1}$ and has mean curvature $n-1$, then (M^n, g) is isometric to the unit ball $\overline{\mathbb{B}^n} \subset \mathbb{R}^n$. (If $n > 7$ we need to assume that M is spin.)*

We sketch the idea of the proof. We glue M with $\mathbb{R}^n \backslash \mathbb{B}^n$ along the boundary $\mathbb{S}^{n-1}$ to obtain an asymptotically flat manifold N with nonnegative scalar curvature. Since it is actually flat near infinity the positive mass theorem implies that N is isometric to $\mathbb{R}^n$ and hence M is isometric to $\overline{\mathbb{B}^n}$ (see [M, ST] for details). There are similar rigidity results for geodesic balls in the hyperbolic space assuming $R \geq -n(n-1)$ by applying the positive mass theorem for asymptotically hyperbolic manifolds.

It is a natural question to consider the hemisphere. The following conjecture was proposed by Min-Oo in 1995.

Conjecture 1. *(Min-Oo) Let (M^n, g) be a compact Riemannian manifold with boundary and scalar curvature $R \geq n(n-1)$. If the boundary is isometric to $\mathbb{S}^{n-1}$ and totally geodesic, then (M^n, g) is isometric to the hemisphere $\mathbb{S}^n_+$.*

The proof of Theorem 1 does not seem to work any more: there is no positive mass theorem providing a miraculous passage from the compact manifold in question to a noncompact manifold. As it stands this conjecture seems quite difficult. There have only been some partial results in [HW] and some recent progress in dimension three in [E].

Motivated by Min-Oo's conjecture, we consider the rigidity of compact Riemannian manifolds with boundary and positive Ricci curvature. Here is our first result.

Theorem 2. *Let (M^n, g) $(n \geq 2)$ be a compact Riemannian manifold with nonempty boundary $\Sigma = \partial M$. Suppose*

- *$Ric \geq (n-1)g$,*
- *$(\Sigma, g|_\Sigma)$ is isometric to the standard sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$,*
- *Σ is convex in M in the sense that its second fundamental form is nonnegative.*

Then (M^n, g) *is isometric to the hemisphere* $\mathbb{S}^n_+ = \{x \in \mathbb{R}^{n+1} : |x| = 1, x_{n+1} \geq 0\} \subset \mathbb{R}^{n+1}$.

Since there are different conventions for the second fundamental form and the mean curvature in the literature, let us explain ours. Let ν be the outer unit normal field of Σ in M. For any $p \in \Sigma$, for any $X, Y \in T_p\Sigma$ the second fundamental form is defined as

$$\Pi(X, Y) = \langle \nabla_X \nu, Y \rangle .$$

The mean curvature is the trace of the second fundamental form.

Put in another way, the theorem says that for a compact manifold with boundary, if we know that the boundary is $\mathbb{S}^{n-1}$(intrinsic geometry on the boundary) and convex (some extrinsic geometry) then we recognize the manifold as the hemisphere $\mathbb{S}^n_+$, provided $\mathrm{Ric} \geq (n-1)\, g$. Theorem 2 can be viewed as the Ricci version of Min-Oo's conjecture. It is a strong evidence that Min-Oo's conjecture should be true.

Shi-Tam [ST] have also studied compact manifolds (M^n, g) whose boundaries isometrically embed in $\mathbb{R}^n$ as a convex hypersurface. In our case we may consider compact Riemannian manifolds whose boundaries isometrically embed as a hypersurface in $\mathbb{S}^n_+$. We prove the following rigidity theorem in this more general case.

Theorem 3. *Let* (M, g) *be a smooth compact Riemannian manifold with boundary* $\partial M = \Sigma$ *and* $\overline{\Omega} \subset \mathbb{S}^n_+$ *is a compact domain with smooth boundary in the open hemisphere. Suppose*

- *Ric*$\geq (n-1)\, g$,
- *there is an isometric embedding* $\iota : (\Sigma, g_\Sigma) \to \partial\overline{\Omega}$,
- $\Pi \geq \Pi_0 \circ \iota$, *here* Π *the second fundamental form of* Σ *in* M *and* Π_0 *is the second fundamental form of* $\partial\overline{\Omega}$ *in* $\mathbb{S}^n_+$.

Then (M, g) *is isometric to* $\left(\overline{\Omega}, g_{\mathbb{S}^n_+}\right)$.

In dimension 2 it turns out that Theorem 2 is essentially equivalent to a result of Toponogov on the length of simple closed geodesics on a strictly convex surface. This connection is discussed in Section 2 in which we also present a different proof working only in dimension 2. This proof may have some independent interest. It is also interesting to compare this two dimensional argument, which is partly geometric and partly analytic, with the unified proof of purely analytic nature presented in Section 4.

To prove Theorem 3, we have to generalize the proof of Theorem 2 to the more general context where we allow the metric to be Lipschitz along a hypersurface. For this purpose we first establish Reilly's theorem on the first eigenvalue in this context in Section. The proof of Theorem 3 is then given.

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2. The two dimensional case

When $n = 2$ we consider a compact surface (M^2, g) with boundary. The boundary then consists of closed curves and there is no intrinsic geometry except the

lengths of these curves. The extrinsic geometry of the boundary is given by the geodesic curvature. Therefore Theorem 2 follows from the following slightly stronger result.

Theorem 4. *Let (M^2, g) be compact surface with boundary and the Gaussian curvature $K \geq 1$. Suppose the geodesic curvature k of the boundary γ satisfies $k \geq c \geq 0$. Then $L(\gamma) \leq 2\pi/\sqrt{1+c^2}$. Moreover equality holds iff (M, g) is isometric to a disc of radius $\cot^{-1}(c)$ in $\mathbb{S}^2$.*

Proof. By Gauss-Bonnet formula

$$2\pi\chi(M) = \int_M K d\sigma + \int_\gamma k ds > 0,$$

where $\chi(M)$ is the Euler number of M. Therefore M is simply connected and in particular γ has only one component. By the Riemann mapping theorem, (M, g) is conformally equivalent to the unit disc $\overline{\mathbb{B}} = \{z \in \mathbb{C} : |z| \leq 1\}$. Without loss of generality, we take (M, g) to be $(\overline{\mathbb{B}}, g = e^{2u}|dz|^2)$ with $u \in C^\infty\left(\overline{\mathbb{B}}, \mathbb{R}\right)$. By our assumptions we have

$$\begin{cases} -\Delta u \geq e^{2u} \text{ on } \overline{\mathbb{B}}, \\ \frac{\partial u}{\partial r} + 1 \geq ce^u \quad \text{on } \mathbb{S}^1 \end{cases}$$

Let $\underline{u} \in C^\infty\left(\overline{\mathbb{B}}, \mathbb{R}\right)$ such that

$$\begin{cases} -\Delta \underline{u} = 0 \text{ on } B, \\ \underline{u}|_{\mathbb{S}^1} = u|_{\mathbb{S}^1}. \end{cases}$$

Then $\underline{u} \leq u$ as u is superharmonic. It follows from sub-sup solution method (see, e.g., [SY, page 187-189]) that we may find a $v \in C^\infty\left(\overline{\mathbb{B}}, \mathbb{R}\right)$ with

$$\begin{cases} -\Delta v = e^{2v} \text{ on } \overline{\mathbb{B}}, \\ \underline{u} \leq v \leq u. \end{cases}$$

Since $v \leq u$ and $v|_{\mathbb{S}^1} = u|_{\mathbb{S}^1}$ we have $\frac{\partial v}{\partial \nu} \geq \frac{\partial u}{\partial \nu}$ and hence $\frac{\partial v}{\partial \nu} + 1 \geq ce^u$ on $\mathbb{S}^1$, i.e. the boundary circle has has geodesic curvature $\geq c$. As the metric $\left(\overline{\mathbb{B}}, e^{2v}|dz|^2\right)$ has curvature 1 and the boundary circle is convex, it can be isometrically embedded as a domain in $\mathbb{S}^2$, say Ω. Denote $\sigma = \partial\Omega$ parametrized by arclength. Notice $L(\sigma) = L(\gamma)$ as $v = u$ on the boundary $\mathbb{S}^1$. Because the boundary has geodesic curvature $\geq c \geq 0$, it is known that the smallest geodesic disc D containing Ω has radius at most $\cot^{-1}(c)$. Hence $L(\gamma) = L(\sigma) \leq 2\pi/\sqrt{1+c^2} = L(\partial D)$. The equality case follows directly from the argument. □

As a corollary we have the following theorem due to Toponogov.

Corollary 1. *(Toponogov [T]) Let (M^2, g) be a closed surface with Gaussian curvature $K \geq 1$. Then any simple closed geodesic in M has length at most 2π. Moreover if there is one with length 2π, then M is isometric to the standard sphere $\mathbb{S}^2$.*

Proof. Suppose γ is a simple close geodesic. We cut M along γ to obtain two compact surfaces with the geodesic γ as their common boundary. The result follows from applying the previous theorem to either of these two compact surfaces with boundary. □

Toponogov's original proof, as presented in Klingenberg [K, page 297] uses his triangle comparison theorem. In applying the triangle comparison theorem, which requires at least two minimizing geodesics, the difficulty is to know how long a geodesic segment is minimizing without assuming an upper bound for curvature. As the proof presented above, this difficulty is overcome by using special features of two dimensional topology.

3. Conformal change of metrics

Of course the conformal method is not very useful in higher dimensions. It may still be of interest to record here what one can prove with it.

Proposition 1. *Assume $\Omega \subset S^n_+$ is a smooth domain, $\widetilde{g} = u^{\frac{4}{n-2}} g_{S^n}$, $u|_{\partial\Omega} = 1$, $\widetilde{R} \geq R_{S^n} = n(n-1)$, then $u \geq 1$, $\widetilde{H} \leq H$. Moreover if equality holds somewhere, then $\widetilde{g} = g_{S^n}$.*

We have

$$\widetilde{R} = u^{-\frac{n+2}{n-2}} \left(-\frac{4(n-1)}{n-2} \Delta u + n(n-1) u \right) \geq n(n-1),$$

in another way it is

$$-\Delta u + \frac{n(n-2)}{4} u \geq \frac{n(n-2)}{4} u^{\frac{n+2}{n-2}}, \quad u|_{\partial\Omega} = 1.$$

Let $\overline{u} = \min\{u, 1\}$, then $\overline{u}$ is Lipschitz and it follows from Kato's inequality that in distribution sense

$$-\Delta \overline{u} \geq -\chi_{u<1} \Delta u \geq \frac{n(n-2)}{4} \left(\overline{u}^{\frac{n+2}{n-2}} - \overline{u} \right), \quad \overline{u}|_{\partial\Omega} = 1.$$

Indeed,

$$\overline{u} = \frac{u+1}{2} - \frac{|u-1|}{2}.$$

for $\varepsilon > 0$, we let $f_\varepsilon(t) = \sqrt{t^2 + \varepsilon^2}$, then $\Delta(f_\varepsilon(u-1)) \geq f'_\varepsilon(u-1) \Delta u$, let $\varepsilon \to 0^+$ it follows that

$$\Delta(|u-1|) \geq \operatorname{sgn}(u-1) \Delta u.$$

Hence

$$-\Delta \overline{u} \geq -\frac{1}{2} \Delta u + \frac{1}{2} \operatorname{sgn}(u-1) \Delta u = -\chi_{u<1} \Delta u$$

in distribution sense. Let $v \in C^2(\overline{\Omega})$ such that

$$\begin{cases} -\Delta \overline{v} + \frac{n(n-2)}{4} \overline{v} = \frac{n(n-2)}{4} \overline{u}^{\frac{n+2}{n-2}}, \\ \overline{v}|_{\partial\Omega} = 1, \end{cases}$$

then $\overline{v} \leq \overline{u} \leq 1$ and hence $\frac{\partial \overline{v}}{\partial \nu} \geq 0$. Define

$$v(x) = \begin{cases} \overline{v}(x), \ x \in \Omega, \\ 1, \ x \notin \Omega. \end{cases}$$

Then v is Lipschitz and in the sense of distribution

$$-\Delta v + \frac{n(n-2)}{4} v \geq \frac{n(n-2)}{4} v^{\frac{n+2}{n-2}}, \quad v|_{S^{n-1}} = 1.$$

Indeed for any nonnegative $\varphi \in C^\infty\left(S_+^n\right)$, $\varphi = 0$ near S^{n-1}, we have

$$\begin{aligned}
&\int_{S_+^n}\left(\nabla v\cdot\nabla\varphi + \frac{n\left(n-2\right)}{4}v\varphi\right)d\mu \\
&= \int_\Omega\left(\nabla\overline{v}\cdot\nabla\varphi + \frac{n\left(n-2\right)}{4}\overline{v}\varphi\right)d\mu + \int_{S_+^n\setminus\Omega}\frac{n\left(n-2\right)}{4}\varphi d\mu \\
&\geq \int_{\partial\Omega}\frac{\partial\overline{v}}{\partial\nu}\varphi dS + \int_{S_+^n}\frac{n\left(n-2\right)}{4}v^{\frac{n+2}{n-2}}\varphi d\mu \\
&\geq \int_{S_+^n}\frac{n\left(n-2\right)}{4}v^{\frac{n+2}{n-2}}\varphi d\mu.
\end{aligned}$$

Let $w \in C^2\left(S_+^n\right)$ satisfy

$$\begin{cases} -\Delta w + \frac{n(n-2)}{4}w = \frac{n(n-2)}{4}v^{\frac{n+2}{n-2}} \\ w|_{S^{n-1}} = 1 \end{cases}.$$

It follows that $0 \leq w \leq v$ and hence $-\Delta w + \frac{n(n-2)}{4}w \geq \frac{n(n-2)}{4}w^{\frac{n+2}{n-2}}$. Using the conformal rigidity result in [HW] (Theorem 3.1 on p99) we see $w = 1$. Hence $v = 1$ and $\overline{u} = 1$. It follows that $u \geq 1$. Note that

$$-\Delta u \geq \frac{n\left(n-2\right)}{4}\left(u^{\frac{n+2}{n-2}} - u\right) \geq 0.$$

Hence u is superharmonic. It follows from strong maximum principle that either $u \equiv 1$ or $u > 1$ in Ω and $\frac{\partial u}{\partial\nu} < 0$. The conclusion follows from

$$\widetilde{H} = \frac{2\left(n-1\right)}{n-2}\frac{\partial u}{\partial\nu} + H.$$

4. The proof of Theorem 2

We now present a proof of Theorem 2 which works in any dimension $n \geq 2$. We first recall the following result due to Reilly.

Theorem 5. *(Reilly* [R]*) Let (M^n, g) be a compact Riemannian manifold with non-empty boundary $\Sigma = \partial M$. Assume that* $\mathrm{Ric} \geq (n-1)\,g$ *and the mean curvature of Σ in M is nonnegative. Then the first (Dirichlet) eigenvalue λ_1 of $-\Delta$ satisfies the inequality $\lambda_1 \geq n$. Moreover $\lambda_1 = n$ iff M is isometric to the standard hemisphere $\mathbb{S}_+^n \subset \mathbb{R}^{n+1}$.*

Therefore to prove Theorem 2, it suffices to show $\lambda_1\left(M\right) = n$. If this were not the case, then $\lambda_1\left(M\right) > n$. Therefore for every $f \in C^\infty\left(\Sigma\right)$ there is a unique $u \in C^\infty\left(M\right)$ solving

$$\begin{cases} -\Delta u = nu & \text{on} \quad M, \\ u = f & \text{on} \quad \Sigma. \end{cases} \tag{4.1}$$

Define

$$\phi = \left|\nabla u\right|^2 + u^2.$$

Lemma 1. *ϕ is subharmonic, i.e. $\Delta\phi \geq 0$.*

Proof. Using the Bochner formula, the equation (4.1) and the assumption $Ric \geq (n-1)\, g$,

$$\begin{aligned}\frac{1}{2}\Delta\phi &= \left|D^2u\right|^2 + \langle\nabla u, \nabla\Delta u\rangle + \mathrm{Ric}(\nabla u, \nabla u) + \left|\nabla u\right|^2 + u\Delta u \\ &\geq \left|D^2u\right|^2 - nu^2 \\ &\geq \frac{(\Delta u)^2}{n} - nu^2 \\ &= 0.\end{aligned}$$

□

Denote $\chi = \frac{\partial u}{\partial \nu}$, the derivative on the boundary in the direction of the outer unit normal field ν. By the assumption of Theorem 2 there is an isometry $F : (\Sigma, g|_\Sigma) \to \mathbb{S}^{n-1} \subset \mathbb{R}^n$. In the following let $f = \sum_{i=1}^n \alpha_i x_i \circ F$, where $x_1, \cdots, x_n$ are the standard coordinate functions on $\mathbb{S}^{n-1}$ and $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{S}^{n-1}$. We have

$$-\Delta_\Sigma f = (n-1)\, f, \quad \left|\nabla_\Sigma f\right|^2 + f^2 = 1.$$

Hence

$$\phi|_\Sigma = \left|\nabla_\Sigma f\right|^2 + \chi^2 + f^2 = 1 + \chi^2. \tag{4.2}$$

On the boundary Σ

$$-nf = \Delta u|_\Sigma = \Delta_\Sigma f + H\chi + D^2u\,(\nu,\nu) = -\,(n-1)\, f + H\chi + D^2u\,(\nu,\nu)\,,$$

whence

$$D^2u\,(\nu,\nu) + f = -H\chi. \tag{4.3}$$

Lemma 2. *On* Σ

$$\frac{1}{2}\frac{\partial\phi}{\partial\nu} = \langle\nabla_\Sigma f, \nabla_\Sigma\chi\rangle - H\chi^2 - \Pi\,(\nabla_\Sigma f, \nabla_\Sigma f)\,.$$

Proof. Indeed

$$\begin{aligned}\frac{1}{2}\frac{\partial\phi}{\partial\nu} &= D^2u\,(\nabla u, \nu) + f\chi \\ &= D^2u\,(\nabla_\Sigma u, \nu) + \chi\left(D^2u\,(\nu,\nu) + f\right) \\ &= D^2u\,(\nabla_\Sigma f, \nu) - H\chi^2,\end{aligned}$$

here we have used (4.3) in the last step. On the other hand

$$\begin{aligned}D^2u\,(\nabla_\Sigma f, \nu) &= \langle\nabla_{\nabla_\Sigma f}\nabla u, \nu\rangle \\ &= \nabla_\Sigma f\,\langle\nabla u, \nu\rangle - \langle\nabla u, \nabla_{\nabla_\Sigma f}\nu\rangle \\ &= \langle\nabla_\Sigma f, \nabla_\Sigma\chi\rangle - \Pi\,(\nabla_\Sigma f, \nabla_\Sigma f)\,.\end{aligned}$$

The lemma follows. □

Lemma 3. *The function* $\phi = \left|\nabla u\right|^2 + u^2$ *is constant and*

$$D^2u = -ug.$$

Moreover $\chi = \frac{\partial u}{\partial\nu}$ *is also constant and* $\Pi\,(\nabla_\Sigma f, \nabla_\Sigma f) \equiv 0$.

Proof. Since ϕ is subharmonic, by the maximum principle ϕ achieves its maximum on Σ, say at $p \in \Sigma$. Obviously we have

$$\nabla_\Sigma \phi(p) = 0, \quad \frac{\partial \phi}{\partial \nu}(p) \geq 0.$$

If $\frac{\partial \phi}{\partial \nu}(p) = 0$, then ϕ must be constant by the strong maximum principle and Hopf lemma (see [GT, page 34-35]). Then the proof of Lemma 1 implies $D^2 u = -ug$. By (4.2) χ is constant. It then follows from Lemma 2 that $\Pi(\nabla_\Sigma f, \nabla_\Sigma f) \equiv 0$.

Suppose $\frac{\partial \phi}{\partial \nu}(p) > 0$. Then $\chi(p) \neq 0$, for otherwise it follows from (4.2) that $\chi \equiv 0$ and hence $\frac{\partial \phi}{\partial \nu}(p) \leq 0$ by Lemma 2, a contradiction. From (4.2) we conclude $\nabla_\Sigma \chi(p) = 0$. By Lemma 2

$$\frac{1}{2}\frac{\partial \phi}{\partial \nu}(p) = \langle \nabla_\Sigma f, \nabla_\Sigma \chi \rangle (p) - H\chi^2 - \Pi(\nabla_\Sigma f, \nabla_\Sigma f) \leq 0,$$

here we have used the assumption that Σ is convex, i.e. $\Pi \geq 0$. This contradicts with $\frac{\partial \phi}{\partial \nu}(p) > 0$ again. □

Recall f depends on a unit vector $\alpha \in \mathbb{S}^{n-1}$. To indicate the dependence on α we will add subscript α to all the quantities. Since $\Pi(\nabla_\Sigma f_\alpha, \nabla_\Sigma f_\alpha) \equiv 0$ on Σ for any $\alpha \in \mathbb{S}^{n-1}$ and $\left\{\nabla_\Sigma f_\alpha : \alpha \in \mathbb{S}^{n-1}\right\}$ span the tangent bundle $T\Sigma$ we conclude that Σ is totally geodesic, i.e. $\Pi = 0$.

We now claim that we can choose α such that $\chi_\alpha \equiv 0$. Indeed, $\alpha \to \chi_\alpha$ is a continuous function on $\mathbb{S}^{n-1}$. Clearly $u_{-\alpha} = -u_\alpha$ and hence $\chi_{-\alpha} = -\chi_\alpha$. Therefore by the intermediate value theorem there exists some $\beta \in \mathbb{S}^{n-1}$ such that $\chi_\beta \equiv 0$. With this particular choice $f = f_\beta, u = u_\beta$ we have

$$\begin{cases} D^2 u = -ug, \\ \frac{\partial u}{\partial \nu} \equiv 0. \end{cases}$$

There is $q \in \Sigma$ such that $f(q) = \max f = 1$. Then $\nabla_\Sigma f(q) = 0$ and hence $\nabla u(q) = 0$ as $\frac{\partial u}{\partial \nu}(q) = 0$. For $X \in T_q M$ such that $\langle X, \nu(q) \rangle \leq 0$ let γ_X be the geodesic with $\dot{\gamma}_X(0) = X$. Note that γ_X lies in Σ if X is tangential to Σ since Σ is totally geodesic. The function $U(t) = u \circ \gamma_X(t)$ then satisfies the following

$$\begin{cases} \ddot{U}(t) = -U, \\ U(0) = 1, \\ \dot{U}(0) = 0. \end{cases}$$

Hence $U(t) = \cos t$. Because Σ is totally geodesic, every point may be connected to q by a minimizing geodesic. Using the geodesic polar coordinates $(r, \xi) \in \mathbb{R}^+ \times \mathbb{S}_+^{n-1}$at q we can write

$$g = dr^2 + h_r$$

where r is the distance function to q and h_r is r-family of metrics on $\mathbb{S}_+^{n-1}$ with

$$\lim_{r \to 0} r^{-2} h_r = h_0,$$

here h_0 is the standard metric on $\mathbb{S}_+^{n-1}$. Then $u = \cos r$. The equation $D^2 u = -ug$ implies

$$\frac{\partial h_r}{\partial r} = 2\frac{\cos r}{\sin r} h_r$$

which can be solved to give $h_r = \sin^2 r h_0$. It follows that (M, g) is isometric to $\mathbb{S}^n_+$. This implies $\lambda_1(M) = n$ and contradicts with the assumption $\lambda_1(M) > n$. Theorem 2 follows.

5. Reilly's theorem and rigidity for certain nonsmooth metrics

To prove Theorem 3, it is natural to try the same argument of Section 4. Namely, we take v to be a linear function on $\mathbb{S}^n$ and then solve

$$\begin{cases} -\Delta u = nu & \text{on} \quad M, \\ u = v \circ \iota & \text{on} \quad \Sigma. \end{cases}$$

As before, $\phi = |\nabla u|^2 + u^2$. But when we apply the strong maximum principle to ϕ, we inevitably have to compare $\frac{\partial u}{\partial \nu}$ with the corresponding quantity $\frac{\partial v}{\partial \nu} \circ \iota$. We have no idea how such a comparison could be established. Instead, we have to take a different route.

First, we generalize Reilly's theorem to the situation where the metric g is only Lipschitz along a hypersurface. To be precise, let M be a smooth compact Riemannian manifold with $\partial M = \Sigma$, $\mathrm{Ric} \geq (n-1)$. Let N be another smooth compact Riemannian manifold with $\partial N = \Sigma \cup \Sigma_1$, Σ and Σ_1 being disjoint components, and $\mathrm{Ric} \geq (n-1)$. Assume $g_M|_\Sigma = g_N|_\Sigma$. Now we glue M and N along Σ to get a smooth manifold P with boundary Σ_1. However the metric on P is only Lipschitz along Σ. Let ν be the outer normal direction of M along Σ. We have two shape $A_M(X) = \nabla^M_X \nu$, $A_N(X) = \nabla^N_X \nu$ for $X \in T\Sigma$. For $X \in T\Sigma_1$, $A(X) = \nabla_X \nu$, here ν is the outer normal direction for N along Σ_1, and $H = \mathrm{tr}\, A$ is the mean curvature.

Theorem 6. *Assume $A_M \geq A_N$ and $H \geq 0$, then $\lambda_1(P) \geq n$. If $\lambda_1(P) = n$, then (P, g) is smooth and (P, g) is isometric to $\left(\mathbb{S}^n_+, g_{\mathbb{S}^n}\right)$.*

Proof. Let $u \in H^1_0(P)$ be the first eigenfunction, then $u \geq 0$ and $-\Delta u = \lambda u$, $\lambda > 0$. It follows from elliptic regularity theory that $u|_M \in C^\infty(M)$, $u|_N \in C^\infty(N)$ and

$$\frac{\partial\, u|_M}{\partial \nu} = \frac{\partial\, u|_N}{\partial \nu} \quad \text{on } \Sigma.$$

In particular $u \in C^{1,1}(P)$. Applying Reilly's formula on M, we get

$$\begin{aligned} &\frac{1}{2}\int_M \left((\Delta u)^2 - \left|D^2 u\right|^2\right) d\mu \\ &= \frac{1}{2}\int_M \mathrm{Ric}(\nabla u, \nabla u)\, d\mu + \int_\Sigma \Delta_\Sigma u \cdot \frac{\partial u}{\partial \nu} dS + \frac{1}{2}\int_\Sigma H_M \left(\frac{\partial u}{\partial \nu}\right)^2 dS \\ &+ \frac{1}{2}\int_\Sigma \langle A_M(\nabla_\Sigma u), \nabla_\Sigma u\rangle\, dS, \end{aligned}$$

here $H_M = \mathrm{tr}\, A_M$. Applying the same formula on N yields

$$\begin{aligned} &\frac{1}{2}\int_N \left((\Delta u)^2 - \left|D^2 u\right|^2\right) d\mu \\ &= \frac{1}{2}\int_N \mathrm{Ric}(\nabla u, \nabla u)\, d\mu - \int_\Sigma \Delta_\Sigma u \cdot \frac{\partial u}{\partial \nu} dS - \frac{1}{2}\int_\Sigma H_N \left(\frac{\partial u}{\partial \nu}\right)^2 dS \\ &- \frac{1}{2}\int_\Sigma \langle A_N(\nabla_\Sigma u), \nabla_\Sigma u\rangle\, dS + \frac{1}{2}\int_{\Sigma_1} H \left(\frac{\partial u}{\partial \nu}\right)^2 dS. \end{aligned}$$

Summing up we get

$$\begin{aligned}
&\frac{1}{2}\int_P \left((\Delta u)^2 - \left|D^2u\right|^2\right) d\mu \\
&= \frac{1}{2}\int_P \mathrm{Ric}\left(\nabla u, \nabla u\right) d\mu + \frac{1}{2}\int_\Sigma \left(H_M - H_0\right)\left(\frac{\partial u}{\partial \nu}\right)^2 dS \\
&+ \frac{1}{2}\int_\Sigma \left\langle \left(A_M - A_N\right)\left(\nabla_\Sigma u\right), \nabla_\Sigma u\right\rangle dS + \frac{1}{2}\int_{\Sigma_1} H\left(\frac{\partial u}{\partial \nu}\right)^2 dS.
\end{aligned}$$

Note that

$$\left|D^2u\right|^2 = \left|D^2u - \frac{\Delta u}{n}g\right|^2 + \frac{(\Delta u)^2}{n} = \left|D^2u - \frac{\lambda u}{n}g\right|^2 + \frac{\lambda^2 u^2}{n}.$$

Hence

$$\begin{aligned}
&\frac{n-1}{n}\lambda^2\int_M u^2 d\mu \\
&\geq (n-1)\int_M |\nabla u|^2 d\mu + \int_M \left|D^2u - \frac{\lambda u}{n}g\right|^2 d\mu + \int_{\Sigma_1} H\left(\frac{\partial u}{\partial \nu}\right)^2 dS \\
&\geq (n-1)\lambda\int_M u^2 d\mu + \int_M \left|D^2u - \frac{\lambda u}{n}g\right|^2 d\mu + \int_{\Sigma_1} H\left(\frac{\partial u}{\partial \nu}\right)^2 dS \\
&\geq (n-1)\lambda\int_M u^2 d\mu.
\end{aligned}$$

Hence $\lambda \geq n$.

If $\lambda = n$, then $D^2u = -ug$ on both M and N and $H\left(\frac{\partial u}{\partial \nu}\right)^2 = 0$ on Σ_1. Since $u > 0$ in $P\backslash\Sigma_1$, it follows from strong maximum principle that $\frac{\partial u}{\partial \nu} < 0$ on Σ_1 and hence $H = 0$ on Σ_1. We aim to show (P, g) is in fact isometric to $\left(\mathbb{S}^n_+, g_{\mathbb{S}^n_+}\right)$. The key is to prove that $g \in C^\infty$.

To continue we build some coordinates along Σ. Note for $r \geq 0$, we have a map $\Sigma \times [0, \varepsilon) \to M : (p, r) \mapsto \exp_p(-r\nu(p))$ which is an smooth embedding when ε is small. If we choose a coordinate locally on Σ, namely $\theta_1, \cdots, \theta_{n-1}$, hence we have a coordinate $r, \theta_1, \cdots, \theta_{n-1}$ near Σ. Similarly using the map $\Sigma \times (-\varepsilon, 0] \to N : (p, r) \mapsto \exp_p(-r\nu(p))$ we have coordinate $r, \theta_1, \cdots, \theta_{n-1}$ near Σ on P. Note that

$$g = dr \otimes dr + b_{ij}(r, \theta)\, d\theta_i \otimes d\theta_j.$$

$b_{ij}(r, \theta)$ is Lipschitz. We will write $u_0 = \partial_r u, u_i = \partial_i u, u_{00} = D^2u(\partial_r, \partial_r), u_{ij} = D^2u(\partial_i, \partial_j)$ etc.

From $D^2u = -ug$ it is easy to see $\nabla\left(|\nabla u|^2 + u^2\right) = 0$ on both M and N. Since $u \in C^1$ we conclude $|\nabla u|^2 + u^2 = \text{const}$. By scaling we may assume $|\nabla u|^2 + u^2 = 1$. We first observe that $u \in C^\infty(P)$. Indeed, to see this we only need to show $\partial_r^m u(0^+, \theta) = \partial_r^m u(0^-, \theta)$ for all m. But since $-u = u_{00} = \partial_r^2 u$, and $u(0, \theta) = a(\theta)$, $\partial_r u(0, \theta) = b(\theta)$, we see $u(r, \theta) = a(\theta)\cos r + b(\theta)\sin r$ for both positive and negative r, hence $\partial_r^m u(0^+, \theta) = \partial_r^m u(0^-, \theta)$ for all m.

Next we observe that if $u(p) = 1$, then at p, $D^2u = -g$, it follows that $u < 1$ for other points near p. Hence the set $\{u = 1\}$ is discrete. On $\{u \neq 1\}$, $|\nabla u| = \sqrt{1 - u^2}$ is smooth too.

Next we claim ∇u is a smooth vector field, though apriori it seems only belongs to Lipschitz. Indeed we have

$$\nabla u = u_0 \partial_r + b^{ij} u_j \partial_i.$$

We need to show $\partial_r^m \left(b^{ij} u_j\right)(0^+, \theta) = \partial_r^m \left(b^{ij} u_j\right)(0^-, \theta)$.

Given $p \in \Sigma$, if $\partial_r u(p) \neq 0$, then it is not zero near p. Note that

$$-u b_{ij} = u_{ij} = \partial_{ij} u + \frac{1}{2} \partial_r b_{ij} u_0 - \Gamma_{ij}^k u_k.$$

Restricting to $r = 0$ on both sides, we see $\partial_r b_{ij}(0^+, \theta) = \partial_r b_{ij}(0^-, \theta)$ for θ near $\theta(p)$. Using

$$\partial_r b_{ij} = \frac{2}{u_0} \left(\Gamma_{ij}^k u_k - u b_{ij} - \partial_{ij} u\right),$$

and the fact $u \in C^\infty$ we see $\partial_r^2 b_{ij}(0^+, \theta) = \partial_r^2 b_{ij}(0^-, \theta)$. By induction we see $\partial_r^m b_{ij}(0^+, \theta) = \partial_r^m b_{ij}(0^-, \theta)$ for all m. Hence g is smooth near p and

$$\partial_r^m \left(b^{ij} u_j\right)\left(0^+, \theta\right) = \partial_r^m \left(b^{ij} u_j\right)\left(0^-, \theta\right)$$

for all m.

Now assume $\partial_r u(p) = 0$. If there exists a sequence $p_i \in \Sigma$ with $\partial_r u(p_i) \neq 0$ such that $p_i \to p$, then by taking limit of what we have at p_i we obtain

$$\partial_r^m \left(b^{ij} u_j\right)\left(0^+, \theta(p)\right) = \partial_r^m \left(b^{ij} u_j\right)\left(0^-, \theta(p)\right)$$

for all m.

If $\partial_r u(q) = 0$ for $q \in \Sigma$ near p. Denote $u(0, \theta) = f(\theta)$. Using $\partial_r^2 = -u$ on both sides, we see $u(r, \theta) = f(\theta) \cos r$. It follows from $u_{0i} = 0$ that

$$\partial_r b_{ik} b^{kj} f_j = -2 f_i \tan r.$$

Hence

$$\partial_r \left(b^{ij} f_j\right) = -b^{ik} \partial_r b_{kl} b^{lj} f_j = -2 b^{ik} f_k \tan r.$$

Note that this is true for both positive and negative r. Hence

$$\left(b^{ij} f_j\right)(r, \theta) = \left(b^{ij} f_j\right)(0, \theta) \cos^2 r.$$

Hence $\left(b^{ij} u_j\right)(r, \theta) = \left(b^{ij} f_j\right)(0, \theta) \cos^3 r$ and it follows that $\partial_r^m \left(b^{ij} u_j\right)(0^+, \theta) = \partial_r^m \left(b^{ij} u_j\right)(0^-, \theta)$ for all m.

In any case we have proved that $\nabla u \in C^\infty(P)$. Let $X = \frac{\nabla u}{|\nabla u|}$, then X is C^∞ on $\{u \neq 1\}$. It follows from $D^2 u = -ug$ on both M and N that $\nabla_X X = 0$ on both M and N. For $p \in \Sigma$, let $F(p, t)$ be given by $\partial_t F(p, t) = X(F(p, t))$ and $F(p, 0) = p$. Then F is C^∞ as long as it is defined, besides when $F(p, t) \in M$ on a time interval, then it is a unit speed geodesic.

Let $\phi(t) = u(F(p, t))$, then $\phi(0) = 0$, $\phi'(t) = |\nabla u(F(p, t))|$, hence $\phi'(0) = 1$. Since $\phi^2 + \phi'^2 = 1$ and $\phi' \neq 0$, we see $\phi'' + \phi = 0$. Hence $u(F(p, t)) = \sin t$. It follows that $F : \Sigma_1 \times \left[0, \frac{\pi}{2}\right) \to \{u \neq 1\}$ is a diffeomorphism. If we choose a local coordinate $\theta_1, \cdots, \theta_{n-1}$ on Σ_1, then note that we have a coordinate $t, \theta_1, \cdots, \theta_{n-1}$ locally on P with $\partial_t = X$, hence $|\partial_t| = 1$ and $\langle \partial_t, \partial_i \rangle = 0$. It follows that

$$g = dt \otimes dt + b_{ij}(t, \theta)\, d\theta_i \otimes d\theta_j,$$

here b_{ij} is locally Lipschitz in (t,θ). If $(t_0,\theta_0)\notin\Sigma$, then we know near (t_0,θ_0), b_{ij} is smooth with $u=\sin t$. Hence

$$\begin{aligned} D^2u &= -\sin t\cdot dt\otimes dt+\frac{1}{2}\cot s\cdot\partial_t b_{ij}d\theta_i\otimes d\theta_j \\ &= -\sin t\cdot dt\otimes dt-\sin t\cdot b_{ij}\left(t,\theta\right)d\theta_i\otimes d\theta_j. \end{aligned}$$

Hence $\partial_t b_{ij}\left(t,\theta\right)=-2\tan t\cdot b_{ij}\left(t,\theta\right)$. For a.e. θ, $(t,\theta)\in\Sigma$ for only a set discrete t's, hence we have $b_{ij}\left(t,\theta\right)=b_{ij}\left(0,\theta\right)\cos^2 t$. By continuity we know this is true for all θ. Therefore g is smooth on $\{u\neq 1\}$. Since the set $\{u=1\}$ is finite, we see g is smooth everywhere (because by taking a limit we see there is no jump in any order of derivatives along Σ). Hence (P,g) must be isometric to the standard upper half sphere. □

We can now prove the following generalized version of Theorem 2.

Theorem 7. *Let (P^n,g) be a compact Riemannian manifold as in Theorem 6. Let $\Sigma_1=\partial P$. Suppose*

- *$Ric\geq(n-1)\,g$,*
- *$(\Sigma_1,g|_{\Sigma_1})$ is isometric to the standard sphere $\mathbb{S}^{n-1}\subset\mathbb{R}^n$,*
- *Σ is convex in M in the sense that its second fundamental form is nonnegative.*

Then (P^n,g) is smooth and isometric to the hemisphere $\mathbb{S}^n_+$.

We first explain how Theorem 3 follows from the above theorem. Let $N=\mathbb{S}^n_+\backslash\Omega$ and let P be the smooth manifold obtained by gluing M and N along Σ via the embedding $\iota:(\Sigma,g_\Sigma)\to\partial\overline{\Omega}$. Clearly P satisfies all the assumptions.We have a Riemannian metric on P which is merely Lipschitz along Σ. And P is spherical near its boundary $\mathbb{S}^{n-1}$. So we can apply Theorem 7.

Proof of Theorem 7. Proof of Theorem 7: By Theorem 6, $\lambda_1\left(P\right)\geq n$. If $\lambda_1\left(P\right)=n$ then we know (P,g) is smooth and is isometric to $\mathbb{S}^n_+$. Assume $\lambda_1\left(P\right)>n$, then we may find $u\in H^1\left(P\right)$ s.t.

$$\begin{cases} -\Delta u=nu & \text{on} \quad P, \\ u=f & \text{on} \quad \mathbb{S}^{n-1}, \end{cases}$$

here f is a linear function on $\mathbb{S}^{n-1}$. By elliptic regularity $u|_M\in C^\infty\left(M\right)$, $u|_N\in C^\infty\left(N\right)$ and $u\in C^1\left(P\right)$. Let $\phi=\left|\nabla u\right|^2+u^2$. We know that ϕ is subharmonic in both M and N. Moreover on Σ, let $\chi=\frac{\partial u}{\partial\nu}$, we have

$$\begin{aligned} \frac{1}{2}\frac{\partial\,\phi|_M}{\partial\nu} &= \langle\nabla_\Sigma u,\nabla_\Sigma\chi\rangle-\langle A_M\left(\nabla_\Sigma u\right),\nabla_\Sigma u\rangle+\chi\left[-\Delta_\Sigma u-\left(n-1\right)u\right]-H_M\chi^2, \\ \frac{1}{2}\frac{\partial\,\phi|_N}{\partial\nu} &= \langle\nabla_\Sigma u,\nabla_\Sigma\chi\rangle-\langle A_0\left(\nabla_\Sigma u\right),\nabla_\Sigma u\rangle+\chi\left[-\Delta_\Sigma u-\left(n-1\right)u\right]-H_0\chi^2. \end{aligned}$$

Hence $\frac{\partial \phi|_M}{\partial \nu} \leq \frac{\partial \phi|_N}{\partial \nu}$. Then for $\varphi \in C_c^\infty(P)$, $\varphi \geq 0$, we have

$$\begin{aligned}
&\int_P \langle \nabla \phi, \nabla \varphi \rangle \, d\mu \\
&= \int_M \langle \nabla \phi, \nabla \varphi \rangle \, d\mu + \int_N \langle \nabla \phi, \nabla \varphi \rangle \, d\mu \\
&= -\int_M \varphi \Delta \phi d\mu + \int_\Sigma \frac{\partial \phi|_M}{\partial \nu} \varphi dS - \int_N \varphi \Delta \phi d\mu - \int_\Sigma \frac{\partial \phi|_N}{\partial \nu} \varphi dS \\
&\leq 0.
\end{aligned}$$

That is ϕ is subharmonic on P in the distribution sense. Hence ϕ achieve a maximum on the boundary $\mathbb{S}^{n-1}$. At this maximum point, we have $\frac{\partial \phi}{\partial \nu} = 0$ by the same argument in Section 4. Hence ϕ must be equal to constant by the strong maximum principle and $D^2 u = -ug$ on both M and N, $u|_N$ =the linear function. We may assume $\frac{\partial u}{\partial \nu} = 0$ on $\mathbb{S}^{n-1}$. Hence $|\nabla u|^2 + u^2 = 1$ on P. It follows from this and $D^2 u = -ug$ on both M and N that $\{u = \pm 1\}$ is finite.

The same argument as before shows that u and ∇u belong to $C^\infty(P)$. Since $|\nabla u| = \sqrt{1 - u^2}$ we see it is smooth on $\{u \neq \pm 1\}$. Let $X = \frac{\nabla u}{|\nabla u|}$, then it generates a smooth flow on $\{u \neq \pm 1\}$, $F(p,t)$ with $\partial_t F(p,t) = X(F(p,t))$, $F(p,0) = p$. Note that here we have used the fact ∇u is tangent to $\mathbb{S}^{n-1}$ on $\mathbb{S}^{n-1}$.

If $p \in P$, $u(p) \neq \pm 1$, let $\phi(t) = u(F(p,t))$, then $\phi'(t) = |\nabla u(F(p,t))| > 0$. Hence $\phi'^2 + \phi^2 = 1$. After differentiation we get $\phi'' + \phi = 0$, hence $\phi(t) = \cos(t + b)$ for some $-\pi < b < 0$. It exists on $(-\pi - b, -b)$. Note that $|\partial_t F(p,t)| = 1$, we see $F(p,t) \to p_+$ as $t \to -b$ from the left and $F(p,t) \to p_-$ as $t \to -\pi - b$ from the right. In particular $u(p_+) = 1$ and $u(p_-) = -1$. It follows that each orbit must have length π and connecting some points with value -1 to another point with value 1.

For every q with $u(q) = 1$, we let

$$U_q = \cup \{\text{all orbits ending at } q\}.$$

Then it is clear that U_q is open and $\cup_{u(q)=1} U_q = P \backslash \{u = \pm 1\}$. It follows from connectivity of $P \backslash \{u = \pm 1\}$ that there is only one q with $u(q) = 1$. Similarly there is only one q with $u(q) = -1$. Let $p_+ \in \mathbb{S}^{n-1}$ with $u(p_+) = 1$, $p_- \in \mathbb{S}^{n-1}$ with $u(p_-) = -1$, then every orbit must start from p_- and end at p_+. Next calculation shows that in the interior of M and N, $D_X X = 0$, hence the orbits in the interior are simply the unit speed geodesics. Let r be the distance to p_-, then near p_- the metric

$$g = dr \otimes dr + \sin^2 r b_{ij}(\theta) \, d\theta_i \otimes d\theta_j.$$

Here $\theta_1, \cdots, \theta_{n-1}$ are local coordinates on $\mathbb{S}_+^{n-1}$ (viewed as in the tangent space of $\mathbb{S}_+^n$ at p_-). Moreover $u(r,\theta) = -\cos r$. We have a diffeomorphism

$$\mathbb{S}_+^{n-1} \times (0,\pi) \to P \backslash \{p_+, p_-\} : (\xi, t) \mapsto F\left(\exp_{p_-}(\varepsilon \xi), t - \varepsilon\right)$$

for some $\varepsilon > 0$ small. Hence we have a coordinate $t, \theta_1, \cdots, \theta_{n-1}$ on $P \backslash \{p_+, p_-\}$. Under this coordinate

$$g = dt \otimes dt + b_{ij}(t,\theta) \, d\theta_i \otimes d\theta_j.$$

It follows from previous calculation that $u(t,\theta) = -\cos t$. Note that if $(t,\theta) \notin \Sigma$, then it follows from $D^2 u = -ug$ that

$$\partial_t b_{ij}(t,\theta) = 2\cot t b_{ij}(t,\theta).$$

For *a.e.* θ, we know $(t,\theta) \notin \Sigma$ except finite many $t's$. Hence using $b_{ij}(t,\theta) = \sin^2 t b_{ij}(\theta)$ for t small we see $b_{ij}(t,\theta) = \sin^2 t b_{ij}(\theta)$ for all t. By continuity argument we see it is true everywhere. Hence g is smooth. The theorem follows. □

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