MTH 310 HW 1 Solutions

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Section 1.1, Question 7

Prove that the square of any integer a is either of the form $3k$ or $3k+1$ for some integer k.

Answer. By the division algorithm, any integer a must have the form $a = 3q + r$ where $0 \leq r \leq 2$. If $r = 0$, then $a = 3q$ and thus $a^2 = (3q)^2 = 9q^2 = 3(3q^2)$, so by letting $k = 3q^2$ we see that $a^2 = 3k$ for some integer k. If $r = 1$, then $a = 3q + 1$ and $a^2 = (3q+1)^2 = 9q^2+6q+1 = 3(3q^2+2q)+1$, so letting $k = 3q^2+2q$ we see that $a^2 = 3k+1$ for some integer k. Finally, if $r = 2$, then $a^2 = (3q+2)^2 = 9q^2+12q+4 = 9q^2+12q+3+1 = 3(3q^2+4q+1)+1$, so letting $k = 3q^2 + 4q + 1$ we see that $a^2 = 3k + 1$ for some integer k.

Section 1.2, Question 11

If $n \in \mathbb{Z}$, what are the possible values of (n, n+2) and (n, n+6)?

Answer. In the first case, let $d = (n, n+2)$. Then $d|n$ and $d|(n+2)$ so there exist integers j, k so that $jd = n$ and $kd = n+2$. This implies that $2 = (n+2) - n = kd - jd = d(k-j)$, so d is a positive integer that divides 2. This says $d = 1$ or $d = 2$, both of which can occur (the former case occurs when n = 1, and the latter case occurs when n = 2).

Similarly, let $d = (n, n+6)$. Similar to above, we have that $d|6$ so since $d > 0$ we have $d \in \{1, 2, 3, 6\}$. All of these occur (respectively, when $n = 1, 2, 3$, and 6).

Section 1.2, Question 14

Find the smallest positive integer in the sets $\{6u + 15v | u, v \in \mathbb{Z}\}$ and $\{12r + 17s | r, s \in \mathbb{Z}\}$.

Answer. By the proof of Theorem 1.2 (Hungerford), the smallest positive integer in the set $\{ax + by | a, b \in \mathbb{Z}\}$ is $gcd(x, y)$. Therefore the smallest positive integer in the first set is $gcd(6, 15) = 3$, and the smallest positive integer in the second set is $gcd(12, 17) = 1$. (To show these actually are the greatest common divisors, one may list all the factors of 6, 15, 12, and 17 and choose the largest).

Section 1.3, Question 27

Prove that if a prime $p > 3$, then $p^2 + 2$ is composite.
Answer. By the division algorithm, p has the form $3k$, $3k+1$, or $3k+2$ for some $k \in \mathbb{Z}$. If $p = 3k$, then $3|p$ and since p is prime the only numbers that divide p are 1 and p. Therefore $p = 3$ which violates our assumption that $p > 3$.

We claim that in the other two cases, $3|p$. Note that since $p > 3, p^2 + 2 > 11$, so 3 is a proper divisor of $p^2 + 2$ (meaning it is neither 1 nor $p^2 + 2$) so it is not prime.

If $p = 3k + 1$, $p^2 + 2 = (3k + 1)^2 + 2 = 9k^2 + 6k + 1 + 2 = 3(3k^2 + 2k + 1)$, so $3|p^2 + 2$.

If $p = 3k + 2$, $p^2 + 2 = (3k + 2)^2 + 2 = 9k^2 + 12k + 4 + 2 = 3(3k^2 + 4k + 2)$ so $3|p^2 + 2$.