In all the problems, we work on a Riemannian manifold \((M^n, g)\).

1. Let \(f \in C^\infty (M)\) and \(\gamma : [0, l] \rightarrow M\) a geodesic. Show that
\[
\frac{d^2}{dt^2} f \circ \gamma (t) = D^2 f \left( \dot{\gamma} (t), \ddot{\gamma} (t) \right).
\]

2. Let \(\Sigma^k\) be a smooth manifold and \(\phi : \Sigma^k \rightarrow M\) a smooth map. Let \(\Gamma (\phi^* TM)\) be the space of vector fields along \(\phi\). We defined the induced connection \(\bar{\nabla}\) on \(\Gamma (\phi^* TM)\) (vector fields along \(\phi\)) satisfying
- for any vector field \(V\) along \(\phi\) and any \(f \in C^\infty (\Sigma)\)
\[
\bar{\nabla} (fV) = df \otimes V + f\bar{\nabla}V.
\]
- for any vector fields \(V, W\) along \(\phi\) and for any \(\xi \in T\Sigma \xi \langle V, W \rangle = \langle \bar{\nabla}_\xi V, W \rangle + \langle V, \bar{\nabla}_\xi W \rangle.
\)
- for any \(X, Y \in \Gamma (T\Sigma)\)
\[
\bar{\nabla}_X (\phi_* Y) - \bar{\nabla}_Y (\phi_* X) = \phi_* [X, Y].
\]
Prove that for \(X, Y \in \Gamma (T\Sigma)\) and for any vector field \(V\) along \(\phi\)
\[
-\bar{\nabla}_X \bar{\nabla}_Y V + \bar{\nabla}_Y \bar{\nabla}_X V + \bar{\nabla}_{[X,Y]} V = R(\phi_* X, \phi_* Y) V,
\]
where \(R\) is the curvature tensor of \((M^n, g)\).

3. (Gauss equation) Suppose \(\Sigma^k \subset M\) is a submanifold. Recall that for \(X, Y \in \Gamma (T\Sigma)\) we have the orthogonal decomposition \(\bar{\nabla}_X Y = \nabla_X Y + \Pi (X, Y)\), where \(\nabla_X Y\) is the Levi-Civita connection of \((\Sigma, g|_{\Sigma})\) and \(\Pi\) is the second fundamental form of \(\Sigma\) in \(M\). Prove that for \(X, Y, Z, W \in \Gamma (T\Sigma)\)
\[
R(X, Y, Z, W) = R^\Sigma (X, Y, Z, W) - \langle \Pi (X, Z), \Pi (Y, W) \rangle + \langle \Pi (X, W), \Pi (Y, Z) \rangle.
\]
Calculate the curvature of \(S^n\) using the above formula.

4. Prove the hyperbolic space is complete. (You can use any of the equivalent models.)

5. Let \((M^n, g)\) be a complete Riemannian manifold and \(X\) a bounded vector field (i.e. there exists \(C > 0\) s.t. \(|X| \leq C\) on \(M\)). Prove that \(X\) generates a global flow, i.e. every integral curve of \(X\) is defined on \(\mathbb{R}\).