# Tailoring Wavelets for Chaos Control

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Chaos is a class of ubiquitous phenomena and controlling chaos is of great interest and importance. In this Letter, we introduce wavelet controlled dynamics as a new paradigm of dynamical control. We find that by modifying a tiny fraction of the wavelet subspaces of a coupling matrix, we could dramatically enhance the transverse stability of the synchronous manifold of a chaotic system. Wavelet controlled Hopf bifurcation from chaos is observed. Our approach provides a robust strategy for controlling chaos and other dynamical systems in nature.

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Chaos is omnipresent in nature. For a nonlinear system of more than 2 degrees of freedom, it is chaotic whenever its evolution sensitively depends on the initial conditions. Mathematically, there must be an infinite number of unstable periodic orbits embedded in the underlying chaotic set and the dynamics in the chaotic attractor is ergodic. Physically, chaos can be found in nonlinear optics (laser), chemistry (Belouzov-Zhabotinski reaction), electronics (Chua-Matsumoto circuit), fluid dynamics (Rayleigh-Bénard convention), meteorology, solar system, and the heart and brain of living organisms. As chaos is intrinsically unpredictable and its trajectories diverge exponentially in the course of time evolution, controlling chaos is apparently of great interest and importance. Ott, Grebogi, and Yorke [1] proposed a successful technique to control low-dimensional chaos. The basic idea is to take advantage of the sensitivity to small disturbances of chaotic systems to stabilize the system in the neighborhood of a desirable unstable periodic orbit naturally embedded in the chaotic motion. Pyragas [2] proposed a more efficient method which makes use of a time-delayed feedback to some dynamical variables of the system. Control of spatiotemporal chaos in partial differential equations was also considered [3,4]. As an alternative control means, chaos synchronization was pioneered by Pecora and Carroll [5]. The theory and application of chaotic synchronization has been extensively studied [6] in various research directions, for instance, electronic circuits, laser experiment, secure communication, biological and chemical systems, shock capturing [7], and wake turbulence [8]. Synchronous stability was studied by Pecora and Carroll [9] and Yang *et al.* [10]. The stability of the synchronous state can be understood from the eigenvalue distribution of the coupling matrix of a nonlinear system. However, possible wavelet subspace control of chaos and chaos synchronization has not been addressed yet.

The theory of wavelets is a new branch of mathematics developed in the last two decades and has had tremendous success and impact on signal/image processing, data compression, computer vision, telecommunication, and a variety of other science and engineering disciplines [11,12]. Mathematically, wavelets are sets of $L^2$ functions generated from a single function by translation and dilation. Compared to the usual orthogonal $L^2$ bases, such as the Fourier transform, wavelets often have much better properties for expanding a function of a physical origin. Some of the most important features of wavelets include time-frequency localization and multiresolution analysis. Physically, wavelet transform can split a function into different frequency bands or components so that each component can be studied with a resolution matched to its scale, thus providing excellent frequency and spatial resolution, and achieving high computational efficiency. Moreover, we can devise a wavelet system for representing physical information at various levels of details, leading to the so-called mathematical microscopy. For many physical systems, due to the multiscale nature, the wavelet multiresolution theory provides perhaps some of the most appropriate analysis tools. Application of wavelets to nonlinear dynamics has been widely studied, and successful examples can be found in time series analysis [13], prediction of low-dimensional dynamics [14], multiscale analysis of turbulence [15–17], spatial hierarchies in measles epidemics [18], North Atlantic oscillation dynamics [19], magnetic flux on the Sun [20], human heartbeat dynamics [21], and pattern characterization [22]. However, to our knowledge, the use of wavelets in all the previous work in the nonlinear dynamics is limited to analysis and/or characterization. The use of wavelets as the basis in the direct control of the system dynamics has not been exploited. The objective of this Letter is to introduce a paradigm of chaos control and synchronization by using wavelets. It is found that the modification of a tiny fraction of wavelet subspaces of a coupling matrix could lead to dramatic change in chaos synchronizing properties.

Let us consider a coupled nonlinear system of $N$ chaotic oscillators

  

$$\frac{d\mathbf{u}}{dt} = (FI + \varepsilon A)\mathbf{u}, \qquad \mathbf{u} = (u_1, u_2, \ldots, u_N)^T, \tag{1}$$

where $Fu_i = f(u_i)$ is a nonlinear function of the $i$th oscillator, which has a state function $u_i \in [0, \infty) \times \mathbb{R}^n$, $I$ is a unit matrix, $\varepsilon$ is a coupling strength, and $A$ is a coupling matrix having the periodic structure at the boundaries.

The synchronous manifold of the chaotic system, which is a subspace of the original coupled system, Eq. (1), can be studied by setting $u_1(t) = u_2(t) = \cdots = u_N(t) = s(t)$, where the chaotic solution $s(t)$ satisfies the single oscillator equation $ds/dt = f[s(t)]$. The stability property of the synchronous manifold can be studied in the space of difference variables $\delta u_i(t) = u_i(t) - s(t)$, which are governed by [5,6,9,10]

$$\frac{d\boldsymbol{\delta}\mathbf{u}}{dt} = (DfI + \varepsilon A)\boldsymbol{\delta}\mathbf{u}, \qquad \boldsymbol{\delta}\mathbf{u} = (\delta u_1, \delta u_2, \ldots, \delta u_N)^T,$$
$$Df = \frac{df(u)}{du}. \tag{2}$$

It turns out that the eigenvalue spectrum of the matrix $A$ determines the stability of the coupled chaotic system. The largest eigenvalue $\lambda_1$ is equal to 0, which governs the motion on the synchronized manifold, and all of other eigenvalues $\lambda_i$ $(i \neq 1)$ control the transverse stability of the chaotic synchronous state. The stability condition can be given by $L_{\max} + \varepsilon\lambda_2 \leq 0$, where $L_{\max} > 0$ is the largest Lyapunov exponent of a single chaotic oscillator. As a consequence, the second largest eigenvalue $\lambda_2$ is dominant in controlling the stability of chaotic synchronization, and the critical coupling strength $\varepsilon_c$ can be determined in terms of $\lambda_2$,

$$\varepsilon_c = \frac{L_{\max}}{-\lambda_2}. \tag{3}$$

For the nearest neighbor coupling, the eigenvalue spectrum of an appropriately normalized $A$ is given by [9,10] $\lambda_i = -4\sin^2\frac{\pi(i-1)}{N}$, $i = 1, 2, \ldots, N$. In general, a larger coupling width gives a smaller nonzero eigenvalue $\lambda_2$. However, very little is known about the reconstruction of matrix $A$ and its eigenvalue reduction for achieving efficient control of chaos synchronization. In the rest of this Letter, we introduce a wavelet approach to enhance synchronous stability and chaos control.

We consider a two-dimensional (2D) multiscale analysis. The 2D subspace $LL_m$ at scale $m$ can be constructed as the tensor product of two 1D subspaces $V_m^x$ and $V_m^y$ [11,12,22],

$$LL_m = V_m^x \otimes V_m^y, \qquad m \in Z. \tag{4}$$

Since $V_m^i$ $(i = x, y)$ admit the decomposition into a lower order resolution scale $m - 1$

$$V_m^i = V_{m-1}^i \oplus W_{m-1}^i, \qquad i = x, y, \tag{5}$$

where $W_{m-1}$ are wavelet subspaces, the 2D subspaces can be decomposed into

$$\begin{aligned} LL_m &= V_m^x \otimes V_m^y \\ &= LL_{m-1} \oplus LH_{m-1} \oplus HL_{m-1} \oplus HH_{m-1}, \end{aligned} \tag{6}$$

where $LL_{m-1} = V_{m-1}^x \otimes V_{m-1}^y$, $LH_{m-1} = V_{m-1}^x \otimes W_{m-1}^y$, $HL_{m-1} = W_{m-1}^x \otimes V_{m-1}^y$, and $HH_{m-1} = W_{m-1}^x \otimes W_{m-1}^y$. Here $L$ and $H$ resemble "low-resolution" and "high-resolution," respectively. A three-scale 2D wavelet decomposition is schematically illustrated in Fig. 1.

For a given matrix $A$, the above wavelet decomposition (transform) allows a perfect reconstruction (inverse wavelet transform), by which there is nothing to gain: $A = \mathcal{W}^{-1}(\mathcal{W}(A))$, where $\mathcal{W}$ and $\mathcal{W}^{-1}$ denote wavelet transform and its inverse, respectively. The advantage of using wavelets is that each wavelet subspace can be independently modified for specific purposes. In this Letter, we consider a simple operation to attain a desirable coupling matrix

$$\tilde{A} = \mathcal{W}^{-1}(O(\mathcal{W}(A))), \tag{7}$$

where $O$ denotes the nontrivial action on selected wavelet subspaces and the identity operator on other subspaces. For a given $O$, the matrix $\tilde{A}$ carries a new relationship

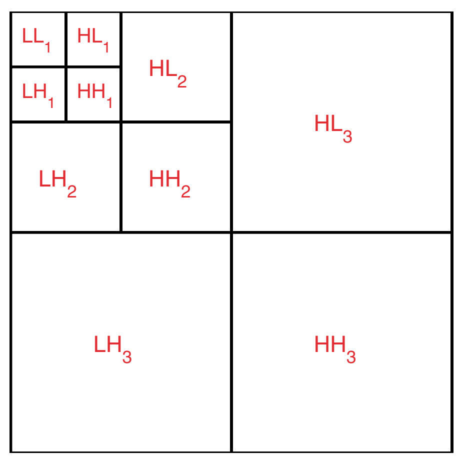


FIG. 1 (color online). Schematic representation of the three-scale wavelet decomposition of Fig. 2. The upper left square labeled by $LL_1$ corresponds to the lowest resolution subspace in both the horizontal and vertical directions. The information contained in this subspace is a coarse approximation of the original matrix. The other nine regions involve higher resolution subspaces, and they constitute the details of the original matrix at different scales. Among them, three diagonal regions labeled by $HH_3$, $HH_2$, $HH_1$ correspond to the highest resolution subspaces at each scale, and they contain the most detailed information of the original matrix in their scales.

among the coupled oscillators, which might not be as simple as the original matrix $A$. Nevertheless, the stability of the synchronous states can be studied with matrix $\tilde{A}$, whose eigenvalue spectrum $\tilde{\lambda}_i$ ($i = 1, 2, \ldots, N$) determines the synchronous stability of the coupled chaotic system.

To illustrate the idea, we choose the matrix $A$ to be the one generated from the nearest neighbor coupling scheme and limit $O$ to be the multiplication of a scalar factor $K$ on the elements of subspaces $LL_1$. An image view of matrix $A$ of size $64^2$ is depicted in Fig. 2(a). The three-scale wavelet transform of $A$ [i.e., $\mathcal{W}(A)$] obtained by using the Daubechies-20 wavelets [11] is plotted in Fig. 2(b). The image of $O(\mathcal{W}(A))$ is displayed in Fig. 2(c). It is seen that only a tiny fraction (1.56%) of $\mathcal{W}(A)$ (the $LL_1$ subspace) is modified. In principle, such a modified fraction can be further minimized in a four-scale or five-scale analysis. However, in the physical space, matrix $\tilde{A}$ exhibits a very interesting nontrivial structure; see Fig. 2(d). It is this wavelet subspace enhanced $\tilde{A}$ that gives rise to spectacular synchronous stability for the coupled chaotic system.

As a proof of principle, we consider a set of coupled Lorenz oscillators $u_i = (x_i, y_i, z_i)$, ($i = 1, 2, \ldots, N$)

$$\frac{dx_i}{dt} = \sigma(y_i - x_i), \qquad \frac{dy_i}{dt} = \gamma x_i - y_i - x_i z_i, \qquad \frac{dz_i}{dt} = x_i y_i - \beta z_i. \tag{8}$$

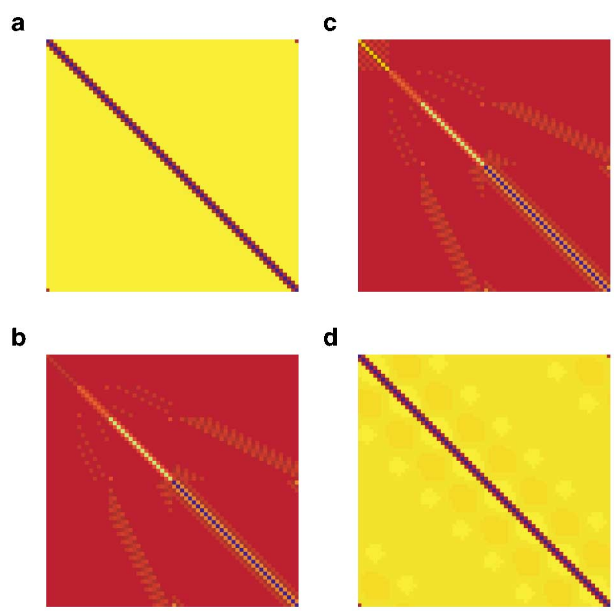


FIG. 2 (color). The impact of modifying a wavelet subspace to the coupling matrix $A$ of chaotic oscillators. (a) The original coupling matrix; (b) the three-scale wavelet decomposition of the original coupling matrix; (c) the modified wavelet decomposition (note the change in subspace $LL_1$); (d) the physical space image of the wavelet enhanced coupling matrix, $\tilde{A}$ obtained by the inverse wavelet transform of (c).

With the classical parameters $\sigma = 10.0$, $\gamma = 28.0$, and $\beta = 8/3$, the system is chaotic as the largest Lyapunov exponent of each single oscillator is $L_{\max} = 0.908$. The size of the system is chosen as $N = 512$. The synchronization of chaos is possible by adding nearest neighbor couplings of the form $\varepsilon(\mathbf{u}_{i-1} - 2\mathbf{u}_i + \mathbf{u}_{i+1})$ to all of three components $x_i$, $y_i$, and $z_i$ as prescribed in Eq. (1).

We demonstrate the wavelet subspace control by examining the relation of the critical coupling strength $\varepsilon_c$ versus the multiplication factor $K$. Without wavelet subspace enhancement, the present chaotic system requires an enormously large critical coupling strength [$\varepsilon_c = 6029$, following Eq. (3)] to synchronize. However, the use of wavelet subspace control leads to a dramatic reduction in $\varepsilon_c$ as indicated in Fig. 3. Obviously, $\varepsilon_c$ decreases linearly with respect to the increase of $K$ until a critical value $K_c = 1011$. The smallest $\varepsilon_c$ is about 6, which is about 1011 times smaller than the original critical coupling strength, indicating the efficiency of the proposed approach.

The $K_c$ value is determined by the wavelet subspace structure and limited by the largest eigenvalue in the high resolution subspace $HH_1$; i.e., $K_c$ is bounded by $\frac{\lambda_{32}}{\lambda_2} = 1011$. For a fixed number of oscillators and a three-scale wavelet analysis, a further increase in $K_c$ is possible, but it requires a different operation $O$, for example, an $O$ that modifies larger subspaces, $LL_2$ (note that $LL_2$ includes $HH_1$). Moreover, other operations that change the elements of a high resolution subspace, such as $HH_1$, or $HH_2$, or $HH_3$ alone do not have any impact on the transverse stability of the synchronous manifold.

It remains to be verified that the proposed wavelet strategy is robust and general for controlling chaos. To

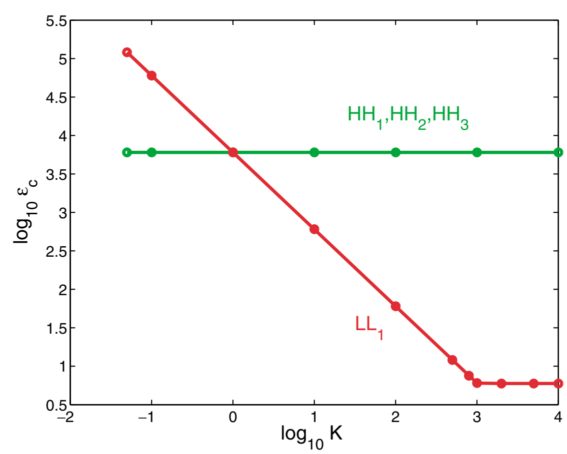


FIG. 3 (color). Critical coupling strength $\varepsilon_c$ vs $K$ with the red line and green line denoting the effects of wavelet control in subspaces (1) $LL_1$ and (2) $HH_1$, $HH_2$, and $HH_3$, respectively. The horizontal green line indicates the nil impact of modifying high resolution subspaces to the transverse stability of synchronous manifold.

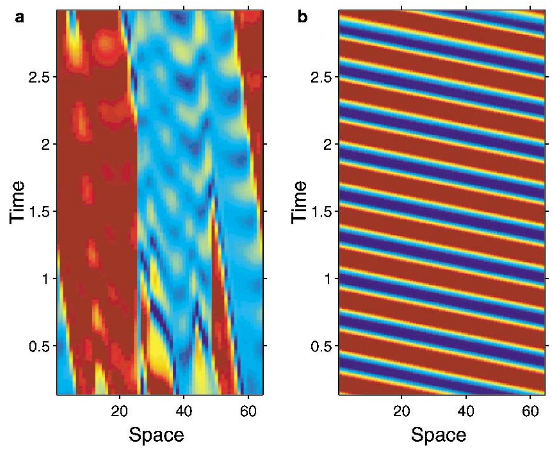


FIG. 4 (color). Wavelet controlled dynamics showing the transition from chaos to periodicity. (a) The original chaotic states; (b) wavelet induced Hopf bifurcation from chaos.

this end, we add nearest neighbor couplings of the type $\varepsilon(y_{i-1} - 2y_i + y_{i+1}) + r(y_{i+1} - y_{i-1})$ to the $y_i$ components of Eq. (8) describing the dynamics of 64 oscillators. With the parameter set $\sigma = 10.0$, $\gamma = 60.0$, and $\beta = 8/3$, the system resides in its chaotic region when the coupling strength is $\varepsilon = 6.0$ and $r = 4.0$, as shown in Fig. 4(a) for the $x_i$ components. We use a two-scale wavelet transform and multiply each element of the $LL_1$ subspace by a factor of $K = 31.8$. We observe the Hopf bifurcation from chaos [23] as indicated in Fig. 4(b). An onset of synchronization is further observed at $K = 32$.

In conclusion, we have presented a novel wavelet subspace approach to the control of chaotic dynamical systems. In contrast to the previous use of wavelets as an analyzing tool, the present study utilizes wavelets as a new efficient strategy for controlling nonlinear dynamics. The control is achieved by modifying the wavelet subspaces of the coupling matrix of chaotic oscillators. We find that the transverse stability of the synchronous manifold is extremely sensitive to the wavelet subspace manipulation of the coupling matrix. Dramatic reduction in the critical coupling strength is achieved with the modification of a tiny fraction of wavelet subspaces. Wavelet controlled Hopf bifurcation from chaos is observed. It is believed that the proposed approach has potential applications to the control of other discrete and continuous dynamical systems, such as coupled map lattices, cellular automata, turbulence, and pattern formation.

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