HW 9

[Hungerford] Section 4.5, #21
For a prime p, $f(x) = x^{p-1} + x^{p-2} + \cdots + x^2 + x + 1$ is irreducible in $\mathbb{Q}[x]$.

Proof. Notice that multiplying $f(x)$ by $x - 1$, and distributing, most of the terms cancel, and you get

$$\begin{aligned}(x-1) \cdot f(x) &= x^p - x^{p-1} + x^{p-1} - x^{p-2} + x^{p-2} + \cdots - x^2 + x^2 - x + x - 1 \\ &= x^p - 1\end{aligned}$$

Making the substitution $x \mapsto x + 1$, this becomes

$$x \cdot f(x+1) = (x+1)^p - 1 \tag{$*$}$$

We can expand $(x + 1)^p$ using the binomial formula:

$$\begin{aligned}(x+1)^p &= \sum_{k=0}^{p} \binom{p}{k} x^p \\ &= 1 + px + \binom{p}{2} x^2 + \cdots + \binom{p}{k} x^k + \cdots + \binom{p}{p-2} x^{p-2} + px^{p-1} + x^p\end{aligned}$$

Looking back at equation $(*)$, we need to subtract 1 from this, which just removes the constant term from the right-hand side. We can thus cancel a factor of x from both sides of equation $(*)$, and obtain

$$f(x+1) = p + \binom{p}{2} x + \cdots + \binom{p}{k} x^{k-1} + \cdots + \binom{p}{p-2} x^{p-3} + px^{p-2} + x^{p-1}$$

The formula for $\binom{p}{k}$ is given by

$$\binom{p}{k} = \frac{p!}{k!(p-k)!}$$

So p divides $\binom{p}{k}$ except when $k = 0$ or $k = p$, in which case it is 1. Therefore p divides all the coefficients of $f(x + 1)$ except the leading coefficient. Furthermore, the constant term is p, and so p^2 does not divide the constant term. By **Eisenstein's Criterion**, $f(x + 1)$ is irreducible. By **Exercise 12** from last homework, this means that the original $f(x)$ was also irreducible. $\square$

[Hungerford] Section 4.6, #6
Let $f(x) = ax^2 + bx + c \in \mathbb{R}[x]$ with $a \neq 0$. Then the roots of $f(x)$ in $\mathbb{C}$ are

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Proof. We wish to solve the equation

$$ax^2 + bx + c = 0$$

Assuming $a \neq 0$, we can divide by a to get

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

Moving the constant term to the other side, and completing the square, we get

$$\begin{aligned}
x^2 + \frac{b}{a}x &= -\frac{c}{a} \\
x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 &= -\frac{c}{a} \\
\left(x + \frac{b}{2a}\right)^2 &= \left(\frac{b}{2a}\right)^2 - \frac{c}{a} \\
\left(x + \frac{b}{2a}\right)^2 &= \frac{b^2 - 4ac}{4a^2} \\
x + \frac{b}{2a} &= \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} \\
x &= -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}
\end{aligned}$$

□

[Hungerford] Section 5.1, #13 Suppose $f, g \in \mathbb{R}[x]$ with $f \equiv g (\mathrm{mod}\ x)$. What can be said about the graphs of $f(x)$ and $g(x)$?

Solution. Since f and g are congruent mod x, this means they differ by a (polynomial) multiple of x. In other words, they have the same constant term. This means that the graphs have the same y-intercept.