A NEW CHARACTERIZATION OF THE CR SPHERE AND THE SHARP EIGENVALUE ESTIMATE FOR THE KOHN LAPLACIAN

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1. Introduction

Let (M, θ) be a pseudohermitian manifold of dimension $2m+1$ and T the Reeb vector field. We always work with a local unitary frame $\{T_\alpha : \alpha = 1, \cdots, m\}$ for $T^{1,0}(M)$ and its dual frame $\{\theta^\alpha\}$. Thus

$$d\theta = \sqrt{-1} \sum_\alpha \theta^\alpha \wedge \overline{\theta^\alpha}. \tag{1.1}$$

We will often denote T by T_0. In [LW], the first and third named authors proved the following Obata type result in CR geometry.

Theorem 1. *Let M be a closed pseudohermitian manifold of dimension $2m+1 \geqslant 5$. Suppose there is a real-valued nonzero function $u \in C^\infty(M)$ satisfying*

$$\begin{aligned} u_{\alpha,\beta} &= 0, \\ u_{\alpha,\overline{\beta}} &= \left(-\frac{\kappa}{2(m+1)} u + \frac{\sqrt{-1}}{2} u_0 \right) \delta_{\alpha\beta}, \end{aligned}$$

for some constant $\kappa > 0$. Then M is equivalent to the sphere $\mathbb{S}^{2m+1}$ with its standard pseudohermitian structure, up to a scaling.

A weaker version of the above theorem is also proved in dimension 3 ($m = 1$) in [LW] which requires an additional condition.

In this paper we prove a variant of the above theorem which characterizes the CR sphere in terms of the existence of a (non-trivial) *complex-valued* function satisfying a certain overdetermined system. The precise statement is the following theorem.

Theorem 2. *Let (M, θ) be closed pseudohermitian manifold with dimension $2m+1 \geqslant 5$. Suppose that there exists a nonzero complex-valued function f on M satisfying*

$$\begin{aligned} f_{\overline{\alpha},\overline{\beta}} &= 0, \\ f_{\overline{\alpha},\beta} &= -cf\delta_{\alpha\beta} \end{aligned}$$

for some constant $c > 0$. Then (M, θ) is CR equivalent to $\mathbb{S}^{2m+1}$ with its standard pseudohermitian structure, up to a scaling.

Theorem 2 is motivated by the recent sharp lower bound for positive eigenvalues of the Kohn Laplacian by Chanillo, Chiu and Yang [CCY], just as Theorem 1 is motivated by Greenleaf's sharp estimate for the first eigenvalue of the sublaplacian Δ_b. Recall that the Kohn Laplacian on a (complex-valued) function f is defined by

$$\Box_b f = \overline{\partial}_b^* \overline{\partial}_b f = -f_{\overline{\alpha},\alpha}. \tag{1.2}$$

and its conjugate $\overline{\Box}_b f = -f_{\alpha,\overline{\alpha}} = \Box_b f - \sqrt{-1} m T f$. We have

$$-\Delta_b = \Box_b + \overline{\Box}_b = 2\Box_b + \sqrt{-1} m T = 2\overline{\Box}_b - \sqrt{-1} m T. \tag{1.3}$$

On a closed pseudohermitian manifold M, the Kohn Laplacian $\Box_b$ defines a non-negative self-adjoint operator on the Hilbert space $L^2(M)$ of all complex-valued functions f with $|f|^2$ is integrable on M, and the inner product on L^2 is defined by

$$\langle f_1, f_2 \rangle = \int_M f_1 \overline{f}_2. \tag{1.4}$$

But unlike Δ_b, it does not satisfy the Hörmander condition and, as a result, its resolvent is not compact. The three dimensional case is more complicated than higher dimensions. Nevertheless, it is proved by Burns and Epstein [BE] that the spectrum of $\Box_b$ in $(0,\infty)$ consists of point eigenvalues of finite multiplicity. In general, there may exist a sequence of eigenvalues rapidly decreasing to zero. Zero is an isolated eigenvalue iff the range of $\Box_b$ is closed.

Motivated by the embedding problem for 3-dimensional CR manifolds, Chanillo, Chiu and Yang [CCY] recently proved the following eigenvalue estimate for the Kohn Laplacian:

Theorem 3. *Let M be a closed 3-dimensional pseudohermitian manifold. If the Paneitz operator P_0 is non-negative and the scalar curvature $R \geqslant \kappa$, with κ being a positive constant, then any nonzero eigenvalue of $\Box_b$ satisfies*

$$\lambda \geqslant \frac{1}{2}\kappa.$$

Recall that the Paneitz operator $P_0 : C^\infty(M) \to C^\infty(M)$ on a closed pseudohermitian manifold is defined by

$$P_0 f = (P_\alpha f)_{,\overline{\alpha}} = f_{\overline{\gamma},\gamma\alpha\overline{\alpha}} + m\sqrt{-1}\left(A_{\alpha\beta} f_{\overline{\beta}}\right)_{,\overline{\alpha}}. \tag{1.5}$$

We say that P_0 is non-negative if for any $f \in C^\infty(M)$

$$\int_M \overline{f} P_0 f \geqslant 0. \tag{1.6}$$

Though Chanillo, Chiu and Yang only proved the eigenvalue estimate for 3-dimensional pseudohermitian manifolds, their argument can be easily generalized to higher dimensions. In fact, since the Paneitz operator P_0 is always non-negative on closed pseudohermitian manifolds of dimension $\geqslant 5$, the statement is even simpler (see Chang and Wu [CW]).

Theorem 4. *Let (M,θ) be a closed pseudohermitian manifold of dimension $2m+1$. Suppose for all $X \in T^{1,0}(M)$*

$$Ric(X,X) \geqslant \kappa |X|^2,$$

where κ is a positive constant. Then any nonzero eigenvalue of $\Box_b$ satisfies

$$\lambda \geqslant \frac{m}{m+1}\kappa.$$

Note that the estimate is sharp as equality holds on the sphere

$$\mathbb{S}^{2m+1} = \left\{ z \in \mathbb{C}^{m+1} : |z| = 1 \right\}$$

with the standard pseudohermitian structure

$$\theta_0 = \left(2\sqrt{-1}\,\overline{\partial}\,|z|^2 \right)|_{\mathbb{S}^{2m+1}}.$$

A natural question is whether the equality case characterizes the CR sphere with the standard pseudohermitian structure up to a scaling. In their preprint [CW] Chang and Wu studied this problem and proved various partial results. One of them states that M is indeed equivalent to the CR sphere $\mathbb{S}^{2m+1}$ if equality holds in Theorem 4, provided that the following identity

$$\int_M A_{\overline{\alpha}\overline{\beta}} f_\alpha \overline{f}_\beta = 0 \tag{1.7}$$

is satisfied for a corresponding eigenfunction f.

As a corollary of Theorem 2, we can resolve this question in the general case. Namely, we have the following rigidity result.

Corollary 1. *If equality holds in Theorem 4, then (M, θ) is equivalent to the CR sphere $\left(\mathbb{S}^{2m+1}, \theta_0\right)$, up to a scaling, i.e. there exists a CR diffeomorphism $F : M \to \mathbb{S}^{2m+1}$ such that $F^*\theta_0 = c\theta$ for some constant $c > 0$.*

We expect that a similar version of Theorem 2 is true in dimension 3 from which the characterization of the equality case in Theorem 3 would follow. But we have not been able to prove it yet. This is due to an additional difficulty that arises only in 3-dimensional case: It is not clear when functions annihilated by the CR Paneitz operator P_0 are CR-pluriharmonic.

Another remark is that despite the similarity between these theorems and Obata's theorem in Riemannian geometry, the proofs are essentially different due to the torsion of the Tanaka-Webster connection. In fact, a crucial step in the proof of Theorem 2 is to show that the torsion must vanish. But unlike the approach in [LW], where the vanishing of torsion was deduced from estimates regarding high powers of the real-valued function u (or an eigenfunction of Δ_b), the vanishing of the torsion in our proof of Theorem 2 is proved by deriving various identities that are satisfied simultaneously only if the torsion is zero.

The paper is organized as follows. In Section 2, we review some basic facts in CR geometry and discuss the Bochner formula for the Kohn Laplacian. In Section 3 we discuss the spectral theory of the Kohn Laplacian. In Section 4 we discuss the eigenvalue estimate of Chanillo, Chiu and Yang and its generalization to higher dimensions. We deduce Corollary 1 from Theorem 2. The proof of Theorem 2 is then presented in Section 5.

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2. Preliminaries

We first review some basic facts in CR geometry. Define the operator $P : C^\infty(M) \to \mathcal{A}^{1,0}(M)$ by

$$Pf = \left(f_{\overline{\gamma},\gamma\alpha} + m\sqrt{-1} A_{\alpha\beta} f_{\overline{\beta}} \right) \theta^\alpha.$$

We write $P_\alpha f = \left(f_{\overline{\gamma},\gamma\alpha} + m\sqrt{-1}A_{\alpha\beta}f_{\overline{\beta}}\right)$. We also write

$$B_{\alpha\overline{\beta}}f = f_{\alpha,\overline{\beta}} - \frac{1}{m}f_{\gamma,\overline{\gamma}}\delta_{\alpha\beta}.$$

The Paneitz operator $P_0 : C^\infty(M) \to C^\infty(M)$ is defined by

$$P_0 f = (P_\alpha f)_{,\overline{\alpha}} = f_{\overline{\gamma},\gamma\alpha\overline{\alpha}} + m\sqrt{-1}\left(A_{\alpha\beta}f_{\overline{\beta}}\right)_{,\overline{\alpha}}.$$

Graham-Lee [GL] proved that P_0 is a real operator. Moreover, P_0 is symmetric, i.e. for $f_1, f_2 \in C^\infty(M)$ with one of them compactly supported

$$\int_M P_0 f_1 \overline{f_2} = \int_M f_1 \overline{P_0 f_2}.$$

Here, the integrals are taken with respect to the volume form $dV = \theta \wedge (d\theta)^m$. They also proved the following identity when M is closed:

$$\int_M |B_{\overline{\alpha},\beta}f|^2 = \int_M \left|B_{\alpha,\overline{\beta}}\overline{f}\right|^2 = \frac{m-1}{m}\int_M \overline{f}P_0 f. \tag{2.1}$$

Therefore, on a closed pseudohermitian manifold of dimension $2m+1 \geqslant 5$, the Paneitz operator is nonnegative, in the sense that for any complex-valued function f, it holds that

$$\int_M \overline{f}P_0 f \geqslant 0.$$

But in dimension 3, there are closed pseudohermitian manifolds whose Paneitz operator is NOT nonnegative.

The following Bochner-type formula for the Kohn Laplacian was established by Chanillo, Chiu and Yang [CCY] (see also Chang and Wu [CW]).

Proposition 1. *Let f be a complex-valued function. Then*

$$\begin{aligned}-\Box_b\left|\overline{\partial}f\right|^2 = &\left(\left|f_{\overline{\alpha},\overline{\beta}}\right|^2 + |f_{\overline{\alpha},\beta}|^2\right) - \frac{m+1}{m}(\Box_b f)_{\overline{\alpha}}\overline{f}_\alpha - \frac{1}{m}f_{\overline{\alpha}}\overline{(\Box_b f)}_\alpha \\ &+ R_{\alpha\overline{\beta}}f_{\overline{\alpha}}\overline{f}_\beta - \frac{1}{m}\overline{f}_\alpha\overline{P_\alpha\overline{f}} + \frac{m-1}{m}f_{\overline{\alpha}}\left(P_\alpha\overline{f}\right).\end{aligned}$$

Integrating over a closed M yields

$$\begin{aligned}0 &= \int_M \left(\left|f_{\overline{\alpha},\overline{\beta}}\right|^2 + |f_{\overline{\alpha},\beta}|^2\right) - \frac{m+1}{m}\int_M (\Box_b f)_{\overline{\alpha}}\overline{f}_\alpha - \frac{1}{m}\int_M f_{\overline{\alpha}}\overline{(\Box_b f)}_\alpha \\ &+ \int_M R_{\alpha\overline{\beta}}f_{\overline{\alpha}}\overline{f}_\beta - \frac{1}{m}\overline{f}_\alpha\overline{P_\alpha\overline{f}} + \frac{m-1}{m}f_{\overline{\alpha}}\left(P_\alpha\overline{f}\right) \\ &= \int_M \left(\left|f_{\overline{\alpha},\overline{\beta}}\right|^2 + |B_{\overline{\alpha},\beta}f|^2\right) + \frac{1}{m}\int_M |\Box_b f|^2 - \frac{m+1}{m}\int_M (\Box_b f)_{\overline{\alpha}}\overline{f}_\alpha - \frac{1}{m}\int_M f_{\overline{\alpha}}\overline{(\Box_b f)}_\alpha \\ &+ \int_M R_{\alpha\overline{\beta}}f_{\overline{\alpha}}\overline{f}_\beta + \frac{m-2}{m}\int_M f_{\overline{\alpha}}\left(P_\alpha\overline{f}\right) \\ &= \int_M \left(\left|f_{\overline{\alpha},\overline{\beta}}\right|^2 + |B_{\overline{\alpha},\beta}f|^2\right) - \frac{m+1}{m}\int_M |\Box_b f|^2 \\ &+ \int_M R_{\alpha\overline{\beta}}f_{\overline{\alpha}}\overline{f}_\beta + \frac{m-2}{m}\int_M f_{\overline{\alpha}}\left(P_\alpha\overline{f}\right)\end{aligned}$$

where we have used the decomposition $f_{\overline{\alpha},\beta} = B_{\overline{\alpha},\beta} f + \frac{1}{m}\Box_b f \delta_{\alpha\beta}$. Therefore, we have

$$\frac{m+1}{m}\int_M |\Box_b f|^2 = \int_M \left|f_{\overline{\alpha},\overline{\beta}}\right|^2 + |B_{\overline{\alpha},\beta} f|^2 + R_{\alpha\overline{\beta}} f_{\overline{\alpha}} \overline{f}_\beta + \frac{m-2}{m}\int_M f_{\overline{\alpha}}\left(P_\alpha \overline{f}\right). \tag{2.2}$$

Integrating by parts yields

$$\begin{aligned}\int_M f_{\overline{\alpha}}\left(P_\alpha \overline{f}\right) &= -\int_M f\left(P_\alpha \overline{f}\right)_{,\overline{\alpha}} \\ &= -\int_M f P_0 \overline{f} \\ &= -\int_M \overline{f} P_0 f,\end{aligned}$$

In the last step, we have used the fact that P_0 is a real operator. Plugging this identity and (2.1) into (2.2) yields

Proposition 2. *Let M be a closed pseudohermitian manifold of dimension $2m+1$ and f a complex function on M. Then*

$$\frac{m+1}{m}\int_M |\Box_b f|^2 = \int_M \left|f_{\overline{\alpha},\overline{\beta}}\right|^2 + \int_M R_{\alpha\overline{\beta}} f_{\overline{\alpha}} \overline{f}_\beta + \frac{1}{m}\int_M \overline{f} P_0 f. \tag{2.3}$$

3. The spectral theory of the Kohn Laplacian

Based on the work of Beals and Greiner [BG], Burn and Epstein [BE] proved the following theorem in dimension 3.

Theorem 5. *Let M be a closed pseudohermitian manifold of dimension 3. The* $\operatorname{spec}(\Box_b)$ *in* $(0,\infty)$ *consists of point eigenvalues of finite multiplicity. Moreover all these eigenfunctions are smooth.*

The spectral theory of the Kohn Laplacian in the higher dimensional case is in fact simpler. This is because the Hodge theory for $(0,1)$-forms is valid for all closed pseudohermitian manifold of dimension $2m+1 \geqslant 5$ by the fundamental work of Kohn [K]. The spectral theory for the Kohn Laplacian can then be deduced from the Hodge decomposition theorem for $(0,1)$-forms. This is known to the experts. But since it is not easily accessible in the literature, we give a detailed presentation, using the Bochner formula as a short cut. For background and a detailed exposition of the Kohn theory we refer to the book [CS] by Chen and Shaw, which is our primary source.

Proposition 3. *Let M be a closed pseudohermitian manifold of dimension $2m+1 \geqslant 5$ and f a complex function on M. Then*

$$\frac{1}{m(m-1)}\int_M |\Box_b f|^2 = \int_M \left|f_{\overline{\alpha},\overline{\beta}}\right|^2 + \frac{1}{m-1}\left|f_{\overline{\alpha},\beta}\right|^2 + \int_M R_{\sigma\overline{\alpha}} f_{\overline{\sigma}} \overline{f}_\alpha.$$

Proof. When $m \geqslant 2$, we have by (2.1)

$$\begin{aligned}\frac{1}{m}\int_M \overline{f} P_0 f &= \frac{1}{m-1}\int_M |B_{\overline{\alpha},\beta} f|^2 \\ &= \frac{1}{m-1}\int_M \left(|f_{\overline{\alpha},\beta}|^2 - \frac{1}{m}|\Box_b f|^2\right).\end{aligned}$$

Plugging this identity into (2.3) yields the desired identity. ∎

Throughout the rest of this section, we assume $m \geqslant 2$. We have from the previous Proposition

$$\|\Box_b f\|^2 \geqslant m\,(m-1)\left\|f_{\overline{\alpha},\overline{\beta}}\right\|^2 + m\,\|f_{\overline{\alpha},\beta}\|^2 - C\left\|\overline{\partial}_b f\right\|^2, \tag{3.1}$$

where $C \geqslant 0$ depends on the pseudohermitian Ricci tensor.

For $f \in Dom\,(\Box_b)$ we have

$$\left\|\overline{\partial}_b f\right\|^2 = \langle \Box_b f, f\rangle \leqslant \|f\|\,\|\Box_b f\|.$$

We will use this inequality implicitly.

Let $\mathcal{H}$ denote the space of L^2 CR holomorphic functions, i.e.

$$\mathcal{H} = \left\{f \in L^2(M) : \overline{\partial}_b f = 0\right\}.$$

Proposition 4. *The range of $\overline{\partial}_b : L^2(M) \to L^2_{(0,1)}(M)$ is closed and more precisely $R\left(\overline{\partial}_b\right) = \overline{\partial}_b\overline{\partial}_b^*\left(Dom\left(\Box_b^{0,1}\right)\right)$. Moreover, for all $\beta \in R\left(\overline{\partial}_b\right)$, there exists a unique $f \in \mathcal{H}^{\perp} \cap Dom\left(\overline{\partial}_b\right)$ such that $\overline{\partial}_b f = \beta$ and*

$$\|f\| \leqslant C\,\|\beta\|.$$

Proof. The first part is Corollary 8.4.11 in [CS]. Suppose $\beta = \overline{\partial}_b u$. By the Hodge decomposition for $\overline{\partial}_b$ on $L^2_{(0,1)}(M)$ (Theorem 8.4.10 in [CS]) there exists $\alpha \in L^2_{(0,1)}(M)$ satisfying $\overline{\partial}_b^*\overline{\partial}_b\alpha = 0$ and

$$\beta = \overline{\partial}_b\overline{\partial}_b^*\alpha.$$

Moreover $\|\alpha\|_1 \leqslant C\,\|\beta\|$. It is easy to check that $f := \overline{\partial}_b^*\alpha$ has all the desired properties. ☐

Proposition 5. *The range of $\Box_b : L^2(M) \to L^2(M)$ is closed and more precisely $R(\Box_b) = \mathcal{H}^{\perp}$. Moreover, for all $\phi \in \mathcal{H}^{\perp}$, there exists a unique $f \in \mathcal{H}^{\perp} \cap Dom\,(\Box_b)$ such that $\Box_b f = \phi$ and*

$$\|f\| \leqslant C\left\|\overline{\partial}_b f\right\|. \tag{3.2}$$

Proof. Clearly $R(\Box_b) \subset \mathcal{H}^{\perp}$. Suppose $\phi = \Box_b u \in R(\Box_b)$. By Proposition 4, there is a unique $f \in \mathcal{H}^{\perp}$ such that $\overline{\partial}_b f = \overline{\partial}_b u$ and $\|f\| \leqslant C\left\|\overline{\partial}_b f\right\|$. Then $\Box_b f = \Box_b u = \phi$. We now prove that $R(\Box_b)$ is closed. Suppose $\phi_k = \Box_b f_k \to \phi$ in $L^2_{(0,1)}(M)$, with each $f_k \in \mathcal{H}^{\perp}$. Then for $k < l$, we have $\Box_b(f_k - f_l) = \phi_k - \phi_l$. Thus,

$$\begin{aligned}\left\|\overline{\partial}_b f_k - \overline{\partial}_b f_l\right\|^2 &\leqslant \|\phi_k - \phi_l\|\,\|f_k - f_l\| \\ &\leqslant C\,\|\phi_k - \phi_l\|\left\|\overline{\partial}_b f - \overline{\partial}_b f_l\right\|.\end{aligned}$$

It follows that $\left\|\overline{\partial}_b f_k - \overline{\partial}_b f_l\right\| \leqslant C\,\|\phi_k - \phi_l\|$. Applying (3.2) again yields $\|f_k - f_l\| \leqslant C^2\,\|\phi_k - \phi_l\|$. Therefore, $\{f_k\}$ is Cauchy in $L^2(M)$. Denote its limit by f. Then $f_k \to f$ and $\Box_b f_k \to \phi$. As $\Box_b$ is a closed operator, we conclude that $f \in Dom\,(\Box_b)$ and $\Box_b f = \phi$. Therefore, $R(\Box_b)$ is closed.

If $R(\Box_b)$ was not the entire $\mathcal{H}^{\perp}$, then there exists a nonzero $\phi \in \mathcal{H}^{\perp}$ that is perpendicular to $R(\Box_b)$. This implies $\phi \in \mathcal{H}$, obviously a contradiction. ☐

Therefore, the operator $\Box_b : \mathcal{H}^\perp \cap Dom\,(\Box_b) \to \mathcal{H}^\perp$ is bijective. The inverse operator exists and is denoted by $T : \mathcal{H}^\perp \to \mathcal{H}^\perp$. Namely, for each $\phi \in \mathcal{H}^\perp$, we define $f = T(\phi) \in \mathcal{H}^\perp$ to be the unique solution to $\Box_b f = \phi$ (which exists and unique by Proposition 5).

Theorem 6. *The operator $T : \mathcal{H}^\perp \to \mathcal{H}^\perp$ is compact.*

Proof. By (3.1) there exists a constant $C > 0$ such that for any $f \in C^\infty(M)$

$$\left\| f_{\overline{\alpha},\overline{\beta}} \right\|^2 + \|f_{\overline{\alpha},\beta}\|^2 \leqslant C\left(\left\|\overline{\partial}_b f\right\|^2 + \|\Box_b f\|^2 \right).$$

It follows, by the Hörmander estimate (see Theorem 8.2.5 in [CS]), that

$$\left\|\overline{\partial}_b f\right\|_{1/2}^2 \leqslant C\left(\left\|\overline{\partial}_b f\right\|^2 + \|\Box_b f\|^2 \right), \tag{3.3}$$

where $\|\cdot\|_{1/2}$ is the norm for the Sobolev space $W^{1/2}_{(0,1)}(M)$. By approximation, this inequality holds for all $f \in Dom\,(\Box_b)$. We can further assume that $f \in \mathcal{H}^\perp$.

Suppose $\{f_k\} \subset \mathcal{H}^\perp \cap Dom\,(\Box_b)$ is a sequence such that $\phi_k = \Box_b f_k$ are bounded in $L^2(M)$. By (3.3), $\overline{\partial}_b f_k$ are bounded in $W^{1/2}_{(0,1)}(M)$. By the Sobolev embedding theorem and passing to a subsequence, we can assume that $\left\{\overline{\partial}_b f_k\right\}$ is Cauchy in $L^2_{(0,1)}(M)$. By (3.2), $\{f_k\}$ is Cauchy in $L^2(M)$. The proof is complete. $\Box$

Theorem 7. *The* $\operatorname{spec}(\Box_b)$ *consists of countably many eigenvalues $\lambda_0 = 0 < \lambda_1 < \lambda_2 < \cdots$ with $\lambda_i \to \infty$ as $i \to \infty$. Moreover, for $i \geqslant 1$, each λ_i is an eigenvalue of finite multiplicity and all the corresponding eigenfunctions are smooth.*

Proof. We have proved, in Proposition, 5 that $R(\Box_b)$ is closed. Thus, $\lambda_0 = 0$ is an eigenvalue whose corresponding eigenspace is $\mathcal{H}$, which is of infinite dimensional. With respect to the orthogonal decomposition $L^2(M) = \mathcal{H} \oplus \mathcal{H}^\perp$, the operator $\lambda I - \Box_b$ is given by the following matrix

$$\begin{bmatrix} \lambda I & 0 \\ 0 & \lambda I - \Box_b|_{\mathcal{H}^\perp} \end{bmatrix}.$$

Therefore, $\lambda > 0$ is in $\operatorname{spec}(\Box_b)$ if and only if λ^{-1} is in $\operatorname{spec}(T)$. As T is compact, $\operatorname{spec}(T) \cap (0, +\infty)$ consists of countably many eigenvalues of finite multiplicities $\mu_1 > \mu_2 > \cdots$ with $\lim \mu_i = 0$. Therefore, $\operatorname{spec}(\Box_b)$ consists of $\lambda_0 = 0 < \lambda_1 < \lambda_2 < \cdots$ with $\lambda_i = 1/\mu_i$ for all $i \geqslant 1$ and the eigenspace of $\Box_b$ corresponding to λ_i equals the eigenspace of T corresponding to μ_i.

Suppose f is an eigenfunction corresponding to an eigenvalue $\lambda > 0$, i.e. $\Box_b f = \lambda f$. Then the $(0,1)$-form $\beta = \overline{\partial}_b f$ satisfies

$$\begin{aligned} \Box_b \beta &= \overline{\partial}_b \Box_b f \\ &= \lambda \overline{\partial}_b f \\ &= \lambda \beta. \end{aligned}$$

By the Hodge theory for $(0,1)$-forms, β is smooth. As $f = \frac{1}{\lambda}\overline{\partial}_b^* \beta$, we see that f is smooth. $\Box$

4. The eigenvalue estimate

With the spectral theory of $\Box_b$ understood, we can now state the following

Theorem 8. *Let M be a closed pseudohermitian manifold of dimension $2m+1$. When $m = 1$, we further assume that the Paneitz operator is non-negative. Suppose*

$$Ric\,(X, X) \geqslant \kappa\,|X|^2\,,$$

where κ is a positive constant. Then any nonzero eigenvalue of $\Box_b$ satisfies

$$\lambda \geqslant \frac{m}{m+1}\kappa.$$

Proof. Suppose f is a nonzero eigenfunction with eigenvalue $\lambda > 0$. By (2.3), under the assumptions

$$\begin{aligned}
\frac{m+1}{m}\lambda^2 \int_M |f|^2 &= \int_M \left|f_{\overline{\alpha},\overline{\beta}}\right|^2 + \int_M R_{\alpha\overline{\beta}} f_{\overline{\alpha}} \overline{f}_{\beta} + \frac{1}{m}\int_M \overline{f} P_0 f \\
&\geqslant \kappa \int_M \left|\overline{\partial} f\right|^2 \\
&= \kappa \int_M \overline{f}\Box_b f \\
&= \lambda\kappa \int_M |f|^2 .
\end{aligned}$$

Thus, $\lambda \geqslant \frac{m}{m+1}\kappa$. ∎

Theorem 8 was first proved by Chanillo, Chiu and Yang [CCY] in the case $m = 1$. We have followed basically the same argument (see also Chang and Wu [CW]).

Proposition 6. *Suppose $\lambda = \frac{m}{m+1}\kappa$ in Theorem 8 and f a corresponding eigenfunction. Then we must have:*

(i) *If $m = 1$, then*

$$f_{\overline{1},\overline{1}} = 0, \quad f_{\overline{1},1} = -\frac{\kappa}{2} f, \quad P_0 f = 0;$$

(ii) *If $m \geqslant 2$, then*

$$f_{\overline{\alpha},\overline{\beta}} = 0, \quad f_{\overline{\alpha},\beta} = -\frac{\kappa}{m+1} f\delta_{\alpha\beta}.$$

Proof. If equality holds, by inspecting the proof of Theorem 8, we must have $f_{\overline{\alpha},\overline{\beta}} = 0$ and $\int_M \overline{f} P_0 f = 0$. As P_0 is nonnegative, it follows easily that $P_0 f = 0$. As P_0 is a real operator, we also have $P_0 \overline{f} = 0$. When $m \geqslant 2$, this implies by (2.1) that

$$\begin{aligned}
f_{\overline{\alpha},\beta} &= -\frac{1}{m}\Box_b f \delta_{\alpha\beta} \\
&= -\frac{\kappa}{m+1} f\delta_{\alpha\beta}.
\end{aligned}$$

∎

Combining this Proposition and Theorem 2, we immediately obtain the following

Corollary 2. *Suppose $\lambda = \frac{m}{m+1}\kappa$ in Theorem 8 and $m \geqslant 2$. Then (M, θ) is CR equivalent to $\mathbb{S}^{2m+1}$ with its standard pseudohermitian structure, up to scaling.*

5. Proof of the Main Theorem

We now prove our main theorem (Theorem 2). By scaling, we may assume $c = 1/2$. Theorem 2 is equivalent to the following

Theorem 9. *Let (M, θ) be closed pseudohermitian manifold with dimension $2m + 1 \geqslant 5$. Suppose that there exists a nonzero complex function f on M satisfying*

$$f_{\overline{\alpha},\overline{\beta}} = 0,$$
$$f_{\overline{\alpha},\beta} = -\frac{1}{2} f \delta_{\alpha\beta}$$

Then (M, θ) is CR equivalent to the $\mathbb{S}^{2m+1}$ with its standard pseudohermitian structure.

Therefore we have

$$f_{\overline{\alpha},\overline{\beta}} = 0, \tag{5.1}$$

$$f_{\overline{\alpha},\beta} = -\frac{1}{2} f \delta_{\alpha\beta}. \tag{5.2}$$

Using (5.1) and (5.2) it is easy to derive

$$\left(\left|\overline{\partial}_b f\right|^2\right)_\alpha = -\frac{1}{2} f \overline{f}_\alpha. \tag{5.3}$$

Proposition 7. *We have*

$$A_{\alpha\beta} \overline{f}_\gamma = A_{\alpha\gamma} \overline{f}_\beta, \tag{5.4}$$

$$f_\gamma = 2\sqrt{-1} A_{\gamma\sigma} f_{\overline{\sigma}}. \tag{5.5}$$

Proof. Differentiating (5.1) yields

$$A_{\alpha\beta} \overline{f}_\gamma = A_{\alpha\gamma} \overline{f}_\beta.$$

Differentiating (5.2) yields

$$\begin{aligned} -\delta_{\alpha\beta} f_\gamma / 2 &= f_{\overline{\alpha},\beta\gamma} \\ &= f_{\overline{\alpha},\gamma\beta} - \sqrt{-1} \left(\delta_{\alpha\beta} A_{\gamma\sigma} - \delta_{\alpha\gamma} A_{\beta\sigma}\right) f_{\overline{\sigma}} \\ &= -\delta_{\alpha\gamma} f_\beta / 2 - \sqrt{-1} \left(\delta_{\alpha\beta} A_{\gamma\sigma} - \delta_{\alpha\gamma} A_{\beta\sigma}\right) f_{\overline{\sigma}}. \end{aligned}$$

Hence

$$\delta_{\alpha\beta} \left(f_\gamma / 2 - \sqrt{-1} A_{\gamma\sigma} f_{\overline{\sigma}}\right) = \delta_{\alpha\gamma} \left(f_\beta / 2 - \sqrt{-1} A_{\beta\sigma} f_{\overline{\sigma}}\right).$$

Therefore $f_\gamma / 2 - \sqrt{-1} A_{\gamma\sigma} f_{\overline{\sigma}} = 0$. □

Let $Q = \sqrt{-1} A_{\alpha\beta} f_{\overline{\alpha}} f_{\overline{\beta}}$. Set

$$K = \left\{p \in M : \overline{\partial}_b f(p) = 0\right\}.$$

On $M \backslash K$ we define

$$\psi = 2Q / \left|\overline{\partial}_b f\right|^2.$$

Note that ψ is smooth and bounded on $M \backslash K$.

Lemma 1. *$M \backslash K$ is open and dense.*

Proof. We need to prove that K has no interior point. Write $f = u + \sqrt{-1}v$ with u and v real. Then, using (5.2)

$$\begin{aligned} u_{\alpha,\overline{\beta}} &= \frac{1}{2}\left(f_{\alpha,\overline{\beta}} + \overline{f}_{\alpha,\overline{\beta}}\right) \\ &= \frac{1}{2}\left(f_{\overline{\beta},\alpha} - \sqrt{-1}f_0\delta_{\alpha\beta} + \overline{f}_{\alpha,\overline{\beta}}\right) \\ &= -\frac{1}{4}\left(f + 2\sqrt{-1}f_0 + \overline{f}\right)\delta_{\alpha\beta}. \end{aligned}$$

Therefore, u is CR pluriharmonic. Similarly, v is also CR pluriharmonic.

Now suppose $\overline{\partial}_b f = 0$ on a connected open set U. By (5.2) $f = 0$ on U. Hence, u and v both vanish on U. Being CR pluriharmonic, u and v then must be identically zero on M. This is a contradiction. □

Proposition 8. *On $M\backslash K$ we have*

$$A_{\alpha\beta} = -\frac{\sqrt{-1}\psi}{2\left|\overline{\partial}_b f\right|^2}\overline{f}_\alpha\overline{f}_\beta, \tag{5.6}$$

$$f_\gamma = \psi\overline{f}_\gamma. \tag{5.7}$$

Proof. Using (5.4), we compute

$$\begin{aligned} A_{\alpha\beta}\left|\overline{\partial}_b f\right|^2 &= A_{\alpha\gamma}\overline{f}_\beta f_{\overline{\gamma}} \\ &= A_{\gamma\alpha}\left|\overline{\partial}_b f\right|^2\frac{\overline{f}_\beta f_{\overline{\gamma}}}{\left|\overline{\partial}_b f\right|^2} \\ &= A_{\gamma\sigma}\overline{f}_\alpha f_{\overline{\sigma}}\frac{\overline{f}_\beta f_{\overline{\gamma}}}{\left|\overline{\partial}_b f\right|^2} \\ &= -\frac{\sqrt{-1}Q}{\left|\overline{\partial}_b f\right|^2}\overline{f}_\alpha\overline{f}_\beta \\ &= -\sqrt{-1}\frac{\psi}{2}\overline{f}_\alpha\overline{f}_\beta. \end{aligned}$$

This proves (5.6). To prove (5.7), we compute using (5.5) and (5.6)

$$\begin{aligned} f_\gamma &= 2\sqrt{-1}A_{\gamma\sigma}f_{\overline{\sigma}} \\ &= \frac{\psi}{\left|\overline{\partial}_b f\right|^2}\overline{f}_\gamma\overline{f}_\sigma f_{\overline{\sigma}} \\ &= \psi\overline{f}_\gamma. \end{aligned}$$

□

Remark 1. *From (5.6), we obtain on $M\backslash K$*

$$\begin{aligned} |A|^2 &:= \sum_{\alpha,\beta}|A_{\alpha\beta}|^2 \\ &= \frac{1}{4}|\psi|^2. \end{aligned}$$

Therefore, $|\psi|^2$ extends smoothly to the entire M.

Proposition 9. *On $M\backslash K$ we have*

$$\overline{\partial}_b\psi = 0 \tag{5.8}$$

and

$$\sqrt{-1}f_0 - \frac{1}{2}f + \frac{1}{2}\psi\overline{f} = 0. \tag{5.9}$$

Proof. Differentiating $f_\alpha = \psi\overline{f}_\alpha$ and using (5.2) yields

$$\begin{aligned} f_{\alpha,\overline{\beta}} &= \psi_{\overline{\beta}}\overline{f}_\alpha + \psi\overline{f}_{\alpha,\overline{\beta}} \\ &= \psi_{\overline{\beta}}\overline{f}_\alpha - \frac{1}{2}\psi\overline{f}\delta_{\alpha\beta}. \end{aligned}$$

By using (5.2) again, we further compute the left hand side

$$\begin{aligned} f_{\alpha,\overline{\beta}} &= f_{\overline{\beta},\alpha} + \sqrt{-1}f_0\delta_{\alpha\beta} \\ &= -\frac{1}{2}f\delta_{\alpha\beta} + \sqrt{-1}f_0\delta_{\alpha\beta}. \end{aligned}$$

Therefore,

$$\psi_{\overline{\beta}}\overline{f}_\alpha = \left(\sqrt{-1}f_0 - \frac{1}{2}f + \frac{1}{2}\psi\overline{f}\right)\delta_{\alpha\beta}.$$

From this, it follows easily (since $m \geqslant 2$) that $\sqrt{-1}f_0 - f/2 + \psi\overline{f}/2 = 0$ and $\psi_{\overline{\beta}} = 0$. $\square$

Proposition 10. *We have*

$$R_{\alpha\overline{\beta}}f_{\overline{\alpha}} = \frac{m+1}{2}f_{\overline{\beta}}. \tag{5.10}$$

Proof. Using (5.1) and (5.2), we compute

$$\begin{aligned} 0 &= \overline{f}_{\alpha,\beta\overline{\gamma}} \\ &= \overline{f}_{\alpha,\overline{\gamma}\beta} + \sqrt{-1}\delta_{\beta\gamma}\overline{f}_{\alpha,0} - R_{\beta\overline{\gamma}\alpha\overline{\sigma}}\overline{f}_\sigma \\ &= -\frac{1}{2}\overline{f}_\beta\delta_{\alpha\gamma} + \sqrt{-1}\delta_{\beta\gamma}\left(\overline{f}_{0,\alpha} - A_{\alpha\sigma}\overline{f}_{\overline{\sigma}}\right) - R_{\beta\overline{\gamma}\alpha\overline{\sigma}}\overline{f}_\sigma. \end{aligned}$$

Differentiating (5.9) and using (5.8) and (5.7) yields

$$\begin{aligned} \sqrt{-1}\overline{f}_{0,\alpha} &= \frac{1}{2}\left(\overline{\psi}f_\alpha - \overline{f}_\alpha\right) \\ &= \frac{1}{2}\left(|\psi|^2 - 1\right)\overline{f}_\alpha. \end{aligned}$$

Plugging this into the previous equation and using (5.7) again, we obtain

$$\begin{aligned} 0 &= -\frac{1}{2}\overline{f}_\beta\delta_{\alpha\gamma} - R_{\beta\overline{\gamma}\alpha\overline{\sigma}}\overline{f}_\sigma + \left[\frac{1}{2}\left(|\psi|^2 - 1\right)\overline{f}_\alpha - \sqrt{-1}\overline{\psi}A_{\alpha\sigma}f_{\overline{\sigma}}\right]\delta_{\beta\gamma} \\ &= -\frac{1}{2}\overline{f}_\beta\delta_{\alpha\gamma} - R_{\beta\overline{\gamma}\alpha\overline{\sigma}}\overline{f}_\sigma - \frac{1}{2}\overline{f}_\alpha\delta_{\beta\gamma}, \end{aligned}$$

where in the last step, we have used (5.6). Therefore,

$$-R_{\beta\overline{\gamma}\alpha\overline{\sigma}}\overline{f}_\sigma = \frac{1}{2}\left(\overline{f}_\beta\delta_{\alpha\gamma} + \overline{f}_\alpha\delta_{\beta\gamma}\right).$$

Taking trace over β and γ yields (5.10). $\square$

Since the Paneitz operator P_0 is real, we have

$$\int_M f P_0 \overline{f} = \overline{\int_M \overline{f} P_0 f} = 0.$$

Applying the Bochner formula to $\overline{f}$ yields

$$\begin{aligned}\frac{m+1}{m}\int_M \left|\Box_b \overline{f}\right|^2 &= \int_M |f_{\alpha,\beta}|^2 + \int_M R_{\alpha\overline{\beta}} \overline{f}_{\overline{\alpha}} f_\beta + \frac{1}{m}\int_M \overline{f} P_0 f \\ &= \int_M |f_{\alpha,\beta}|^2 + \frac{m+1}{2}\int_M |\psi|^2 \left|\overline{\partial}_b f\right|^2.\end{aligned}$$

We compute on $M\backslash K$, using (5.7) and (5.1)

$$\begin{aligned} f_{\alpha,\beta} &= \psi_\beta \overline{f}_\alpha + \psi \overline{f}_{\alpha,\beta} \\ &= \psi_\beta \overline{f}_\alpha. \end{aligned} \tag{5.11}$$

From this, we get on $M\backslash K$

$$|\partial_b \psi|^2 \left|\overline{\partial}_b f\right|^2 = \sum_{\alpha,\beta} |f_{\alpha\beta}|^2. \tag{5.12}$$

Notice that the right hand side is smooth on M. Therefore, $|\partial_b \psi|^2 \left|\overline{\partial}_b f\right|^2$ extends smoothly to the entire M and the above inequality holds on M. Similarly, using (5.8) as well

$$\Box_b \overline{f} = -\overline{f}_{\overline{\alpha},\alpha} = -\left(\overline{\psi} f_{\overline{\alpha}}\right)_{,\alpha} = -\overline{\psi} f_{\overline{\alpha},\alpha} = \frac{m}{2}\overline{\psi} f.$$

Plugging these into the integral identity, we obtain

$$\begin{aligned} &\frac{m(m+1)}{4}\int_M |\psi|^2 |f|^2 \\ &= \int_M |\partial_b \psi|^2 \left|\overline{\partial}_b f\right|^2 + \frac{m+1}{2}\int_M |\psi|^2 \left|\overline{\partial}_b f\right|^2, \end{aligned}$$

i.e.

$$\int_M |\partial_b \psi|^2 \left|\overline{\partial}_b f\right|^2 = \frac{m+1}{2}\int_M |\psi|^2 \left(\frac{m}{2}|f|^2 - \left|\overline{\partial}_b f\right|^2\right). \tag{5.13}$$

Lemma 2. *We have on $M\backslash K$*

$$\partial_b \psi = 0.$$

Proof. By (5.11), we have $\psi_\alpha \overline{f}_\beta = \psi_\beta \overline{f}_\alpha$. Therefore, on $M\backslash K$

$$\psi_\alpha = \left(\sum_\beta \psi_\beta f_{\overline{\beta}}\right) \overline{f}_\alpha \left|\overline{\partial}_b f\right|^{-2}. \tag{5.14}$$

From this, we get

$$|\partial_b \psi|^2 \left|\overline{\partial}_b f\right|^2 = \left|\sum_\beta \psi_\beta f_{\overline{\beta}}\right|^2.$$

Since $\overline{\partial}_b\psi = 0$, we have $\psi_{\alpha,\overline{\beta}} = \sqrt{-1}\psi_0\delta_{\alpha\beta}$. For each α we compute using (5.14) and Proposition 6

$$\begin{aligned}\psi_{\alpha\overline{\alpha}} &= \left[\left(\sum_\beta \psi_{\beta,\overline{\alpha}} f_{\overline{\beta}}\right)\overline{f}_\alpha + \left(\sum_\beta \psi_\beta f_{\overline{\beta}}\right)\overline{f}_{\alpha,\overline{\alpha}}\right]\left|\overline{\partial}_b f\right|^{-2} \\ &\quad - \left(\sum_\beta \psi_\beta f_{\overline{\beta}}\right)\overline{f}_\alpha\left|\overline{\partial}_b f\right|^{-4}\sum_\gamma f_{\overline{\gamma}}\overline{f}_{\gamma,\overline{\alpha}} \\ &= \left[\psi_{\alpha,\overline{\alpha}}\left|f_{\overline{\alpha}}\right|^2 - \frac{1}{2}\left(\sum_\beta \psi_\beta f_{\overline{\beta}}\right)\overline{f}\right]\left|\overline{\partial}_b f\right|^{-2} + \frac{1}{2}\left(\sum_\beta \psi_\beta f_{\overline{\beta}}\right)\overline{f}\left|f_{\overline{\alpha}}\right|^2\left|\overline{\partial}_b f\right|^{-4}.\end{aligned}$$

Hence,

$$\psi_{\alpha\overline{\alpha}}\left(1 - \left|\overline{\partial}_b f\right|^{-2}\left|f_{\overline{\alpha}}\right|^2\right) = -\frac{1}{2}\left(\sum_\beta \psi_\beta f_{\overline{\beta}}\right)\overline{f}\left|\overline{\partial}_b f\right|^{-2}\left(1 - \left|\overline{\partial}_b f\right|^{-2}\left|f_{\overline{\alpha}}\right|^2\right).$$

It follows that on $M\backslash K$

$$\psi_{\alpha\overline{\alpha}} = -\frac{1}{2}\left(\sum_\beta \psi_\beta f_{\overline{\beta}}\right)\left|\overline{\partial}_b f\right|^{-2}$$

Set

$$B = \sum_\beta |\psi|^2_\beta f_{\overline{\beta}}.$$

Note that B is a smooth function on M. Then on $M\backslash K$, as $\overline{\partial}_b\psi = 0$

$$\overline{\psi}\psi_{\alpha\overline{\alpha}} = -\frac{1}{2}\left(\sum_\beta \overline{\psi}\psi_\beta f_{\overline{\beta}}\right)\left|\overline{\partial}_b f\right|^{-2} = -\frac{1}{2}B\overline{f}\left|\overline{\partial}_b f\right|^{-2}. \tag{5.15}$$

We compute on $M\backslash K$, using Proposition 6 and (5.15)

$$\begin{aligned}&\left(\overline{f}\,|\psi|^2 f_{\overline{\alpha}} + \left|\overline{\partial}_b f\right|^2 |\psi|^2_{\overline{\alpha}}\right)_{,\alpha} \\ &= \left(\overline{f}\,|\psi|^2 f_{\overline{\alpha}} + \left|\overline{\partial}_b f\right|^2 \psi\overline{\psi}_{\overline{\alpha}}\right)_{,\alpha} \\ &= \overline{f}B + |\psi|^2\left|\overline{\partial}_b f\right|^2 - \frac{m}{2}|\psi|^2|f|^2 - f\psi\overline{f}_\alpha\overline{\psi}_{\overline{\alpha}} + \left|\overline{\partial}_b f\right|^2\left(\psi\overline{\psi}_{\overline{\alpha},\alpha} + |\partial_b\psi|^2\right) \\ &= \frac{1}{2}\left(\overline{f}B - f\overline{B}\right) + |\psi|^2\left(\left|\overline{\partial}_b f\right|^2 - \frac{m}{2}|f|^2\right) + \left|\overline{\partial}_b f\right|^2|\partial_b\psi|^2.\end{aligned}$$

Since both sides are smooth on M and $M\backslash K$ is open and dense, the above identity holds on the entire M. Integrating over M and taking the real part yields

$$\int_M \left|\overline{\partial}_b f\right|^2|\partial_b\psi|^2 = \int_M |\psi|^2\left(\frac{m}{2}|f|^2 - \left|\overline{\partial}_b f\right|^2\right).$$

Combining this identity with (5.13), we obtain $\int_M \left|\overline{\partial}_b f\right|^2|\partial_b\psi|^2 = 0$. Therefore, $\partial_b\psi = 0$ on $M\backslash K$. □

Lemma 3. $\psi = 0$ *and therefore, the torsion vanishes.*

Proof. By Proposition 9 and Proposition 2, $\overline{\partial}_b\psi = 0$ and $\partial_b\psi = 0$ on $M\backslash K$. Therefore, ψ is locally constant on $M\backslash K$. Since $|\psi|^2$ extends smoothly to M, $|\psi|^2$ is constant on M. Differentiating (5.6) on $M\backslash K$ using Proposition 6 and (5.3), we get on $M\backslash K$

$$2A_{\alpha\beta,\gamma}\left|\overline{\partial}_b f\right|^2 = A_{\alpha\beta} f\overline{f}_{\gamma}.$$

Hence,

$$\begin{aligned}2A_{\alpha\beta,\gamma} f_{\overline{\alpha}} f_{\overline{\beta}} f_{\overline{\gamma}}\left|\overline{\partial}_b f\right|^2 &= A_{\alpha\beta} f_{\overline{\alpha}} f_{\overline{\beta}} f\left|\overline{\partial}_b f\right|^2\\ &= -\sqrt{-1}Qf\left|\overline{\partial}_b f\right|^2.\end{aligned}$$

Thus, on $M\backslash K$, we have $Qf = 2\sqrt{-1}A_{\alpha\beta,\gamma} f_{\overline{\alpha}} f_{\overline{\beta}} f_{\overline{\gamma}}$, i.e.

$$\psi f = 4\sqrt{-1}A_{\alpha\beta,\gamma} f_{\overline{\alpha}} f_{\overline{\beta}} f_{\overline{\gamma}} / \left|\overline{\partial}_b f\right|^2.$$

From this, we obtain (first on $M\backslash K$ and then, by continuity, on the whole M as $|\psi|$ is continuous on M and $M\backslash K$ is open and dense)

$$|\psi|\,|f| \leqslant 4C\left|\overline{\partial}_b f\right|, \tag{5.16}$$

where, $C = \max_M \sqrt{\sum |A_{\alpha\beta,\gamma}|^2}$.

Let $p_0 \in M$ be a point where $|f|^2$ achieves its maximum. Suppose $\overline{\partial}_b f(p_0) \neq 0$, i.e $p_0 \in M\backslash K$. Then differentiating at p_0 and using (5.7), we have

$$\begin{aligned}0 &= f\overline{f}_{\alpha} + f_{\alpha}\overline{f}\\ &= \overline{f}_{\alpha}\left(f + \psi\overline{f}\right).\end{aligned}$$

Hence,

$$\psi(p_0) = -\frac{f(p_0)}{\overline{f(p_0)}}. \tag{5.17}$$

As $|\psi|$ is constant, we have $|\psi| \equiv 1$. By (5.9) we have $\sqrt{-1}f_0 = \frac{1}{2}\left(f - \psi\overline{f}\right)$on $M\backslash K$. Differentiating and using (5.7) yields

$$\begin{aligned}\sqrt{-1}f_{0\alpha} &= \frac{1}{2}\left(f_a - \psi\overline{f}_{\alpha}\right) = 0,\\ \sqrt{-1}f_{0\overline{\alpha}} &= \frac{1}{2}\left(f_{\overline{a}} - \psi\overline{f}_{\overline{\alpha}}\right)\\ &= \frac{1}{2}(1 - |\psi|^2)f_{\overline{\alpha}}\\ &= 0.\end{aligned}$$

Therefore, f_0 is constant. As $\int_M f_0 = 0$, we must have $f_0 \equiv 0$. Then (5.9) reduces to $f = \psi\overline{f}$. At p_0 this yields $\psi(p_0) = f(p_0)/\overline{f(p_0)}$. This is contradictory to (5.17). Therefore, $\overline{\partial}_b f(p_0) = 0$, then the inequality (5.16) implies $|\psi(p_0)| = 0$. Consequently, $\psi \equiv 0$. ∎

Since $\psi \equiv 0$, we have, in view of (5.6) (5.7) and (5.9)

$$\begin{aligned} A_{\alpha\beta} &= 0, \\ \partial_b f &= 0, \\ \sqrt{-1} f_0 &= \frac{1}{2} f. \end{aligned}$$

Write $f = u + \sqrt{-1}v$ in terms of its real part u and imaginary part v. From the above identities and Proposition 6 it is easy to prove the following

Proposition 11. *We have $v = 2u_0$, while u satisfies*

$$\begin{aligned} u_{\alpha\beta} &= 0, \\ u_{\alpha,\overline{\beta}} &= \left(-\frac{1}{4} u + \frac{\sqrt{-1}}{2} u_0 \right) \delta_{\alpha\beta}. \end{aligned}$$

With this proved, we can now apply the result of Li-Wang [LW] (Theorem 5 therein) to conclude that M is CR equivalent to $\mathbb{S}^{2m+1}$ with its standard pseudohermitian structure. In fact, we do not need that result in its full generality as we have the additional condition $A = 0$ at our disposal. Since the argument there under the additional condition $A = 0$ is simple, we give an outline for completeness. Let g_θ be the adapted Riemannian metric and $D^2 u$ the Riemannian Hessian. Then from the above identities we obtain by standard calculation (see Proposition 2 in [LW])

$$D^2 u = -\frac{u}{4} g_\theta,$$

here we used the fact that the torsion $A = 0$.

By Obata's theorem [O], (M, g_θ) is isometric to the sphere $\mathbb{S}^{2m+1}$ with the metric $g_0 = 4g_c$, where g_c is the canonical metric. Without loss of generality, we can take (M, g_θ) to be $(\mathbb{S}^{2m+1}, g_0)$. Then θ is a pseudohermitian structure on $\mathbb{S}^{2m+1}$ whose adapted metric is g_0 and the associated Tanaka-Webster connection is torsion-free. It is a well known fact that the Reeb vector field T is then a Killing vector field for g_0 (see Remark 1 in [LW]). Therefore there exists a skew-symmetric matrix A such that for all $X \in \mathbb{S}^{2m+1}, T(X) = AX$, here we use the obvious identification between $z = (z_1, \ldots, z_{m+1}) \in \mathbb{C}^{m+1}$ and $X = (x_1, y_1, \ldots, x_{m+1}, y_{m+1}) \in \mathbb{R}^{2m+2}$. Changing coordinates by an orthogonal transformation we can assume that A is of the following form

$$A = \begin{bmatrix} 0 & a_1 & & & \\ -a_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & a_{m+1} \\ & & & -a_{m+1} & 0 \end{bmatrix}$$

where $a_i \geqslant 0$. Therefore

$$T = \sum_i a_i \left(y_i \frac{\partial}{\partial x_i} - x_i \frac{\partial}{\partial y_i} \right)$$

Since T is of unit length we must have

$$4 \sum_i a_i^2 (x_i^2 + y_i^2) = 1$$

on $\mathbb{S}^{2m+1}$. Therefore all the a_i's are equal to $1/2$. It follows that

$$\theta = g_0(T, \cdot) = 2\sqrt{-1}\,\overline{\partial}(|z|^2 - 1).$$

This finishes the proof of Theorem 2.

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