Exercise 3.2

19. Let R and S be rings with identity. What are the units in the ring $R \times S$?

Proof: Assume $(r, s)$ is a unit in $R \times S$ (with $r \in R$ and $s \in S$). We have $(r, s)(u, v) = (1_R, 1_S)$, then $ru = 1_R, sv = 1_s$. So $r$ is a unit in $R$ and $s$ is a unit in $S$. The unit is $(r, s)$ in the ring $R \times S$.

24. Let $R$ be a ring and $a, b \in R$. Let $m$ and $n$ be positive integers.

(a) Show that $a^m a^n = a^{m+n}$ and $(a^m)^n = a^{mn}$.

\begin{align*}
a^m &= a \ast a \ast a \ast \ldots \ast a \text{(m factors)} \\
a^n &= a \ast a \ast a \ast \ldots \ast a \text{(n factors)} \\
a^m a^n &= (a \ast a \ast a \ast \ldots \ast a) \text{(m factors)} \ast (a \ast a \ast a \ast \ldots \ast a) \text{(n factors)} \\
      &= (a \ast a \ast a \ast \ldots \ast a) \ast (a \ast a \ast a \ast \ldots \ast a) \text{(m factors)} \ast (a \ast a \ast a \ast \ldots \ast a) \ast (m \text{ factors}) \ast \ldots \ast (a \ast a \ast a \ast \ldots \ast a) \text{(n factors)}, \text{ and we have number n of m factors.} \\
\end{align*}

Hence $(a^m)^n = a^{mn}$

(b) $(ab)^n = abaaba\ldots ab(a \ast b \ast b \ast b \ast \ldots \ast b = a^nb^n(a \ast b \ast b \ast b \ast \ldots \ast b) \text{(n factors each)}$

29. Let $R$ be a commutative ring with identity. Prove that $R$ is an integral domain if and only if cancellation holds in $R$.

Proof: $\Rightarrow$ If $R$ is an integral domain, then whenever $a, b \in R$ and $ab = 0_R$, then $a = 0_R$ or $b = 0_R$.

Assume $R$ is an integral domain and $a \neq 0$, and $ab = ac$
\[ ab = ac \]
\[ (ab - ac) = 0_R \]
\[ a(b - c) = 0_R \]
\[ a = 0 \text{ or } b - c = 0 \]

since \( a \neq 0 \), \( b - c = 0 \) then \( b = c \)

Hence, the cancellation holds.

\[ \Leftarrow \] Assume the cancellation holds with the ring \( R \) be a commutative ting with identity. Also, we assume \( a \neq 0 \) and \( ab = 0 \).

Since

\[ ab = 0 \]
\[ ab = a \cdot 0 \]
\[ b = 0 \]

Therefore, it has integral domain.

44. (a) Let \( a \) and \( b \) be nilpotent elements in a commutative ring \( R \). Prove that \( a + b \) and \( ab \) are also nilpotent.

Since \( a \) and \( b \) be nilpotent element, then \( a^n = 0_R \) and \( b^n = 0_R \)

\[
(a + b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + ... + \binom{n}{n-1}ab^{n-1} + b^n
\]

\[
= a^n + \binom{n}{1}a^{-1}a^n b + \binom{n}{2}a^{-2}a^n b + ... + \binom{n}{n-1}ab^{-1}b^n + b^n
\]

\[
= 0 + 0 + 0 + ... + 0
\]

\[
= 0
\]

\( a + b \) is nilpotent, and since \( a^n = 0 \), \( b^n = 0 \), \( (ab)^n = a^n b^n = 0 \), \( ab \) is niloptent.
(b) Let N be the set of all nilpotent elements of R. Show that N is a subring.

Assume $a, b \in N$. And we know $a^n = 0$ and $b^n = 0$. Proved by definition that:

$$a^n + b^n = 0 + 0 = 0 \in N$$
$$a^n \ast b^n = 0 \ast 0 = 0 \in N$$
$$0_R \in N$$
$$a + x = 0, \text{ then } x = -a \in N$$

Then, N is a subring of R.

**Exercise 3.3**

4. If $\vec{0} \rightarrow 0, \vec{1} \rightarrow 2, \vec{2} \rightarrow 4, \vec{2} \rightarrow 6, \vec{4} \rightarrow 8$.

Then, $3 \vec{4} = \vec{2}$ which is $6 \times 8 = 48 = 8$

but $\vec{2} \not\rightarrow 8$

Hence, $\mathbb{Z}_5$ to S is not an isomorphism.

9. If $f : \mathbb{Z} \rightarrow \mathbb{Z}$ is an isomorphism, prove that $f$ is the identity map.

Since $f$ is an isomorphism, then $f(1) = 1, f(0) = 0, f(1 + 1) = f(1) + f(1) = 1 + 1 = 2$. We claim that $f(n) = n$.

Prove by induction, for $n = 0$ it’s true. Assume $n = k$, then $f(k) = k \rightarrow f(k + 1) = f(k) + f(1) = k + 1$, then, it’s true for $n = k + 1$.

For negative numbers $-n$ (with $n > 0$), we have $f(-n) = -f(n) = -n$.

Therefore, $f$ must be the identity.