[Hungerford] Section 4.1, #4 In each part, give an example of polynomials \( f, g \in \mathbb{Q}[x] \) that satisfy the given condition:

(a) The degree of \( f + g \) is less than the maximum of the degrees of \( f \) and \( g \).

Solution. We just need the leading terms to cancel. For example, \( f = x^2 + 1 \) and \( g = 1 - x^2 \). In this case, \( \deg(f) = \deg(g) = 2 \), but \( \deg(f + g) = 0 \).

(b) The degree of \( f + g \) is equal to the maximum of the degrees of \( f \) and \( g \).

Solution. Perhaps the easiest examples would be where \( \deg(f) \neq \deg(g) \). For example, \( f = x^2 \) and \( g = x^3 \). Then \( \deg(f) = 2 \), \( \deg(g) = 3 \), and \( \deg(f + g) = 3 \).

[Hungerford] Section 4.1, #11 Show that \( 1 + 3x \) is a unit in \( \mathbb{Z}_9[x] \).

Solution. The multiplicative inverse of \( 1 + 3x \) is \( 1 - 3x = 1 + 6x \), since

\[
(1 + 3x)(1 - 3x) = 1 - 9x = 1 - 0x = 1
\]

[Hungerford] Section 4.1, #21 Let \( h: R \to S \) be a ring homomorphism, and define \( \overline{h}: R[x] \to S[x] \) by

\[
\overline{h} \left( \sum_{i=0}^{n} a_i x^i \right) = \sum_{i=0}^{n} h(a_i) x^i
\]

(a) \( \overline{h} \) is a ring homomorphism.
Proof. First, we'll see that $h$ is additive. Take two polynomials $f$ and $g$ in $R[x]$, given by

$$f = \sum_{i=0}^{n} a_i x^i$$
$$g = \sum_{i=0}^{m} b_i x^i$$

Assume (without loss of generality) that $n > m$, and that $b_i = 0$ for $i > m$. Let's evaluate $h$ on the sum $f + g$. By definition, we have

$$h(f + g) = h\left(\sum_{i=0}^{n} a_i x^i + \sum_{i=0}^{m} b_i x^i\right)$$
$$= h\left(\sum_{i=0}^{n} (a_i + b_i) x^i\right)$$
$$= \sum_{i=0}^{n} h(a_i + b_i) x^i$$

Now, since $h$ is assumed to be a homomorphism from $R$ to $S$, we know $h$ is additive, and so it distributes over the addition $a_i + b_i$, and we get

$$h(f + g) = \sum_{i=0}^{n} (h(a_i) + h(b_i)) x^i$$
$$= \sum_{i=0}^{n} h(a_i) x^i + \sum_{i=0}^{m} h(b_i) x^i$$
$$= h(f) + h(g)$$

Now let's check that $h$ is multiplicative. Let $f$ and $g$ be as above. Then we have

$$h(f \cdot g) = h\left(\left(\sum_{i=0}^{n} a_i x^i\right) \cdot \left(\sum_{j=0}^{m} b_j x^j\right)\right)$$
$$= h\left(\sum_{k=0}^{n+m} \sum_{i+j=k} a_i b_j x^k\right)$$
$$= \sum_{k=0}^{n+m} h\left(\sum_{i+j=k} a_i b_j\right) x^k$$

Now we use that $h$ is a homomorphism twice in a row. First we use that $h$ is additive to get that

$$h(f \cdot g) = \sum_{k=0}^{n+m} \sum_{i+j=k} h(a_i b_j) x^k$$

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Then we use that $h$ is multiplicative to get that
\[
\overline{h}(f \cdot g) = \sum_{k=0}^{n+m} \sum_{i+j=k} h(a_i)h(b_j)x^k
\]

Finally, we factor this polynomial to get that
\[
\overline{h}(f \cdot g) = \sum_{k=0}^{n+m} \sum_{i+j=k} h(a_i)h(b_j)x^k = \left( \sum_{i=0}^n h(a_i)x^i \right) \cdot \left( \sum_{j=0}^m h(b_j)x^j \right) = \overline{h}(f) \cdot \overline{h}(g)
\]

This proves that $\overline{h}$ is multiplicative, and so it is a homomorphism. \qed

(b) $\overline{h}$ is injective if and only if $h$ is injective.

Proof. “$\Leftarrow$”: Suppose first that $h$ is injective, and suppose that $\overline{h}(f) = \overline{h}(g)$ for some polynomials $f,g \in R[x]$. We want to show that $f = g$. Let’s generically write them as in part (a):
\[
f = \sum_{i=0}^n a_i x^i
\]
\[
g = \sum_{i=0}^m b_i x^i
\]
Assume that $n \geq m$. We are assuming that $\overline{h}(f) = \overline{h}(g)$ in $S[x]$, which means that all of their coefficients are equal. This gives us that $h(a_i) = h(b_i)$ for all $i \leq n$. But since $h$ is injective, this tells us that $a_i = b_i$ for all $i$. Thus the coefficients of $f$ and $g$ are all equal, and so $f = g$. This proves that $\overline{h}$ is injective.

$\Rightarrow$”: Now suppose that $\overline{h}$ is injective, and that $h(a) = h(b)$ for some $a$ and $b$ in $R$. Although $a$ and $b$ are constants in $R$, we can think of them as constant polynomials in $R[x]$. For constant polynomials, $h$ and $\overline{h}$ are essentially the same. So $h(a) = h(b)$ means that $\overline{h}(a) = \overline{h}(b)$ if we consider $a$ and $b$ as constant polynomials. But since we assumed that $\overline{h}$ is injective, this tells us that $a$ and $b$ are equal as polynomials. But this also tells us that $a$ and $b$ are equal as elements of $R$. This proves that $h$ is injective. \qed

(c) $\overline{h}$ is surjective if and only if $h$ is surjective.

Proof. “$\Leftarrow$”: Suppose that $h$ is surjective, and let $f \in S[x]$ be given by
\[
f = \sum_{i=0}^n \alpha_i x^i
\]
for constants $\alpha_i \in S$. Since $h$ is surjective, there is some $a_i \in R$ for each $\alpha_i \in S$ so that $h(a_i) = \alpha_i$. If we let $g \in R[x]$ be given by $g = \sum a_i x^i$, then $\overline{h}(g) = f$. This shows that $\overline{h}$ is surjective.

$\Rightarrow$”: Suppose now that $\overline{h}$ is surjective, and let $\alpha \in S$. As in part (b), we can think of $\alpha \in S$ as a
constant polynomial in $S[x]$. Then since $h$ is surjective, there is some polynomial $f \in R[x]$ so that $h(f) = \alpha$. We can see that $f$ must, in fact, also be a constant polynomial, corresponding to some constant $a \in R$. Then $h(a) = \alpha$, and so $h$ is also surjective.

(d) If $R \cong S$, then $R[x] \cong S[x]$.

Proof. If we have some isomorphism $\varphi : R \to S$, then $\varphi$ is both injective and surjective. By parts (b) and (c), we know that $\overline{\varphi}$ is also injective and surjective, and so $\overline{\varphi}$ is an isomorphism between $R[x]$ and $S[x]$.  \qed