(1) Let \((M^n, g)\) be a Riemannian manifold. For \(p \in M\), pick \(\delta > 0\) s.t. \(\exp_p : B_\delta \to B(p, \delta)\) is a diffeomorphism, where 

\[
B_\delta = \{ v \in T_pM : |v| < \delta \}, B(p, \delta) = \{ q \in M : d(p, q) < \delta \}.
\]

Let \(\{e_i\}\) be an orthonormal basis for \(T_pM\) and \(\phi : \mathbb{R}^n \to T_pM\) the isometry \(\phi(x) = \sum_i x_i e_i\). We can introduce a local chart \(\Phi : B(p, \delta) \to \mathbb{R}^n\) by \(x = \Phi(q) = \phi^{-1} \circ \exp_p^{-1}(q)\). Write \(g = g_{ij}(x) dx_i \otimes dx_j\) in these coordinates.

Prove 
\[
g_{ij}(0) = \delta_{ij}, \Gamma^k_{ij}(0) = 0;\]
\[
g_{ij}(x) x_j = x_i.\]

(Hint: \(t \to (tx_1, \cdots, tx_n)\) is a geodesic.)

(2) Consider the conformal ball model of the hyperbolic space: \(B^n\) with \(g = 4/(1-|x|^2)^2\). Prove

\[
d(0, x) = \log \frac{1 + |x|}{1 - |x|}.
\]

(3) Let \(M^n\) be a smooth manifold with a pseudo-Riemannian metric \(g\), i.e. \(g\) is a symmetric 2-tensor which is nondegenerate at every point. (It is Riemannian if it is positive definite.) Prove that there is a unique connection \(\nabla\), called the Levi-Civita connection associated to \(g\), s.t.

- \(Zg(X, Y) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y)\);
- \(\nabla_X Y - \nabla_Y X = [X, Y]\).

(4) The Minkowski space \(\mathbb{R}^{n,1}\) is \(\mathbb{R}^{n+1}\) with the pseudo-Riemannian metric \(g_0 = dx_1 \otimes dx_1 + \cdots + dx_n \otimes dx_n - dx_{n+1} \otimes dx_{n+1}\). Recall that the hyperbolic space is

\[
\mathbb{H}^n = \{ \xi \in \mathbb{R}^{n,1} : g(\xi, \xi) = -1, \xi_{n+1} > 0 \}
\]

with the metric \(g_0|_{\mathbb{H}^n}\).

- Show that the Levi-Civita connection \(\nabla\) of \(g_0\) on \(\mathbb{R}^{n,1}\) is the ordinary differentiation on vector-valued functions.
- For \(X, Y \in \Gamma(T\mathbb{H}^n)\) we can decompose the induced connection \(\nabla_X Y\) into the tangential part and the normal part. Show that the tangential part is the Levi-Civita connection on \(\mathbb{H}^n\).
- Determine the normal part.
- Write down the geodesic equation in \(\mathbb{H}^n\) and solve it.
- For \(\xi, \eta \in \mathbb{H}^n\), derive a formula for the distance \(d(\xi, \eta)\) in \(\mathbb{H}^n\).

(The whole calculation is parallel to that for \(\mathbb{S}^n \subset \mathbb{R}^{n+1}\).)