[Hungerford] Section 3.1, #5. Which of the following are subrings of $M_2(\mathbb{R})$ (the ring of all 2-by-2 matrices over $\mathbb{R}$)?

$$(a)\quad \left\{ \begin{pmatrix} 0 & r \\ 0 & 0 \end{pmatrix} \in M_2(\mathbb{R}) \,\middle|\, r \in \mathbb{Q} \right\}$$

$$(b)\quad \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in M_2(\mathbb{R}) \,\middle|\, a, b, c \in \mathbb{Z} \right\}$$

$$(c)\quad \left\{ \begin{pmatrix} a & a \\ b & b \end{pmatrix} \in M_2(\mathbb{R}) \,\middle|\, a, b \in \mathbb{R} \right\}$$

$$(d)\quad \left\{ \begin{pmatrix} a & 0 \\ a & 0 \end{pmatrix} \in M_2(\mathbb{R}) \,\middle|\, a \in \mathbb{R} \right\}$$

$$(e)\quad \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \in M_2(\mathbb{R}) \,\middle|\, a \in \mathbb{R} \right\}$$

$$(f)\quad \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \in M_2(\mathbb{R}) \,\middle|\, a \in \mathbb{R} \right\}$$

Note that by the **Subring Theorem** (Theorem 2.6.2 in the notes, Theorem 3.2 in the book), we only need to check the following axioms:

- Contains zero
- Closure of addition
- Closure under negatives (additive inverses)
- Closure of multiplication

It is straightforward to see that all the examples contain the zero matrix. Addition of matrices is done coordinate-wise, and so because $\mathbb{Z}, \mathbb{Q}$, and $\mathbb{R}$ are rings, they are closed under addition (and negatives), and so the sum and negatives of matrices in each case will be of the same form. All that is left in each case is to check that the set is closed under multiplication (and whether it contains an identity element).

(a) The subset is $\left\{ \begin{pmatrix} 0 & r \\ 0 & 0 \end{pmatrix} \in M_2(\mathbb{R}) \,\middle|\, r \in \mathbb{Q} \right\}$. The product of two such matrices is:

$$\begin{pmatrix} 0 & r \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & s \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

We have already noted that the zero matrix is in this set, so it is closed under multiplication, and therefore a subring of $M_2(\mathbb{R})$. It *does not* have an identity, since the product of any two elements is zero.

(b) The subset is $\left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in M_2(\mathbb{R}) \,\middle|\, a, b, c \in \mathbb{Z} \right\}$. The product of two such matrices is:

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} = \begin{pmatrix} ax & ay + bz \\ 0 & cz \end{pmatrix}$$

Since $\mathbb{Z}$ is a ring, ax, cz, and $ay + bz$ are all integers, and so the product is of the same form. The subset is thus a subring of $M_2(\mathbb{R})$. It contains the identity matrix ($a = c = 1$ and $b = 0$).

(c) The subset is $\left\{ \begin{pmatrix} a & a \\ b & b \end{pmatrix} \in \mathrm{M}_2(\mathbb{R}) \;\middle|\; a, b \in \mathbb{R} \right\}$. The product of two such matrices is:

$$\begin{pmatrix} a & a \\ b & b \end{pmatrix} \begin{pmatrix} x & x \\ y & y \end{pmatrix} = \begin{pmatrix} a(x+y) & a(x+y) \\ b(x+y) & b(x+y) \end{pmatrix}$$

Since $\mathbb{R}$ is a ring, $a(x+y)$ and $b(x+y)$ are real numbers, and the product matrix is of the same form. So this is a subring of $\mathrm{M}_2(\mathbb{R})$. One can check that any right-identity element must be the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$. But this fails to be a left-identity element.

(d) The subset is $\left\{ \begin{pmatrix} a & 0 \\ a & 0 \end{pmatrix} \in \mathrm{M}_2(\mathbb{R}) \;\middle|\; a \in \mathbb{R} \right\}$. The product of two such matrices is:

$$\begin{pmatrix} a & 0 \\ a & 0 \end{pmatrix} \begin{pmatrix} b & 0 \\ b & 0 \end{pmatrix} = \begin{pmatrix} ab & 0 \\ ab & 0 \end{pmatrix}$$

The product ab is a real number, since $\mathbb{R}$ is a ring, and so the product matrix is of the same form. This is then a subring of $\mathrm{M}_2(\mathbb{R})$ (isomorphic to $\mathbb{R}$ itself!). It is easy to see from the formula for the product that the matrix $\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ is the multiplicative identity.

(e) The subset is $\left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \in \mathrm{M}_2(\mathbb{R}) \;\middle|\; a \in \mathbb{R} \right\}$. The product of two such matrices is:

$$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} ab & 0 \\ 0 & ab \end{pmatrix}$$

The product ab is a real number, since $\mathbb{R}$ is a ring, and so the product matrix is of the same form. This is then a subring of $\mathrm{M}_2(\mathbb{R})$ (isomorphic to $\mathbb{R}$). The regular identity matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the identity.

(f) The subset is $\left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \in \mathrm{M}_2(\mathbb{R}) \;\middle|\; a \in \mathbb{R} \right\}$. The product of two such matrices is:

$$\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} ab & 0 \\ 0 & 0 \end{pmatrix}$$

The product ab is a real number, since $\mathbb{R}$ is a ring, and so the product matrix is of the same form. This is then a subring of $\mathrm{M}_2(\mathbb{R})$ (isomorphic to $\mathbb{R}$). The matrix with $a = 1$ is the identity.

[Hungerford] Appendix D, #2. Define a relation on $\mathbb{Q}$ by: $r \sim s$ if and only if $r - s \in \mathbb{Z}$. Prove that $\sim$ is an equivalence relation.

Reflexivity: Since for any $q \in \mathbb{Q}$, we have $q - q = 0 \in \mathbb{Z}$, we get that $\sim$ is reflexive.

Symmetry: If $q, r \in \mathbb{Q}$ with $q - r = n \in \mathbb{Z}$, then $r - q = -n \in \mathbb{Z}$, so $\sim$ is symmetric.

Transitivity: If $q, r, s \in \mathbb{Q}$ with $q - r = n \in \mathbb{Z}$ and $r - s = m \in \mathbb{Z}$, then $q - s = q - r + (r - s) = n + m \in \mathbb{Z}$, and so $\sim$ is transitive.

[Hungerford] Appendix D, #13. In the set $\mathbb{R}[x]$ of all polynomials with real coefficients, define $f(x) \sim g(x)$ if and only if $f'(x) = g'(x)$, where $'$ denotes the derivative. Prove that $\sim$ is an equivalence relation on $\mathbb{R}[x]$.

Reflexivity: Certainly $f'(x) = f'(x)$, and so $f \sim f$.

Symmetry: If $f(x) \sim g(x)$, $f'(x) = g'(x)$, and so $g'(x) = f'(x)$, since $=$ is symmetric.

Transitivity: If $f(x) \sim g(x)$ and $g(x) \sim h(x)$, then we have $f'(x) = g'(x)$ and $g'(x) = h'(x)$. Since $=$ is transitive, $f'(x) = h'(x)$.