[Hungerford] Section 4.1, #4 In each part, give an example of polynomials $f, g \in \mathbb{Q}[x]$ that satisfy the given condition:

(a) The degree of $f + g$ is less than the maximum of the degrees of f and g.

Solution. We just need the leading terms to cancel. For example, $f = x^2 + 1$ and $g = 1 - x^2$. In this case, $\deg(f) = \deg(g) = 2$, but $\deg(f + g) = 0$.

(b) The degree of $f + g$ is equal to the maximum of the degrees of f and g.

Solution. Perhaps the easiest examples would be where $\deg(f) \neq \deg(g)$. For example, $f = x^2$ and $g = x^3$. Then $\deg(f) = 2$, $\deg(g) = 3$, and $\deg(f + g) = 3$.

[Hungerford] Section 4.1, #11 Show that $1 + 3x$ is a unit in $\mathbb{Z}_9[x]$.

Solution. The multiplicative inverse of $1 + 3x$ is $1 - 3x = 1 + 6x$, since

$$(1 + 3x)(1 - 3x) = 1 - 9x = 1 - 0x = 1$$

[Hungerford] Section 4.1, #21 Let $h\colon R \to S$ be a ring homomorphism, and define $\overline{h}\colon R[x] \to S[x]$ by

$$\overline{h}\left(\sum_{i=0}^{n} a_i x^i\right) = \sum_{i=0}^{n} h(a_i) x^i$$

(a) $\overline{h}$ is a ring homomorphism.

Proof. First, we'll see that $\overline{h}$ is additive. Take two polynomials f and g in $R[x]$, given by

$$f = \sum_{i=0}^{n} a_i x^i$$

$$g = \sum_{i=0}^{m} b_i x_i$$

Assume (without loss of generality) that $n > m$, and that $b_i = 0$ for $i > m$. Let's evaluate $\overline{h}$ on the sum $f + g$. By definition, we have

$$\begin{aligned} \overline{h}(f+g) &= \overline{h}\left(\sum_{i=0}^{n} a_i x^i + \sum_{i=0}^{m} b_i x^i\right) \\ &= \overline{h}\left(\sum_{i=0}^{n} (a_i + b_i) x^i\right) \\ &= \sum_{i=0}^{n} h(a_i + b_i) x^i \end{aligned}$$

Now, since h is assumed to be a homomorphism from R to S, we know h is additive, and so it distributes over the addition $a_i + b_i$, and we get

$$\begin{aligned} \overline{h}(f+g) &= \sum_{i=0}^{n} \left(h(a_i) + h(b_i)\right) x^i \\ &= \sum_{i=0}^{n} h(a_i) x^i + \sum_{i=0}^{m} h(b_i) x^i \\ &= \overline{h}(f) + \overline{h}(g) \end{aligned}$$

Now let's check that $\overline{h}$ is multiplicative. Let f and g be as above. Then we have

$$\begin{aligned} \overline{h}(f \cdot g) &= \overline{h}\left(\left(\sum_{i=0}^{n} a_i x^i\right) \cdot \left(\sum_{j=0}^{m} b_j x^j\right)\right) \\ &= \overline{h}\left(\sum_{k=0}^{n+m} \sum_{i+j=k} a_i b_j x^k\right) \\ &= \sum_{k=0}^{n+m} h\left(\sum_{i+j=k} a_i b_j\right) x^k \end{aligned}$$

Now we use that h is a homomorphism twice in a row. First we use that h is additive to get that

$$\overline{h}(f \cdot g) = \sum_{k=0}^{n+m} \sum_{i+j=k} h(a_i b_j) x^k$$

Then we use that h is multiplicative to get that

$$\overline{h}(f \cdot g) = \sum_{k=0}^{n+m} \sum_{i+j=k} h(a_i)h(b_j)x^k$$

Finally, we factor this polynomial to get that

$$\overline{h}(f \cdot g) = \sum_{k=0}^{n+m} \sum_{i+j=k} h(a_i)h(b_j)x^k = \left(\sum_{i=0}^{n} h(a_i)x^i\right) \cdot \left(\sum_{j=0}^{m} h(b_j)x^j\right) = \overline{h}(f) \cdot \overline{h}(g)$$

This proves that $\overline{h}$ is multiplicative, and so it is a homomorphism. □

(b) $\overline{h}$ is injective if and only if h is injective.

Proof. "$\Longleftarrow$": Suppose first that h is injective, and suppose that $\overline{h}(f) = \overline{h}(g)$ for some polynomials $f, g \in R[x]$. We want to show that $f = g$. Let's generically write them as in part (a):

$$f = \sum_{i=0}^{n} a_i x^i$$

$$g = \sum_{i=0}^{m} b_i x^i$$

Assume that $n \geq m$. We are assuming that $\overline{h}(f) = \overline{h}(g)$ in $S[x]$, which means that all of their coefficients are equal. This gives us that $h(a_i) = h(b_i)$ for all $i \leq n$. But since h is injective, this tells us that $a_i = b_i$ for all i. Thus the coefficients of f and g are all equal, and so $f = g$. This proves that $\overline{h}$ is injective.

"$\Longrightarrow$": Now suppose that $\overline{h}$ is injective, and that $h(a) = h(b)$ for some a and b in R. Although a and b are constants in R, we can think of them as constant polynomials in $R[x]$. For constant polynomials, h and $\overline{h}$ are essentially the same. So $h(a) = h(b)$ means that $\overline{h}(a) = \overline{h}(b)$ if we consider a and b as constant polynomials. But since we assumed that $\overline{h}$ is injective, this tells us that a and b are equal *as polynomials*. But this also tells us that a and b are equal as elements of R. This proves that h is injective. □

(c) $\overline{h}$ is surjective if and only if h is surjective.

Proof. "$\Longleftarrow$": Suppose that h is surjective, and let $f \in S[x]$ be given by

$$f = \sum_{i=0}^{n} \alpha_i x^i$$

for constants $\alpha_i \in S$. Since h is surjective, there is some $a_i \in R$ for each $\alpha_i \in S$ so that $h(a_i) = \alpha_i$. If we let $g \in R[x]$ be given by $g = \sum a_i x^i$, then $\overline{h}(g) = f$. This shows that $\overline{h}$ is surjective.

"$\Longrightarrow$": Suppose now that $\overline{h}$ is surjective, and let $\alpha \in S$. As in part (b), we can think of $\alpha \in S$ as a constant polynomial in $S[x]$. Then since $\overline{h}$ is surjective, there is some polynomial $f \in R[x]$ so that $\overline{h}(f) = \alpha$. We can see that f must, in fact, also be a constant polynomial, corresponding to some constant $a \in R$. Then $h(a) = \alpha$, and so h is also surjective. $\square$

(d) If $R \cong S$, then $R[x] \cong S[x]$.

Proof. If we have some isomorphism $\varphi\colon R \to S$, then φ is both injective and surjective. By parts (b) and (c), we know that $\overline{\varphi}$ is also injective and surjective, and so $\overline{\varphi}$ is an isomorphism between $R[x]$ and $S[x]$. $\square$