[Hungerford] Section 3.1, #15. Define a new multiplication on $\mathbb{Z}$ by the rule $ab = 1$ for all $a, b \in \mathbb{Z}$. With ordinary addition and this new multiplication, is $\mathbb{Z}$ a ring?

The answer is ***no***, this is *not* a ring. The reason is the distributive law fails. By definition of our new multiplication, for any $a, b, c \in \mathbb{Z}$, we have

$$a(b + c) = 1$$

On the other hand, we have

$$ab + ac = 1 + 1 = 2$$

[Hungerford] 3.1, #17. Show that the subset $S = \{0, 2, 4, 6, 8\} \subset \mathbb{Z}_{10}$ is a subring. Does S have an identity?

By the **Subring Theorem**, we only need to check 4 conditions:

1. [Closure Under Addition] These are all the even numbers in $\mathbb{Z}_{10}$. The sum of two even numbers is even, and when we take its remainder mod 10, we still get an even number (since 10 is even), and so S is closed under addition.

4. [Zero Element] Zero is in S by definition.

5. [Existence of Negatives] $-2 \equiv 8$, $-4 \equiv 6$, $-6 \equiv 4$, and $-8 \equiv 2$, so S is closed under negatives.

6. [Closure Under Multiplication] The product of two even numbers is even, and when we reduce mod 10, it will still be even, so S is closed under multiplication.

This verifies that S is a subring of $\mathbb{Z}_{10}$. It *does* have an identity element, which is 6:

$$6 \cdot 0 = 0$$

$$6 \cdot 2 = 12 \equiv 2$$

$$6 \cdot 4 = 24 \equiv 4$$

$$6 \cdot 6 = 36 \equiv 6$$

$$6 \cdot 8 = 48 \equiv 8$$

[Hungerford] Section 3.1, #18. Define a new addition $\oplus$ and multiplication $\odot$ on $\mathbb{Z}$ by

$$a \oplus b = a + b - 1$$

$$a \odot b = a + b - ab$$

where the operations on the right-hand sides are ordinary addition, subtraction, and multiplication. Prove that, with the new operations $\oplus$ and $\odot$, $\mathbb{Z}$ is an integral domain.
First, we must show that $\mathbb{Z}$ is, in fact, a ring with these operations. Let's check all the axioms:

1. [**Closure Under Addition**] Obviously if $a, b \in \mathbb{Z}$, then $a + b - 1 \in \mathbb{Z}$ since $\mathbb{Z}$ is a ring with respect to regular addition.

2. [**Associativity of Addition**] $(a \oplus b) \oplus c = (a+b-1)+c-1 = a+b+c-2 = a+(b+c-1)-1 = a \oplus (b \oplus c)$

3. [**Commutativity of Addition**] $a \oplus b = a + b - 1 = b + a - 1 = b \oplus a$

4. [**Zero Element**] It turns out that 1 is the additive identity, since for any $a \in \mathbb{Z}$, we have

$$a \oplus 1 = a + 1 - 1 = a = 1 + a - 1 = 1 \oplus a$$

5. [**Existence of Negatives**] For any $a \in \mathbb{Z}$ its additive inverse is $2 - a$ (with the usual subtraction), since

$$a \oplus (2 - a) = a + 2 - a - 1 = 1$$

6. [**Closure Under Multiplication**] Again, this follows from the fact that $\mathbb{Z}$ is a ring under the regular operations, since $a \odot b = a + b - ab$ is an integer if a and b are.

7. [**Associativity of Muliplication**] For any $a, b, c \in \mathbb{Z}$, we have

$$\begin{aligned}
a \odot (b \odot c) &= a + (b \odot c) - a(b \odot c) \\
&= a + (b + c - bc) - a(b + c - bc) \\
&= a + b + c - (ab + bc + ac) - abc \\
&= (a + b - ab) + c - (a + b - ab)c \\
&= (a \odot b) + c - (a \odot b)c \\
&= (a \odot b) \odot c
\end{aligned}$$

8. [**Distributivity**] For any $a, b, c \in \mathbb{Z}$, we have

$$\begin{aligned}
a \odot (b \oplus c) &= a + (b \oplus c) - a(b \oplus c) \\
&= a + (b + c - 1) - a(b + c - 1) \\
&= a + b + c - ab - ac + a - 1 \\
&= 2a + b + c - a(b + c) - 1
\end{aligned}$$

On the other hand:

$$\begin{aligned}(a \odot b) \oplus (a \odot c) &= (a \odot b) + (a \odot c) - 1 \\ &= a + b - ab + a + c - ac - 1 \\ &= 2a + b + c - a(b + c) - 1\end{aligned}$$

This verifies left-distributivity. Now, to check right-distributivity:

$$\begin{aligned}(a \oplus b) \odot c &= (a \oplus b) + c - (a \oplus b)c \\ &= (a + b - 1) + c - (a + b - 1)c \\ &= a + b + c - 1 - ac - bc + c \\ &= a + b + 2c - (a + b)c - 1\end{aligned}$$

On the other hand, we have

$$\begin{aligned}(a \odot c) \oplus (b \odot c) &= (a \odot c) + (b \odot c) - 1 \\ &= (a + c - ac) + (b + c - bc) - 1 \\ &= a + b + 2c - (a + b)c - 1\end{aligned}$$

All of the axioms have been verified, so we conclude that $\mathbb{Z}$ is a ring with these operations. To show that it is an integral domain, we must also show that it is commutative, has a multiplicative identity, and has no zero divisors. First, multiplication is commutative, since

$$a \odot b = a + b - ab = b + a - ba = b \odot a$$

The mulitplicative identity element is 0, since

$$a \odot 0 = a + 0 - a \cdot 0 = a = 0 + a - 0 \cdot a = 0 \odot a$$

Suppose that $a \odot b = 1$ (the additive identity/zero element). Then this means that

$$\begin{aligned}a \odot b &= 1 \\ a + b - ab &= 1\end{aligned}$$

From this you can derivat that $a(1 - b) = 1 - b$ and that $b(1 - a) = 1 - a$. From these two equations, you can see that either a or b must be equal to 1 (which is the zero element).