[Hungerford] Section 3.1, #15. Define a new multiplication on \( \mathbb{Z} \) by the rule \( ab = 1 \) for all \( a, b \in \mathbb{Z} \).

With ordinary addition and this new multiplication, is \( \mathbb{Z} \) a ring?

The answer is no, this is not a ring. The reason is the distributive law fails. By definition of our new multiplication, for any \( a, b, c \in \mathbb{Z} \), we have

\[
a(b + c) = 1
\]

On the other hand, we have

\[
ab + ac = 1 + 1 = 2
\]

[Hungerford] 3.1, #17. Show that the subset \( S = \{0, 2, 4, 6, 8\} \subset \mathbb{Z}_{10} \) is a subring. Does \( S \) have an identity?

By the Subring Theorem, we only need to check 4 conditions:

1. **[Closure Under Addition]** These are all the even numbers in \( \mathbb{Z}_{10} \). The sum of two even numbers is even, and when we take its remainder mod 10, we still get an even number (since 10 is even), and so \( S \) is closed under addition.

4. **[Zero Element]** Zero is in \( S \) by definition.

5. **[Existence of Negatives]** \(-2 \equiv 8, -4 \equiv 6, -6 \equiv 4, \) and \(-8 \equiv 2\), so \( S \) is closed under negatives.

6. **[Closure Under Multiplication]** The product of two even numbers is even, and when we reduce mod 10, it will still be even, so \( S \) is closed under multiplication.

This verifies that \( S \) is a subring of \( \mathbb{Z}_{10} \). It does have an identity element, which is 6:

\[
6 \cdot 0 = 0
\]

\[
6 \cdot 2 = 12 \equiv 2
\]

\[
6 \cdot 4 = 24 \equiv 4
\]

\[
6 \cdot 6 = 36 \equiv 6
\]

\[
6 \cdot 8 = 48 \equiv 8
\]
[Hungerford] Section 3.1, #18. Define a new addition $\oplus$ and multiplication $\odot$ on $\mathbb{Z}$ by

\[
a \oplus b = a + b - 1 \\
a \odot b = a + b - ab
\]

where the operations on the right-hand sides are ordinary addition, subtraction, and multiplication. Prove that, with the new operations $\oplus$ and $\odot$, $\mathbb{Z}$ is an integral domain.

First, we must show that $\mathbb{Z}$ is, in fact, a ring with these operations. Let’s check all the axioms:

1. [Closure Under Addition] Obviously if $a, b \in \mathbb{Z}$, then $a + b - 1 \in \mathbb{Z}$ since $\mathbb{Z}$ is a ring with respect to regular addition.

2. [Associativity of Addition] $(a \oplus b) \oplus c = (a + b - 1) + c - 1 = a + b + c - 2 = a + (b + c - 1) - 1 = a \oplus (b \oplus c)$

3. [Commutativity of Addition] $a \oplus b = a + b - 1 = b + a - 1 = b \oplus a$

4. [Zero Element] It turns out that 1 is the additive identity, since for any $a \in \mathbb{Z}$, we have

\[
a \oplus 1 = a + 1 - 1 = a = 1 + a - 1 = 1 \oplus a
\]

5. [Existence of Negatives] For any $a \in \mathbb{Z}$ its additive inverse is $2 - a$ (with the usual subtraction), since

\[
a \oplus (2 - a) = a + 2 - a - 1 = 1
\]

6. [Closure Under Multiplication] Again, this follows from the fact that $\mathbb{Z}$ is a ring under the regular operations, since $a \odot b = a + b - ab$ is an integer if $a$ and $b$ are.

7. [Associativity of Multiplication] For any $a, b, c \in \mathbb{Z}$, we have

\[
a \odot (b \odot c) = a + (b \odot c) - a(b \odot c) \\
= a + (b + c - bc) - a(b + c - bc) \\
= a + b + c - (ab + bc + ac) - abc \\
= (a + b - ab) + c - (a + b - ab)c \\
= (a \odot b) + c - (a \odot b)c \\
= (a \odot b) \odot c
\]

8. [Distributivity] For any $a, b, c \in \mathbb{Z}$, we have

\[
a \odot (b \oplus c) = a + (b \oplus c) - a(b \oplus c) \\
= a + (b + c - 1) - a(b + c - 1) \\
= a + b + c - ab - ac + a - 1 \\
= 2a + b + c - a(b + c) - 1
\]
On the other hand:

\[(a \odot b) \oplus (a \odot c) = (a \odot b) + (a \odot c) - 1 \]
\[= a + b - ab + a + c - ac - 1 \]
\[= 2a + b + c - a(b + c) - 1 \]

This verifies left-distributivity. Now, to check right-distributivity:

\[(a \oplus b) \odot c = (a \oplus b) + c - (a \oplus b)c \]
\[= (a + b - 1) + c - (a + b - 1)c \]
\[= a + b + c - 1 - ac - bc + c \]
\[= a + b + 2c - (a + b)c - 1 \]

On the other hand, we have

\[(a \odot c) \oplus (b \odot c) = (a \odot c) + (b \odot c) - 1 \]
\[= (a + c - ac) + (b + c - bc) - 1 \]
\[= a + b + 2c - (a + b)c - 1 \]

All of the axioms have been verified, so we conclude that \( Z \) is a ring with these operations. To show that it is an integral domain, we must also show that it is commutative, has a multiplicative identity, and has no zero divisors. First, multiplication is commutative, since

\[a \odot b = a + b - ab = b + a - ba = b \odot a \]

The multiplicative identity element is 0, since

\[a \odot 0 = a + 0 - a \cdot 0 = a = 0 + a - 0 \cdot a = 0 \odot a \]

Suppose that \( a \odot b = 1 \) (the additive identity/zero element). Then this means that

\[a \odot b = 1 \]
\[a + b - ab = 1 \]

From this you can derivat that \( a(1 - b) = 1 - b \) and that \( b(1 - a) = 1 - a \). From these two equations, you can see that either \( a \) or \( b \) must be equal to 1 (which is the zero element).