MTH 930 HOMEWORK ASSIGNMENT 1

DUE SEPT. 14 IN CLASS

(1) Let \((S^n, g_0)\) be the standard sphere. Let \(\phi : S^n \setminus \{N\} \to \mathbb{R}^n\) be the stereographic projection from the north pole \(N = (0, \cdots, 0, 1)\), i.e. for \(\xi \in S^n \setminus \{N\}\) the line joining \(N\) and \(\xi\) intersects the equator hyperplane \(\xi_{n+1} = 0\) at \(\phi(\xi)\).

- Find explicitly \(\phi\) and \(\phi^{-1}\).
- Show that \((\phi^{-1})^* g_0 = \frac{4}{(1 + |x|^2)^2} dx^2\).

(2) In class we defined the hyperbolic space as

\[
\mathbb{H}^n = \left\{ \xi \in \mathbb{R}^{n+1} : \xi_{n+1} = \sqrt{1 + \sum_{i=1}^{n} \xi_i^2} \right\}
\]

in the Minkowski space \(\mathbb{R}^{n,1}\). Define the stereographic projection \(\phi : \mathbb{H}^n \to B^n = \{x \in \mathbb{R}^n : |x| < 1\}\) as following: \(\phi(\xi)\) is the intersection point of the line joining \(\xi\) and \(-e_{n+1}\) with the hyperplane \(\xi_{n+1} = 0\).

- Find explicitly \(\phi\) and \(\phi^{-1}\) and show that \(\phi : \mathbb{H}^n \to B^n\) is a diffeomorphism.
- Show that \((\phi^{-1})^* g_0 = \frac{4}{(1 - |x|^2)^2} dx^2\).

\(B^n\) with the metric \(\frac{4}{(1 - |x|^2)^2} dx^2\) is called the conformal ball model of the hyperbolic space.

(3) Let \(\pi : E \to M\) be a vector bundle over a smooth manifold. Recall that a connection on \(E\) is a map

\[\nabla : \Gamma(M) \times \Gamma(E) \to \Gamma(E)\]

satisfying

- for any \(X_1, X_2 \in \Gamma(M), f_1, f_2 \in C^\infty(M)\) and \(\sigma \in \Gamma(E)\)

\[\nabla_{f_1 X_1 + f_2 X_2} \sigma = f_1 \nabla_{X_1} \sigma + f_2 \nabla_{X_2} \sigma.\]

- \(\nabla_X (f \sigma) = X f \sigma + f \nabla_X \sigma.\)
- \(\nabla_X (\sigma_1 + \sigma_2) = \nabla_X \sigma_1 + \nabla_X \sigma_2.\)

Prove

(a) If \(X\) vanishes at a point \(p\), so does \(\nabla_X \sigma\). Therefore for \(v \in T_p M, \sigma \in \Gamma(E)\), we can define \(\nabla_v \sigma \in E_p.\)
(b) If $\sigma$ vanishes in a neighborhood of $p$, then $\nabla_v \sigma = 0$ for any $v \in T_p M$. Therefore we can define $\nabla_v \sigma$ if $\sigma$ is only a smooth section on a neighborhood of $p$.

(4) Let $g$ be a Riemannian metric on $M$ and $\tilde{g} = e^{2f}g$ another metric conformal to $g$, where $f$ is a smooth function on $M$. Give the relation between the Levi-Civita connection $\nabla$ of $g$ and the Levi-Civita connection $\tilde{\nabla}$ of $\tilde{g}$. 