MTH 310

Homework 04

Exercise 3.2

19. Let R and S be rings with identity, What are the units in the ring RxS?

Proof: Assume (r, s) is a unit in RxS(with r in R and s in S). We have $(r, s)(u, v) = (1_R, 1_R)$, then $ru = 1_R, sv = 1_s$. So r is a unit in R and s is a unit in S. The unit is (r, s) in the ring RxS.

24. Let R be a ring and $a, b \in R$. Let m and n be positive integers.

(a) Show that $a^m a^n = a^{m+n}$ and $(a^m)^n = a^{mn}$.

$$a^m = a * a * a * a * ...... * a(\text{m factors})$$

$$a^n = a * a * a * a * ...... * a(\text{n factors})$$

$$a^m a^n = (a * a * a * a * ...... * a)(\text{m factors})*(a * a * a * a * ...... * a)(\text{n factors})$$

$$(a * a * a * a * ...... * a) = a^(m + n)(m + n \text{ factors})$$

$(a^m)^n = (a * a * a * a * ...... * a)$(m factors)*$(a * a * a * a * ...... * a)$(m factors)*.....*$(a * a * a * a * ...... * a)$(m factors), and we have number n of m factors.

Hence $(a^m)^n = a^{mn}$

(b) $(ab)^n = ababab...ab$(n factors), If R is a commutative ring and $a, b \in R$, then $(ab)^n = a * a * a * a * ... * a * b * b * b * b * .. * b = a^n b^n$(n factors each)

29. Let R be a commutative ring with identity. Prove that R is an integral domain if and only if cancellation holds in R.

Proof: $\Rightarrow$ If R is an integral domain, then whenever $a, b \in R$ and $ab = 0_R$, then $a = 0_R$ or $b = 0_R$.

Assume R is an integral domain and $a \neq 0$, and $ab = ac$

$$ab = ac$$

$$(ab - ac) = 0_R$$

$$a(b - c) = 0_R$$

$$a = 0 \text{ or } b - c = 0$$

$$\text{since } a \neq 0,\ b - c = o$$

$$\text{then } b = c$$

Hence, the cancellation holds.

$\Leftarrow$ Assume the cancellation holds with the ring R be a commutative ting with identity. Also, we assume $a \neq 0$ and $ab = 0$.

Since

$$ab = 0$$

$$ab = a * 0$$

$$b = 0$$

Therefore, it has integral domain.

44.(a) Let a and b be nilpotent elements in a commutative ring R. Prove that $a + b$ and ab are also nilpotent.

Since a and b be nilpotent element,then $a^n = 0_R$ and $b^n = 0_R$

$(a+b)^n = a^n + \binom{n}{1}a^(n-1)b + \binom{n}{2}a^(n-2)b^2 + ... + \binom{n}{n-1}ab^(n-1) + b^n$

$= a^n + \binom{n}{1}a^{-1}a^n b + \binom{n}{2}a^{-2}a^n b + ... + \binom{n}{n-1}ab^{-1}b^n + b^n$

$= 0 + 0 + 0 + ... + 0$

$= 0$

$a + b$ is nilpotent, and since $a^n = 0$, $b^n = 0$, $(ab)^n = a^n b^n = 0$, ab is niloptent.

(b) Let N be the set of all nilpotent elements of R. Show that N is a subring.

Assume $a, b \in$ N. And we know $a^n = 0$ and $b^n = 0$.proved by definition that:

$$a^n + b^n = 0 + 0 = 0 \in \text{N}$$

$$a^n * b^n = 0 * 0 = 0 \in\text{N}$$

$$0_R \in \text{N}$$

$$a + x = 0, \text{ then } x = -a \in \text{N}$$

Then, N is a subring of R.

Exercise 3.3

4. If $\vec{0} \to 0, \vec{1} \to 2, \vec{2} \to 4, \vec{2} \to 6, \vec{4} to 8$.

Then, $\vec{3}$x$\vec{4} = \vec{2}$ which is 6x8 $= 48 = 8$

but $\vec{2} \not\to 8$

Hence, $\mathbb{Z}_5$ to S is not an isomorphism.

9.If f : $\mathbb{Z} \to \mathbb{Z}$ is an isomorphism, prove that f is the identity map.

Since f is an isomorphism, then $f(1) = 1, f(0) = 0, f(1 + 1) = f(1) + f(1) = 1 + 1 = 2$. We claim that $f(n) = n$.

Prove by induction, for $n = 0$ it's true. Assume $n = k$, then $f(k) = k \to f(k + 1) = f(k) + f(1) = k + 1$, then, it's true for $n = k + 1$.

For negative numbers -n (with $n > 0$), we have $f(-n) = -f(n) = -n$.

Therefore, f must be the identity.