# New bounds for equiangular line sets

Wei-Hsuan Yu

Michigan State University

Aug 10, 2015

WPI Systems of Line Workshop

## Definition

If $X = \{x_1, x_2, \cdots, x_N\} \subset \mathbb{S}^{n-1}$ (unit sphere in $\mathbb{R}^n$) and $\langle x_i, x_j \rangle = a$ or $b$ for all $i \neq j$, then we call $X$ is a spherical two-distance set.

Q : What is the maximum cardinality of a spherical two-distance set?

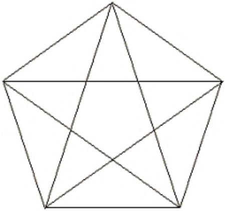

Figure: The maximum spherical two-distance set in $\mathbb{R}^2$: Pentagon.

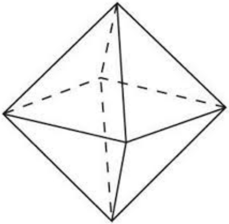

Figure: The maximum spherical two-distance set in $\mathbb{R}^3$: Octahedron.

# Musin's result

Musin used Delsarte's linear programming method to prove that

$$g(n) = \frac{n(n+1)}{2} \quad \text{if } 7 \leq n \leq 39, n \neq 22, 23$$

and $g(23) = 276$ or $277$.

# Barg and Yu 2013

We use the semidefinite programming (SDP) method showing that

$$g(n) = \frac{n(n+1)}{2}, \quad 7 \leq n \leq 93, n \neq 22, 46, 78. \tag{1}$$

In particular, $g(23) = 276$.

**Definition**

A set of lines in $\mathbb{R}^n$ is called *equiangular* if the angle between each pair of lines is the same.

- An equiangular line set can be defined as an unit vectors set $S = \{x_i\}_{i=1}^{M}$ such that $|\langle x_i, x_j \rangle| = c, 1 \leq i < j \leq M$ for some $c > 0$.
- A equiangular line set can be defined as a spherical two-distance set with inner product value $c$ and $-c$.
- Question : What is the maximum cardinality of an equiangular line set in $\mathbb{R}^n$?

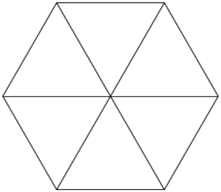

Figure: Maximum equiangular lines in $\mathbb{R}^2$: 3 lines through opposite vertices of a regular hexagon.

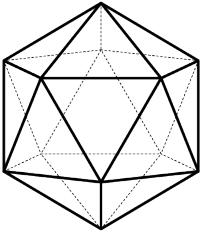

Figure: Maximum equiangular lines in $\mathbb{R}^3$: 6 lines through opposite vertices of the icosahedron.

# Known results

Let $M(n)$ denote the maximum size of an equiangular line set in $\mathbb{R}^n$

- Hanntjes found $M(n)$ for $n = 2$ and $3$ in $1948$.
- Van Lint and Seidel found the largest number of equiangular lines for $4 \leq n \leq 7$ in 1966.
- Lemmens and Seidel used linear-algebraic methods to determine $M(n)$ for most values of $n$ in the region $8 \leq n \leq 23$ in 1973.

| $n$ | $M(n)$ | $1/\alpha$ | | $n$ | $M(n)$ | $1/\alpha$ |
|---|---|---|---|---|---|---|
| 2 | 3 | 2 | | 17 | 48-50 | 5 |
| 3 | 6 | $\sqrt{5}$ | | 18 | 48-61 | 5 |
| 4 | 6 | 3; $\sqrt{5}$ | | 19 | 72-76 | 5 |
| 5 | 10 | 3 | | 20 | 90-96 | 5 |
| 6 | 16 | 3 | | 21 | 126 | 5 |
| $7 \leq n \leq 13$ | 28 | 3 | | 22 | 176 | 5 |
| 14 | 28-29 | 3; 5 | | 23 | 276 | 5 |
| 15 | 36 | 5 | | $24 \leq n \leq 42$ | $\geq$ 276 | 5 |
| 16 | 40-41 | 5 | | 43 | $\geq$ 344 | 7 |

Table: Known bounds on $M(n)$ in small dimensions

# Our results

**Theorem (Barg and Yu 2014)**

*We use the semidefinite programming (SDP) method to show that* $M(n) = 276$ *for* $24 \leq n \leq 41$ *and* $M(43) = 344$.

| $n$ | $M(n)$ | SDP bound | | $n$ | $M(n)$ | SDP bound |
|---|---|---|---|---|---|---|
| 3 | 6 | 6 | | 18 | 48-61 | 61 |
| 4 | 6 | 6 | | 19 | 72-76 | 76 |
| 5 | 10 | 10 | | 20 | 90-96 | 96 |
| 6 | 16 | 16 | | 21 | 126 | 126 |
| $7 \leq n \leq 13$ | 28 | 28 | | 22 | 176 | 176 |
| 14 | 28-29 | 30 | | 23 | 276 | 276 |
| 15 | 36 | 36 | | $24 \leq n \leq 41$ | 276 | 276 |
| 16 | 40-41 | 42 | | 42 | $\geq 276$ | 288 |
| 17 | 48-50 | 51 | | 43 | 344 | 344 |

Table: Bounds on $M(n)$ including new results

# Gegenbauer polynomials

Let $G_k^{(n)}(t), k = 0, 1, \ldots$ denote the Gegenbauer polynomials of degree $k$. They are defined recursively as follows: $G_0^{(n)} \equiv 1, G_1^{(n)}(t) = t$, and

$$G_k^{(n)}(t) = \frac{(2k+n-4)tG_{k-1}^{(n)}(t) - (k-1)G_{k-2}^{(n)}(t)}{k+n-3}, \quad k \geq 2.$$

Define a matrix $Y_k^n(u,v,t), k \geq 0$

$$(Y_k^n(u,v,t))_{ij} = u^i v^j ((1-u^2)(1-v^2))^{k/2} G_k^{(n-1)}\Big(\frac{t-uv}{\sqrt{(1-u^2)(1-v^2)}}\Big)$$

and a matrix $S_k^n(u,v,t)$ by setting

$$S_k^n(u,v,t) = \frac{1}{6} \sum_{\sigma \in S_3} Y_k^n(\sigma(u,v,t)), \qquad (2)$$

S. Bochner (1941) proved that

$$\sum_{(x,y,z) \in C^3} S_k^n(x \cdot y, x \cdot z, y \cdot z) \succeq 0.$$

I.J. Schoenberg (1942) proved that

$$\sum_{x,y \in C^2} G_k^n(\langle x, y \rangle) \geq 0.$$

# Semidefinite programming

$$\min c^T x$$

$$\text{subject to} \quad F_0 + \sum_{i=1}^{m} F_i x_i \succeq 0$$

where $c, x \in \mathbb{R}^m$ and $F_i$ is an $n$ by $n$ symmetric matrix $\forall i$. The sign "$\succeq$" means that the matrix is positive semidefinite.

### Theorem (Gerzon, absolute bounds)

*If there are* $M$ *equiangular lines in* $\mathbb{R}^n$*, then* $M \leq \frac{n(n+1)}{2}$.

Gerzon bounds are known to be attained only for $n = 2, 3, 7,$ and $23$.

### Theorem (Neumann)

*If there are* $M$ *equiangular lines in* $\mathbb{R}^n$ *with angle* $\arccos \alpha$ *and* $M > 2n$*, then* $1/\alpha$ *is an odd integer.*

### Theorem (Lemmens and Seidel)

$M_{1/3}(n) = 2(n-1)$ *for* $n \geq 16$*, where* $M_\alpha(n)$ *is the maximum size of an equiangular line set when the value of the angle is* $\arccos \alpha$.

### Theorem (Relative bounds)

$$M_\alpha(n) \leq \frac{n(1-\alpha^2)}{1-n\alpha^2} \qquad (3)$$

*valid for all* $\alpha$ *such that the denominator is positive.*

# SDP bound

**Theorem**

*Let $\mathcal{C}$ be an equiangular line set with inner product values either $a$ or $-a$. Let $p$ be a positive integer. The cardinality $|\mathcal{C}|$ is bounded above by the solution of the following semi-definite programming problem :*

$$1 + \frac{1}{3}\max(x_1 + x_2) \tag{4}$$

*subject to*

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}(x_1 + x_2) + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}(x_3 + x_4 + x_5 + x_6) \succeq 0 \tag{5}$$

$$\begin{aligned} &S_k^n(1,1,1) + S_k^n(a,a,1)x_1 + S_k^n(-a,-a,1)x_2 + S_k^n(a,a,a)x_3 \\ &+ S_k^n(a,a,-a)x_4 + S_k^n(a,-a,-a)x_5 + S_k^n(-a,-a,-a)x_6 \succeq 0 \end{aligned} \tag{6}$$

$$3 + G_k^{(n)}(a)x_1 + G_k^{(n)}(-a)x_2 \geq 0, \tag{7}$$

*where $k = 0, 1, \cdots, p$ and $x_j \geq 0$, $j = 1, \cdots, 6$.*

# SDP bound table

| $n$ | 1/5 | 1/7 | 1/9 | 1/11 | 1/13 | 1/15 | max | Gerzon | angle |
|---|---|---|---|---|---|---|---|---|---|
| 22 | 176 | 39 | 29 | 26 | 25 | 24 | 176 | 253 | 1/5 |
| 23 | 276 | 42 | 31 | 28 | 26 | 25 | 276 | 276 | 1/5 |
| 24 | 276 | 46 | 33 | 29 | 27 | 26 | 276 | 300 | 1/5 |
| 25 | 276 | 50 | 35 | 31 | 29 | 28 | 276 | 325 | 1/5 |
| 26 | 276 | 54 | 37 | 32 | 30 | 29 | 276 | 351 | 1/5 |
| 27 | 276 | 58 | 40 | 34 | 31 | 30 | 276 | 378 | 1/5 |
| 28 | 276 | 64 | 42 | 36 | 33 | 31 | 276 | 406 | 1/5 |
| 29 | 276 | 69 | 44 | 37 | 34 | 33 | 276 | 435 | 1/5 |
| 30 | 276 | 75 | 47 | 39 | 36 | 34 | 276 | 465 | 1/5 |
| 31 | 276 | 82 | 49 | 41 | 37 | 35 | 276 | 496 | 1/5 |
| 32 | 276 | 90 | 52 | 43 | 39 | 37 | 276 | 528 | 1/5 |
| 33 | 276 | 99 | 55 | 45 | 40 | 38 | 276 | 561 | 1/5 |
| 34 | 276 | 108 | 57 | 46 | 42 | 39 | 276 | 595 | 1/5 |
| 35 | 276 | 120 | 60 | 48 | 43 | 41 | 276 | 630 | 1/5 |
| 36 | 276 | 132 | 64 | 50 | 45 | 42 | 276 | 666 | 1/5 |
| 37 | 276 | 148 | 67 | 52 | 47 | 44 | 276 | 703 | 1/5 |
| 38 | 276 | 165 | 70 | 54 | 48 | 45 | 276 | 741 | 1/5 |
| 39 | 276 | 187 | 74 | 57 | 50 | 46 | 276 | 780 | 1/5 |
| 40 | 276 | 213 | 78 | 59 | 52 | 48 | 276 | 820 | 1/5 |
| 41 | 276 | 246 | 82 | 61 | 53 | 49 | 276 | 861 | 1/5 |
| 42 | 276 | 288 | 86 | 63 | 55 | 51 | 288 | 903 | 1/7 |
| 43 | 276 | 344 | 90 | 66 | 57 | 52 | 344 | 946 | 1/7 |
| 44 | 276 | 422 | 95 | 68 | 59 | 54 | 422 | 990 | 1/7 |
| 45 | 276 | 540 | 100 | 71 | 60 | 56 | 540 | 1035 | 1/7 |
| 46 | 276 | 736 | 105 | 73 | 62 | 57 | 736 | 1081 | 1/7 |
| 47 | 276 | 1128 | 110 | 76 | 64 | 59 | 1128 | 1128 | 1/7 |
| 48 | 276 | 1128 | 116 | 78 | 66 | 60 | 1128 | 1176 | 1/7 |
| 49 | 276 | 1128 | 122 | 81 | 68 | 62 | 1128 | 1225 | 1/7 |
| 50 | 276 | 1128 | 129 | 84 | 70 | 64 | 1128 | 1275 | 1/7 |
| 51 | 276 | 1128 | 136 | 87 | 72 | 65 | 1128 | 1326 | 1/7 |
| 52 | 276 | 1128 | 143 | 90 | 74 | 67 | 1128 | 1378 | 1/7 |
| 53 | 276 | 1128 | 151 | 93 | 76 | 69 | 1128 | 1431 | 1/7 |
| 54 | 276 | 1128 | 160 | 96 | 78 | 70 | 1128 | 1485 | 1/7 |
| 55 | 276 | 1128 | 169 | 100 | 81 | 72 | 1128 | 1540 | 1/7 |
| 56 | 276 | 1128 | 179 | 103 | 83 | 74 | 1128 | 1596 | 1/7 |
| 57 | 276 | 1128 | 190 | 106 | 85 | 76 | 1128 | 1653 | 1/7 |
| 58 | 276 | 1128 | 201 | 110 | 87 | 77 | 1128 | 1711 | 1/7 |
| 59 | 276 | 1128 | 214 | 114 | 90 | 79 | 1128 | 1770 | 1/7 |
| 60 | 276 | 1128 | 228 | 118 | 92 | 81 | 1128 | 1830 | 1/7 |
| 61 | 279 | 1128 | 244 | 122 | 94 | 83 | 1128 | 1891 | 1/7 |
| 62 | 290 | 1128 | 261 | 126 | 97 | 85 | 1128 | 1953 | 1/7 |
| 63 | 301 | 1128 | 280 | 130 | 99 | 87 | 1128 | 2016 | 1/7 |
| 64 | 313 | 1128 | 301 | 134 | 102 | 89 | 1128 | 2080 | 1/7 |

# SDP bound table (Cont.)

| $n$ | 1/5 | 1/7 | 1/9 | 1/11 | 1/13 | 1/15 | max | Gerzon | angle |
|---|---|---|---|---|---|---|---|---|---|
| 65 | 326 | 1128 | 325 | 139 | 105 | 91 | 1128 | 2145 | 1/7 |
| 66 | 339 | 1128 | 352 | 144 | 107 | 92 | 1128 | 2211 | 1/7 |
| 67 | 353 | 1128 | 382 | 148 | 110 | 94 | 1128 | 2278 | 1/7 |
| 68 | 367 | 1128 | 418 | 153 | 113 | 97 | 1128 | 2346 | 1/7 |
| 69 | 382 | 1128 | 460 | 159 | 115 | 99 | 1128 | 2415 | 1/7 |
| 70 | 398 | 1128 | 509 | 164 | 118 | 101 | 1128 | 2485 | 1/7 |
| 71 | 416 | 1128 | 568 | 170 | 121 | 103 | 1128 | 2556 | 1/7 |
| 72 | 434 | 1128 | 640 | 176 | 124 | 105 | 1128 | 2628 | 1/7 |
| 73 | 453 | 1128 | 730 | 182 | 127 | 107 | 1128 | 2701 | 1/7 |
| 74 | 473 | 1128 | 845 | 188 | 130 | 109 | 1128 | 2775 | 1/7 |
| 75 | 494 | 1128 | 1000 | 195 | 134 | 112 | 1128 | 2850 | 1/7 |
| 76 | 517 | 1128 | 1216 | 202 | 137 | 114 | 1216 | 2926 | 1/9 |
| 77 | 542 | 1128 | 1540 | 210 | 140 | 116 | 1540 | 3003 | 1/9 |
| 78 | 568 | 1128 | 2080 | 217 | 144 | 118 | 2080 | 3081 | 1/9 |
| 79 | 596 | 1128 | 3160 | 225 | 147 | 121 | 3160 | 3160 | 1/9 |
| 80 | 626 | 1128 | 3160 | 234 | 151 | 123 | 3160 | 3240 | 1/9 |
| 81 | 658 | 1128 | 3160 | 243 | 154 | 126 | 3160 | 3321 | 1/9 |
| 82 | 693 | 1128 | 3160 | 252 | 158 | 128 | 3160 | 3403 | 1/9 |
| 83 | 731 | 1128 | 3160 | 262 | 162 | 130 | 3160 | 3486 | 1/9 |
| 84 | 772 | 1128 | 3160 | 272 | 166 | 133 | 3160 | 3570 | 1/9 |
| 85 | 816 | 1128 | 3160 | 283 | 170 | 136 | 3160 | 3655 | 1/9 |
| 86 | 866 | 1128 | 3160 | 294 | 174 | 138 | 3160 | 3741 | 1/9 |
| 87 | 920 | 1128 | 3160 | 307 | 178 | 141 | 3160 | 3828 | 1/9 |
| 88 | 979 | 1128 | 3160 | 320 | 182 | 143 | 3160 | 3916 | 1/9 |
| 89 | 1046 | 1128 | 3160 | 333 | 186 | 146 | 3160 | 4005 | 1/9 |
| 90 | 1120 | 1128 | 3160 | 348 | 191 | 149 | 3160 | 4095 | 1/9 |
| 91 | 1203 | 1128 | 3160 | 364 | 196 | 152 | 3160 | 4186 | 1/9 |
| 92 | 1298 | 1128 | 3160 | 380 | 200 | 154 | 3160 | 4278 | 1/9 |
| 93 | 1406 | 1128 | 3160 | 398 | 205 | 157 | 3160 | 4371 | 1/9 |
| 94 | 1515 | 1128 | 3160 | 417 | 210 | 160 | 3160 | 4465 | 1/9 |
| 95 | 1556 | 1128 | 3160 | 438 | 215 | 163 | 3160 | 4560 | 1/9 |
| 96 | 1599 | 1128 | 3160 | 460 | 220 | 166 | 3160 | 4656 | 1/9 |
| 97 | 1644 | 1128 | 3160 | 485 | 226 | 169 | 3160 | 4753 | 1/9 |
| 98 | 1691 | 1128 | 3160 | 511 | 231 | 172 | 3160 | 4851 | 1/9 |
| 99 | 1739 | 1128 | 3160 | 540 | 237 | 176 | 3160 | 4950 | 1/9 |
| 100 | 1790 | 1128 | 3160 | 571 | 243 | 179 | 3160 | 5050 | 1/9 |
| 101 | 1842 | 1128 | 3160 | 606 | 249 | 182 | 3160 | 5151 | 1/9 |
| 102 | 1897 | 1128 | 3160 | 644 | 255 | 185 | 3160 | 5253 | 1/9 |
| 103 | 1954 | 1128 | 3160 | 686 | 262 | 189 | 3160 | 5356 | 1/9 |
| 104 | 2014 | 1128 | 3160 | 734 | 268 | 192 | 3160 | 5460 | 1/9 |
| 105 | 2077 | 1128 | 3160 | 787 | 275 | 196 | 3160 | 5565 | 1/9 |
| 106 | 2142 | 1128 | 3160 | 848 | 282 | 199 | 3160 | 5671 | 1/9 |
| 107 | 2211 | 1128 | 3160 | 917 | 289 | 203 | 3160 | 5778 | 1/9 |

# SDP bound table (Cont.)

| $n$ | 1/5 | 1/7 | 1/9 | 1/11 | 1/13 | 1/15 | max | Gerzon | angle |
|---|---|---|---|---|---|---|---|---|---|
| 108 | 2282 | 1128 | 3160 | 997 | 297 | 206 | 3160 | 5886 | 1/9 |
| 109 | 2358 | 1128 | 3160 | 1090 | 305 | 210 | 3160 | 5995 | 1/9 |
| 110 | 2437 | 1128 | 3160 | 1200 | 313 | 214 | 3160 | 6105 | 1/9 |
| 111 | 2521 | 1128 | 3160 | 1332 | 321 | 218 | 3160 | 6216 | 1/9 |
| 112 | 2609 | 1128 | 3160 | 1493 | 330 | 222 | 3160 | 6328 | 1/9 |
| 113 | 2702 | 1128 | 3160 | 1695 | 339 | 226 | 3160 | 6441 | 1/9 |
| 114 | 2800 | 1128 | 3160 | 1954 | 348 | 230 | 3160 | 6555 | 1/9 |
| 115 | 2904 | 1128 | 3160 | 2300 | 357 | 234 | 3160 | 6670 | 1/9 |
| 116 | 3015 | 1128 | 3160 | 2784 | 367 | 238 | 3160 | 6786 | 1/9 |
| 117 | 3132 | 1128 | 3160 | 3510 | 378 | 242 | 3510 | 6903 | 1/11 |
| 118 | 3257 | 1128 | 3160 | 4720 | 388 | 247 | 4720 | 7021 | 1/11 |
| 119 | 3390 | 1128 | 3160 | 7140 | 399 | 251 | 7140 | 7140 | 1/11 |
| 120 | 3532 | 1128 | 3160 | 7140 | 411 | 256 | 7140 | 7260 | 1/11 |
| 121 | 3684 | 1128 | 3160 | 7140 | 423 | 260 | 7140 | 7381 | 1/11 |
| 122 | 3848 | 1128 | 3160 | 7140 | 436 | 265 | 7140 | 7503 | 1/11 |
| 123 | 4024 | 1128 | 3160 | 7140 | 449 | 270 | 7140 | 7626 | 1/11 |
| 124 | 4214 | 1128 | 3160 | 7140 | 462 | 275 | 7140 | 7750 | 1/11 |
| 125 | 4419 | 1128 | 3160 | 7140 | 477 | 280 | 7140 | 7875 | 1/11 |
| 126 | 4643 | 1128 | 3160 | 7140 | 492 | 285 | 7140 | 8001 | 1/11 |
| 127 | 4887 | 1128 | 3160 | 7140 | 508 | 290 | 7140 | 8128 | 1/11 |
| 128 | 5153 | 1128 | 3160 | 7140 | 524 | 295 | 7140 | 8256 | 1/11 |
| 129 | 5447 | 1128 | 3160 | 7140 | 541 | 301 | 7140 | 8385 | 1/11 |
| 130 | 5770 | 1128 | 3160 | 7140 | 560 | 306 | 7140 | 8515 | 1/11 |
| 131 | 6130 | 1128 | 3160 | 7140 | 579 | 312 | 7140 | 8646 | 1/11 |
| 132 | 6531 | 1130 | 3160 | 7140 | 599 | 317 | 7140 | 8778 | 1/11 |
| 133 | 6982 | 1158 | 3160 | 7140 | 620 | 323 | 7140 | 8911 | 1/11 |
| 134 | 7493 | 1187 | 3160 | 7140 | 643 | 329 | 7493 | 9045 | 1/5 |
| 135 | 8075 | 1218 | 3160 | 7140 | 667 | 336 | 8075 | 9180 | 1/5 |
| 136 | 8747 | 1249 | 3160 | 7140 | 692 | 342 | 8747 | 9316 | 1/5 |
| *137 | 9528 | 1282 | 3160 | 7140 | 719 | 348 | 9528 | 9453 | 1/5 |
| *138 | 10450 | 1315 | 3160 | 7140 | 747 | 355 | 10450 | 9591 | 1/5 |
| *139 | 11553 | 1350 | 3160 | 7140 | 778 | 362 | 11553 | 9730 | 1/5 |

# Our results

**Theorem (Barg and Yu)**

*We use the semidefinite programming method to show that* $M(n) = 276$ *for* $24 \leq n \leq 41$ *and* $M(43) = 344$ *and we get tighter upper bounds for* $M(n)$ *when* $n \leq 136$.

# Spherical $t$-design

**Definition (Delsarte at el. 77')**

Let $t$ be a natural number. A finite subset $X$ of the unit sphere $S^{n-1}$ is called a *spherical $t$-design* if, for any polynomial $f(x) = f(x_1, x_2, \ldots, x_n)$ of degree at most $t$, the following equality holds :

$$\frac{1}{|S^{n-1}|}\int_{S^{n-1}} f(x)d\sigma(x) = \frac{1}{|X|}\sum_{x\in X} f(x). \tag{8}$$

The set $X$ is a spherical design if

$$\sum_{x\in X} f(x) = 0 \quad \forall f(x) \in \mathrm{Harm}_j(\mathbb{R}^n), 1 \leq j \leq t. \tag{9}$$

Delsarte at el 77' proved that the cardinality of a spherical $t$-design $X$ is bounded below by

$$|X| \geq \binom{n+e-1}{n-1} + \binom{n+e-2}{n-1}, \quad |X| \geq 2\binom{n+e-1}{n-1}$$

for $t = 2e$ and $t = 2e+1$.

The spherical $t$-design is called tight if any one of these bound is attained.

# Spherical designs of harmonic index $t$

**Definition**

A spherical design of harmonic index $t$ is a finite subset $X \subset S^{n-1}$ such that

$$\sum_{x \in X} f(x) = 0 \quad \forall f(x) \in \text{Harm}_t(\mathbb{R}^n). \tag{10}$$

1. A lower bound of the size of a spherical design of harmonic index $t$ has been derived by Bannai, Okuda and Tagami 13'.
2. A lower bound of a spherical design of harmonic index 4 in $S^{n-1}$ is $\frac{(n+1)(n+2)}{6}$. If the lower bound on the cardinality is attained, we call it a tight spherical design of harmonic index 4.

# Tight spherical designs of harmonic index 4

1. Bannai, Okuda and Tagami 13' showd that a tight design of harmonic index $4$ has cardinality $\frac{(n+1)(n+2)}{6}$ and give arise to an equiangular line set in $\mathbb{R}^n$ with angle $\arccos \sqrt{\frac{3}{n+4}}$.
2. If tight harmonic index $4$-designs on $S^{n-1}$ exist, then $n = 2$ or $n$ must be of the form $n = 3(2k-1)^2 - 4 = 12k^2 - 12k - 1$ for some integer $k \geq 3$, and the configuration is a set of equiangular lines with the angle $\arccos \frac{1}{2k-1}$ in the $(12k^2 - 12k - 1)$-dimensional Euclidean space.

# New relative bounds

## Theorem (Okuda, Yu 2014)

*Let $n \geq 3$. Then the following holds:*

1. $$M_\alpha(n) \leq 2 + (n-2)\frac{(1-\alpha)^3}{\alpha((n-2)\alpha^2+6\alpha-3)}$$

   *for each $\alpha \in (0,1)$ with*

   $$(1-\alpha)^3(-(n-2)\alpha^2+6\alpha+3) \geq (1+\alpha)^3((n-2)\alpha^2+6\alpha-3) \geq 0.$$

2. $$M_\alpha(n) \leq 2 + (n-2)\frac{(1+\alpha)^3}{\alpha(-(n-2)\alpha^2+6\alpha+3)}$$

   *for each $\alpha \in (0,1)$ with*

   $$(1+\alpha)^3((n-2)\alpha^2+6\alpha-3) \geq (1-\alpha)^3(-(n-2)\alpha^2+6\alpha+3) \geq 0,$$

*where $M_\alpha(n)$ is the maximum size of an equiangular line set in $\mathbb{R}^n$ and the value of the angle is $\arccos\alpha$.*

## Example

Let us consider the cases where $n_k := 3(2k-1)^2 - 4$ and $\alpha_k := 1/(2k-1)$ for an integer $k \geq 2$. Note that in such cases, Lemmens–Seidel's relative bound (3) does not work since $1 - n_k\alpha_k^2 = -2(4k^2-4k-1)/(2k-1)^2 < 0$. One can compute that

$$(1-\alpha_k)^3(-(n_k-2)\alpha_k^2 + 6\alpha_k + 3) = \frac{96(k-1)^3 k}{(2k-1)^5},$$
$$(1+\alpha_k)^3((n_k-2)\alpha_k^2 + 6\alpha_k - 3) = \frac{96(k-1)k^3}{(2k-1)^5},$$

and hence

$$(1+\alpha_k)^3((n_k-2)\alpha_k^2+6\alpha_k-3) \geq (1-\alpha_k)^3(-(n_k-2)\alpha_k^2+6\alpha_k+3) \geq 0.$$

Therefore, by our new relative bounds Theorem, we have

$$M_{\alpha_k}(n_k) \leq 2 + (n-2)\frac{(1+\alpha)^3}{\alpha(-(n-2)\alpha^2+6\alpha+3)}$$
$$= 2(k-1)(4k^3-k-1).$$

Observe that

$$\frac{(n_k+1)(n_k+2)}{6} - 2(k-1)(4k^3-k-1) = 2(k-1)(2k-1)(4k^2-4k-1) > 0$$

when $k \geq 2$.

Therefore, we have $\frac{(n_k+1)(n_k+2)}{6} > 2(k-1)(4k^3-k-1)$, when $k \geq 2$.

**Corollary**

*For each $n \geq 3$, there does not exist a tight harmonic index $4$-design on $S^{n-1}$.*

# Spherical designs of harmonic index $T$

**Definition**

Let $T$ be a subset of $\mathbb{N}$. A spherical design of harmonic index $T$ is a finite subset $X \subset S^{n-1}$ such that

$$\sum_{x \in X} f(x) = 0 \quad \forall f(x) \in \mathrm{Harm}_t(\mathbb{R}^n), \text{with } t \in T.$$

Y. Zhu, E. Bannai, E. Bannai, K-T. Kim and W-H. Yu completely discuss the non-existence of tight spherical designs of harmonic index 6, 8, asymptotic results for all even numbers, {6,2},{6,4},{8,6},{8,4},{8,2}, {10,6,2},{12,8,4}. The paper is vailable on arxiv:1507.05373.

# Open problems

1. $M(14) = 28$ or 29. $M(16) = 40$ or 41. Can we determine them?
2. The constructions and upper bounds for complex equiangular lines.
3. How to prove that $M_\alpha(n)$ has long stable ranges in general?

$$\begin{array}{lll} \frac{1}{5} & 23-60 & 276 \\ \frac{1}{7} & 47-131 & 1128 \\ \frac{1}{9} & 79-227 & 3160 \\ \frac{1}{11} & 119-349 & 7140 \\ \ldots & & \end{array}$$