In all the problems, we work on a Riemannian manifold \((M^n, g)\).

1. Let \(X\) be a vector field. Its divergence is defined by \(\text{div} X = \text{tr} \nabla X\). For a smooth function \(f\) its gradient is the vector field \(\nabla f\) s.t. \(\langle \nabla f, \cdot \rangle = df\). The Laplace operator is defined by \(\Delta f = \text{div} \nabla f\). Prove the following formulas in local coordinates.
   - \(\nabla f = g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^j}\).
   - If we write \(X = a^i \frac{\partial}{\partial x^i}\), then
     \[\text{div} X = \frac{1}{\sqrt{G}} \frac{\partial}{\partial x^i} \left( \sqrt{G} a^i \right),\]
     where \(G = \det [g_{ij}]\).
   - \(\Delta f = \frac{1}{\sqrt{G}} \frac{\partial}{\partial x^i} \left( \sqrt{G} g^{ij} \frac{\partial f}{\partial x^j} \right)\).

2. Exercises 2.65 on P75.

3. Suppose \(M\) is oriented. Then there is a natural volume form \(\Omega\) on \(M\) s.t. for any positive orthonormal basis \(\{e_1, \cdots, e_n\}\) of \(T_p M\)
   \[\Omega(e_1, \cdots, e_n) = 1.\]
   - Prove that in a positive local chart
     \[\Omega = \sqrt{G} dx_1 \wedge \cdots \wedge dx_n.\]
   - Prove that for any \(C^1\) vector field \(X\)
     \[\text{div} X = diX \Omega/\Omega,\]

4. Consider the Riemannian metric \(g = f(r)^2 dx^2\) on some open set \(U \subset \mathbb{R}^n \setminus \{0\}\), where \(f\) is a smooth and positive function of \(r = \sqrt{x_1^2 + \cdots + x_n^2}\). Calculate the connection and curvature explicitly. Check your answer on the following examples: \(f = \frac{2}{1+r^2}\) (the sphere) and \(f = \frac{2}{1-r^2}\) (the hyperbolic space).

5. Suppose \(\phi : (M, g) \to (M', g')\) is an isometry. For vector fields \(X, Y, \cdots\) on \(M\), let \(X' = \phi_* X, Y' = \phi_* Y, \cdots\) be the corresponding vector fields on \(M'\). Prove
   - \(\phi_* (\nabla_X Y) = \nabla_{X'} Y'.\)
   - \(R'(X', Y', Z', W') = R(X, Y, Z, W) \circ \phi.\)