# Exact Solutions for Vibration of Multi-Span Rectangular Mindlin Plates

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*This paper presents the first-known exact solutions for the vibration of multi-span rectangular Mindlin plates with two opposite edges simply supported. The Levy type solution method and the state-space technique are employed to develop an analytical approach to deal with the vibration of rectangular Mindlin plates of multiple spans. Exact vibration frequencies are obtained for two-span square Mindlin plates with varying span ratios and two-, three- and four-equal-span rectangular Mindlin plates. The influence of the span ratios, the number of spans and plate boundary conditions on the vibration behavior of square and rectangular Mindlin plates is examined. The presented exact vibration results may serve as benchmark solutions for such plates.* [DOI: 10.1115/1.1501083]

## 1 Introduction

Multi-span rectangular plates are important structural components that are widely used in various engineering applications, *i.e.*, aeroplane surfaces, slabs in house construction, bridge decks and glass window panels. The problem of free vibration of multi-span rectangular plates or plates with internal line supports has attracted the attention of many researchers. Veletsos and Newmark [1] pioneered the research in this area by studying the vibration behavior of a two-span rectangular plate. Since then, many researchers have carried out investigations on the topic using various analytical and numerical [2–14] methods. Among these researchers, Azimi et al. [7] obtained exact solutions for simply supported multi-span plates by using the receptance method. Most recently, Xiang et al. [15] proposed an analytical approach for the vibration analysis of multi-span rectangular plates with two opposite edges simply supported. By utilizing the Levy solution method and the state-space technique, their approach is able to find exact solutions for the vibration of multi-span rectangular plates involving free and clamped edges [15].

All the aforementioned studies are for plates based on the classical thin plate theory (the Kirchhoff plate theory). The investigations on the vibration of multi-span thick rectangular plates are relatively scarce [16]. It is well-known that the classical thin plate theory overestimates the vibration frequencies when a plate becomes thick or a plate vibrates at higher frequencies. This overestimation is due to the fact that the thin plate theory neglects the effects of transverse shear deformation and rotary inertia in a plate. To overcome this problem, one may employ the first order shear-deformable plate theory (the Mindlin plate theory [17]) or the higher order plate theory (the Reddy plate theory [18]) for analysis of thick plates. Using the $pb$-2 Ritz method, Liew et al. [19] studied the vibration of rectangular Mindlin plates with internal line supports being either parallel to the edges or in diagonal directions. They extended their studies later to the vibration of rectangular Mindlin plates with intermediate stiffeners [20] and skew Mindlin plates with internal line supports [21]. Kong and Cheung [22] investigated the vibration of shear-deformable plates with internal line supports by the finite layer approach.

It is apparent that there are no exact solutions for the vibration of multi-span thick rectangular plates in the open literature. The purpose of this work is to fill the current gap in this area by introducing an analytical method for dealing with multi-span rectangular Mindlin plates and providing exact vibration solutions for these plates. We shall consider a multi-span rectangular Mindlin plate with two opposite edges simply supported while the other two edges may take an arbitrary combination of boundary conditions. The Levy solution method and the state-space technique [15,23–27] are employed to establish the proposed analytical approach for the vibration analysis of the Mindlin plate. This approach is used to generate exact vibration solutions. These solutions are indeed valuable and are tabulated as they serve as important benchmark values for checking the convergence, validity and accuracy of potential numerical methods for the analysis of multi-span thick plates.

## 2 Mathematical Modelling

Consider an isotropic rectangular Mindlin plate of length $aL$, width $L$ and thickness $h$ as shown in Fig. 1. The plate is simply supported on the two edges parallel to the $x$ axis. There are $(n-1)$ internal line supports in the $x$ direction that divide the plate into $n$ spans (see Fig. 1). The internal line supports enforce zero transverse displacement along the line supports in the plate. The origin of the coordinates $(x, y)$ is set at the middle point of the plate bottom edge. The problem at hand is to determine the vibration frequencies of the multi-span rectangular Mindlin plate.

We take a typical span in the plate to derive the Levy type solution. The governing differential equations based on the Mindlin plate theory [14] for the $i$-th span in harmonic vibration can be derived as:

$$\kappa^2 Gh\left[\frac{\partial}{\partial x}\left(\frac{\partial w^i}{\partial x}+\theta_x^i\right)+\frac{\partial}{\partial y}\left(\frac{\partial w^i}{\partial y}+\theta_y^i\right)\right]+\rho h\omega^2 w^i=0 \quad (1)$$

$$D\left[\frac{\partial}{\partial x}\left(\frac{\partial \theta_x^i}{\partial x}+\upsilon\frac{\partial \theta_i^y}{\partial y}\right)\right]+\frac{(1-\upsilon)D}{2}\left[\frac{\partial}{\partial y}\left(\frac{\partial \theta_y^i}{\partial x}+\frac{\partial \theta_x^i}{\partial y}\right)\right]$$
$$-\kappa^2 Gh\left(\frac{\partial w^i}{\partial x}+\theta_x^i\right)+\frac{\rho h^3}{12}\omega^2\theta_x^i=0 \quad (2)$$

$$D\left[\frac{\partial}{\partial y}\left(\frac{\partial \theta_y^i}{\partial y}+\upsilon\frac{\partial \theta_x^i}{\partial x}\right)\right]+\frac{(1-\upsilon)D}{2}\left[\frac{\partial}{\partial x}\left(\frac{\partial \theta_y^i}{\partial x}+\frac{\partial \theta_x^i}{\partial y}\right)\right]$$
$$-\kappa^2 Gh\left(\frac{\partial w^i}{\partial y}+\theta_y^i\right)+\frac{\rho h^3}{12}\omega^2\theta_y^i=0 \quad (3)$$

where the superscript $i(=1,2,\ldots,n)$ denotes the $i$-th span in the plate, $E$ is Young's modulus, $G=E/[2(1+\upsilon)]$ is the shear modulus, $\upsilon$ is Poisson's ratio, $\kappa^2$ is the shear correction factor, $D=Eh^3/[12(1-\upsilon^2)]$ is the flexural rigidity of the plate, $\rho$ is the

Contributed by the Technical Committee on Vibration and Sound for publication in the JOURNAL OF VIBRATION AND ACOUSTICS. Manuscript received April 1 2001; Revised May 2002. Associate Editor: T. Wickert.

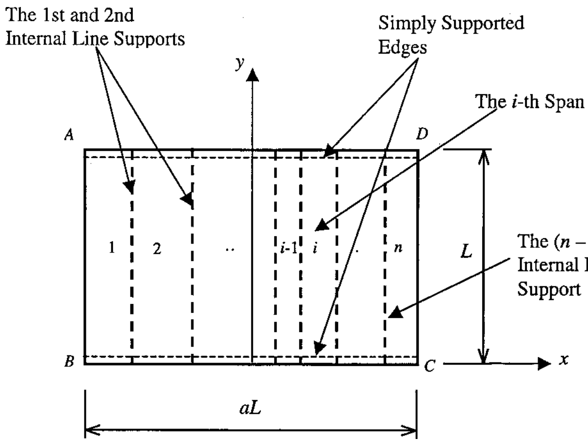



**Fig. 1 Geometry and coordinate system for a Levy plate with $(n-1)$ internal line supports**

mass density of the plate, $\omega$ is the vibration frequency of the plate, $w$ is the transverse displacement, and $\theta_x$ and $\theta_y$ are rotations in the $y$ and $x$ directions.

The essential and natural boundary conditions for the two parallel edges (at $y=0$ and $y=L$) in the $i$-th span are

$$w^i=0, \tag{4a}$$

$$M_y^i=0, \tag{4b}$$

$$\theta_x^i=0 \tag{4c}$$

where $M_y^i$ is the bending moment and is defined by

$$M_y^i=D\left(\frac{\partial\theta_y^i}{\partial y}+\upsilon\frac{\partial\theta_x^i}{\partial x}\right) \tag{5}$$

The general Levy-type solution approach is employed to solve the governing differential equations for the $i$-th span [15,26,27]. The displacement fields can be expressed as

$$\begin{Bmatrix} w^i(x,y)\\ \theta_x^i(x,y)\\ \theta_y^i(x,y)\end{Bmatrix}=\begin{bmatrix}\phi_w^i(x)\sin\frac{m\pi y}{L}\\ \phi_x^i(x)\sin\frac{m\pi y}{L}\\ \phi_y^i(x)\cos\frac{m\pi y}{L}\end{bmatrix} \tag{6}$$

where $\phi_w^i(x)$, $\phi_x^i(x)$ and $\phi_y^i(x)$ are unknown functions to be determined. Equation (6) satisfies the simply supported boundary conditions on edges $y=0$ and $y=L$ as defined in Eqs. (4a–4c).

Substituting Eq. (6) into Eqs. (1–3), the following differential equation system can be derived:

$$(\boldsymbol{\psi}^i)'=\mathbf{H}^i\boldsymbol{\psi}^i \tag{7}$$

where $\boldsymbol{\psi}^i=[\phi_w^i\ (\phi_w^i)'\ \phi_x^i\ (\phi_x^i)'\ \phi_y^i\ (\phi_y^i)']^T$, the prime $'$ represents the derivative with respect to $x$ and $\mathbf{H}^i$ is a $6\times6$ matrix with the following non-zero elements:

$$H_{12}^i=H_{34}^i=H_{56}^i=1 \tag{8}$$

$$H_{21}^i=\frac{-(m\pi/L)^2\kappa^2Gh+\rho h\omega^2}{-\kappa^2Gh} \tag{9}$$

$$H_{24}^i=-1, \tag{10}$$

$$H_{25}^i=\frac{m\pi}{L} \tag{11}$$

$$H_{42}^i=\frac{\kappa^2Gh}{D}, \tag{12}$$

$$H_{46}^i=\frac{(m\pi/L)(1+\upsilon)}{2} \tag{13}$$

$$H_{43}^i=\frac{D(1-\upsilon)(m\pi/L)^2/2+\kappa^2Gh-\rho h^3\omega^2/12}{D} \tag{14}$$

$$H_{61}^i=\frac{(m\pi/L)\kappa^2Gh}{[D(1-\upsilon)/2]}, \tag{15}$$

$$H_{64}^i=-\frac{(m\pi/L)(1+\upsilon)}{1-\upsilon} \tag{16}$$

$$H_{65}^i=\frac{D(m\pi/L)^2+\kappa^2Gh-\rho h^3\omega^2/12}{[D(1-\upsilon)/2]} \tag{17}$$

A general solution of Eq. (7) can be obtained as

$$\boldsymbol{\psi}^i=\mathbf{e}^{\mathbf{H}x}\mathbf{c}^i \tag{18}$$

where $\mathbf{c}^i$ is a constant column vector that can be determined by the plate boundary conditions of the two edges parallel to the $y$ axis and/or interface conditions between adjacent spans and $\mathbf{e}^{\mathbf{H}x}$ is the general matrix solution of Eq. (7). The detailed procedure in determining Eq. (18) has been given in [26] and [27].

Each of the two edges parallel to the $y$ axis may have the following edge conditions

$$M_x^i=0, \tag{19a}$$

$$M_{xy}^i=0, \tag{19b}$$

$$Q_x^i=0,\quad \text{if the edge is free} \tag{19c}$$

$$w^i=0, \tag{20a}$$

$$M_x^i=0, \tag{20b}$$

$$\theta_y^i=0,\quad \text{if the edge is simply supported} \tag{20c}$$

$$w^i=0, \tag{21a}$$

$$\theta_x^i=0, \tag{21b}$$

$$\theta_y^i=0,\quad \text{if the edge is clamped} \tag{21c}$$

where $i$ takes the value 1 or $n$, $M_x^i$, $M_{xy}^i$ and $Q_x^i$ are bending moment, twist moment and transverse shear force in the plate, respectively, and defined by

$$M_x^i=D\left(\frac{\partial\theta_x^i}{\partial x}+\upsilon\frac{\partial\theta_y^i}{\partial y}\right) \tag{22}$$

$$M_{xy}^i=D\,\frac{1-\upsilon}{2}\left(\frac{\partial\theta_x^i}{\partial y}+\frac{\partial\theta_y^i}{\partial x}\right) \tag{23}$$

$$Q_x^i=\kappa^2Gh\left(\frac{\partial w^i}{\partial x}+\theta_x^i\right) \tag{24}$$

To ensure the continuity and the internal line support conditions, the essential and natural boundary conditions for the interface between the $i$-th and the $(i+1)$-th spans are defined as:

$$w^i=0, \tag{25a}$$

$$w^{(i+1)}=0, \tag{25b}$$

$$\theta_x^i=\theta_x^{(i+1)}, \tag{25c}$$

$$\theta_y^i=\theta_y^{(i+1)}, \tag{25d}$$

$$M_x^i=M_x^{(i+1)}, \tag{25e}$$

**Table 1 Comparison studies of frequency parameters for simply supported rectangular Mindlin plates with two-equal spans ($h/L=0.2$)**

| a | Sources | Mode Sequences 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| 1 | [19] | 3.8656 | 4.6102 | 5.5879 | 5.9975 | 7.9737 | 8.2209 | 9.6018 | 10.385 |
| | Present | 3.8656 | 4.4839 | 5.5879 | 5.8608 | 7.9737 | 8.0696 | 9.6018 | 9.9392 |
| 2 | [19] | 1.7679 | 2.0151 | 3.8656 | 3.8656 | 3.9689 | 4.3107 | 5.5879 | 5.8685 |
| | Present | 1.7679 | 1.9869 | 3.8656 | 3.8656 | 3.9319 | 4.1811 | 5.5879 | 5.7238 |

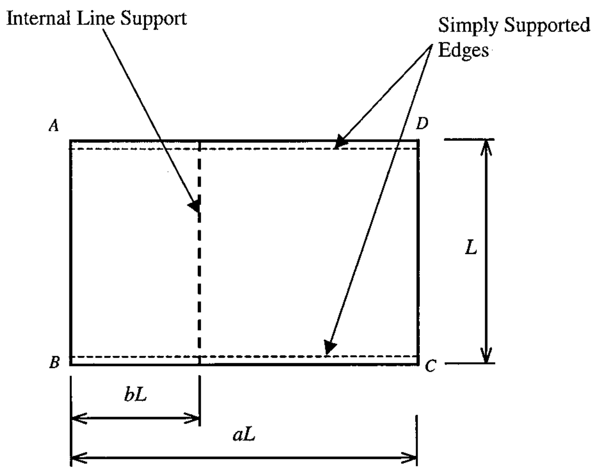



**Fig. 2 A Levy plate with an internal line support**

$$M_{xy}^{i}=M_{xy}^{(i+1)} \tag{25f}$$

In view of Eq. (18), a homogeneous system of equations can be derived by implementing the boundary conditions of the plate along the two edges parallel to the $y$ axis [Eqs. (19)–(21)] and the interface conditions between spans [Eq. (25)] when assembling the spans to the whole plate

$$\mathbf{K}\begin{Bmatrix}\mathbf{c}^{1}\\ \mathbf{c}^{2}\\ \vdots\\ \mathbf{c}^{(i-1)}\\ \mathbf{c}^{i}\\ \mathbf{c}^{(i+1)}\\ \vdots\\ \mathbf{c}^{n}\end{Bmatrix}=\{0\} \tag{26}$$

The vibration frequency $\omega$ is determined when the determinant of $\mathbf{K}$ in Eq. (26) is equal to zero. As the vibration frequency $\omega$ is imbedded in matrix $\mathbf{H}$, it cannot be obtained directly from Eq. (26). A numerical iteration procedure has been developed to carry out the calculation [26,27].

## 3 Results and Discussions

In this section, the proposed analytical approach is employed to study the vibration of multi-span rectangular Mindlin plates with various edge support conditions. Although the method can be applied to obtain Levy solutions for rectangular plates of arbitrary number of spans along the $x$ direction, we limit our study to two span square plates with varying span ratios and two-, three- and four-equal-span rectangular plates in the present study. The natural frequency of a plate is expressed in terms of a non-dimensional frequency parameter $\lambda=(\omega L^2/\pi^2)\sqrt{\rho h/D}$. For brevity we shall use the letters $F$ for a free edge, $S$ for a simply supported edge and $C$ for a clamped edge and a two-letter designation to represent the edge conditions for a Levy plate. For instance, an $FS$ plate will have edge $AB$ free and edge $CD$ simply supported (see Fig. 1), respectively. The Poisson ratio $\upsilon=0.3$ and the shear correction factor $\kappa^2=5/6$ are used in this study.

**3.1 Comparison Studies.** To confirm the correctness of the proposed analytical approach, Table 1 presents the vibration results for a simply supported two-span rectangular Mindlin plate generated by the present analytical approach and by the $pb$-2 Ritz method [16]. We observe that the Ritz results [16] are in close agreement with the present exact solutions in general. However, a few Ritz results in Table 1 are not yet quite converged to the exact solutions.

**3.2 Square Mindlin Plates of Two-Unequal Spans.** The layout of a two-span Levy plate is shown in Fig. 2. The location of the internal line support is determined by the location parameter $b$ (see Fig. 2).

Tables 2–4 present the exact frequency parameters of the first ten modes for two-span square Mindlin plates with symmetric Levy-type supporting edges (*i.e.*, *SS*, *FF* and *CC* plates). The location parameter of the internal line support $b$ varies from 0.1, 0.3 to 0.5 and the plate thickness ratio $h/L$ is set to be 0.01, 0.05 and 0.1. Tables 5–7 show the exact frequency parameters of the first ten modes for two-span square Mindlin plates with asymmetric Levy-type supports (*i.e.*, *SF*, *CF* and *CS* plates). In this case, the location parameter of the internal line support $b$ changes from 0.3, 0.5 to 0.7. All results in Tables 2 to 7 are presented with 5 significant digits. As expected, we observe that the frequency parameters for all six plates decrease with increasing plate thickness ratio $h/L$ due to the increased influence of transverse shear deformation and rotary inertia.

**Table 2 Vibration frequency parameters $\lambda=(\omega L^2/\pi^2)\sqrt{\rho h/D}$ for an $SS$ square plate with an internal line support**

| h/L | b | Mode 1 | Mode 2 | Mode 3 | Mode 4 | Mode 5 | Mode 6 | Mode 7 | Mode 8 | Mode 9 | Mode 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.01 | 0.1 | 2.6392 | 5.4604 | 6.7781 | 9.5650 | 10.362 | 13.259 | 14.376 | 16.075 | 17.284 | 20.862 |
| | 0.3 | 3.5328 | 6.3571 | 9.6807 | 11.243 | 12.620 | 14.898 | 17.538 | 17.599 | 18.151 | 21.476 |
| | 0.5 | 4.9955 | 7.0108 | 7.9884 | 9.5607 | 12.969 | 14.161 | 16.948 | 19.928 | 19.928 | 20.854 |
| 0.05 | 0.1 | 2.5843 | 5.3124 | 6.5118 | 9.1169 | 9.9078 | 12.425 | 13.492 | 14.931 | 16.119 | 19.088 |
| | 0.3 | 3.4630 | 6.1682 | 9.2836 | 10.713 | 11.964 | 13.833 | 16.165 | 16.326 | 16.869 | 19.456 |
| | 0.5 | 4.8907 | 6.7106 | 7.7267 | 9.0653 | 12.305 | 13.229 | 15.845 | 18.433 | 18.433 | 18.938 |
| 0.10 | 0.1 | 2.4416 | 4.9380 | 5.8821 | 8.1036 | 8.8524 | 10.675 | 11.633 | 12.637 | 13.722 | 15.762 |
| | 0.3 | 3.2732 | 5.6883 | 8.3308 | 9.4966 | 10.485 | 11.613 | 13.348 | 13.835 | 14.271 | 15.791 |
| | 0.5 | 4.6084 | 5.9863 | 7.0716 | 7.9475 | 10.809 | 11.300 | 13.539 | 15.174 | 15.462 | 15.462 |

**Table 3 Vibration frequency parameters $\lambda=(\omega L^2/\pi^2)\sqrt{\rho h/D}$ for an *FF* square plate with an internal line support**

| h/L | b | Mode 1 | Mode 2 | Mode 3 | Mode 4 | Mode 5 | Mode 6 | Mode 7 | Mode 8 | Mode 9 | Mode 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.01 | 0.1 | 1.2456 | 3.2596 | 4.2351 | 6.5319 | 7.4051 | 9.1903 | 10.891 | 11.599 | 13.489 | 16.119 |
| | 0.3 | 1.4263 | 3.2920 | 4.4249 | 5.4786 | 6.4685 | 8.5657 | 9.3695 | 11.456 | 12.469 | 13.542 |
| | 0.5 | 1.6309 | 2.3050 | 4.7253 | 5.1271 | 7.6042 | 9.7036 | 9.9757 | 10.068 | 11.195 | 13.320 |
| 0.05 | 0.1 | 1.2324 | 3.1863 | 4.1481 | 6.3087 | 7.1414 | 8.8283 | 10.330 | 11.008 | 12.738 | 15.062 |
| | 0.3 | 1.4082 | 3.2051 | 4.3260 | 5.2780 | 6.2026 | 8.1429 | 8.9870 | 10.797 | 11.773 | 12.677 |
| | 0.5 | 1.6067 | 2.2520 | 4.6094 | 4.9625 | 7.3277 | 9.2886 | 9.4560 | 9.5028 | 10.600 | 12.312 |
| 0.10 | 0.1 | 1.2046 | 3.0298 | 3.9283 | 5.8065 | 6.5417 | 7.9786 | 9.1555 | 9.7234 | 11.136 | 12.857 |
| | 0.3 | 1.3707 | 3.0266 | 4.0825 | 4.8245 | 5.6518 | 7.2601 | 8.1024 | 9.4666 | 10.271 | 10.951 |
| | 0.5 | 1.5593 | 2.1387 | 4.3358 | 4.5951 | 6.7071 | 8.1537 | 8.3498 | 8.4725 | 9.3836 | 10.390 |

**Table 4 Vibration frequency parameters $\lambda=(\omega L^2/\pi^2)\sqrt{\rho h/D}$ for a *CC* square plate with an internal line support**

| h/L | b | Mode 1 | Mode 2 | Mode 3 | Mode 4 | Mode 5 | Mode 6 | Mode 7 | Mode 8 | Mode 9 | Mode 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.01 | 0.1 | 3.3169 | 5.8735 | 8.1316 | 10.628 | 10.667 | 15.230 | 15.277 | 17.473 | 17.871 | 21.885 |
| | 0.3 | 4.6841 | 7.1453 | 11.779 | 12.023 | 14.614 | 18.541 | 19.160 | 20.344 | 22.724 | 25.621 |
| | 0.5 | 7.0108 | 9.5608 | 9.6209 | 11.690 | 14.161 | 15.775 | 20.854 | 21.011 | 22.072 | 23.639 |
| 0.05 | 0.1 | 3.1958 | 5.6619 | 7.6536 | 9.9975 | 10.114 | 13.976 | 14.121 | 16.247 | 16.251 | 19.909 |
| | 0.3 | 4.5184 | 6.8505 | 11.138 | 11.238 | 13.543 | 17.147 | 17.511 | 18.165 | 20.057 | 22.259 |
| | 0.5 | 6.7130 | 8.9583 | 9.0738 | 10.797 | 13.244 | 14.427 | 18.944 | 19.091 | 19.877 | 21.135 |
| 0.10 | 0.1 | 2.9489 | 5.1901 | 6.7054 | 8.6714 | 8.9777 | 11.606 | 11.982 | 13.358 | 13.789 | 16.265 |
| | 0.3 | 4.1108 | 6.1636 | 9.5728 | 9.7472 | 11.394 | 14.202 | 14.408 | 14.431 | 15.467 | 17.049 |
| | 0.5 | 5.9992 | 7.5511 | 7.9854 | 9.0159 | 11.354 | 11.928 | 15.197 | 15.785 | 16.086 | 16.707 |

**Table 5 Vibration frequency parameters $\lambda=(\omega L^2/\pi^2)\sqrt{\rho h/D}$ for an *SF* square plate with an internal line support**

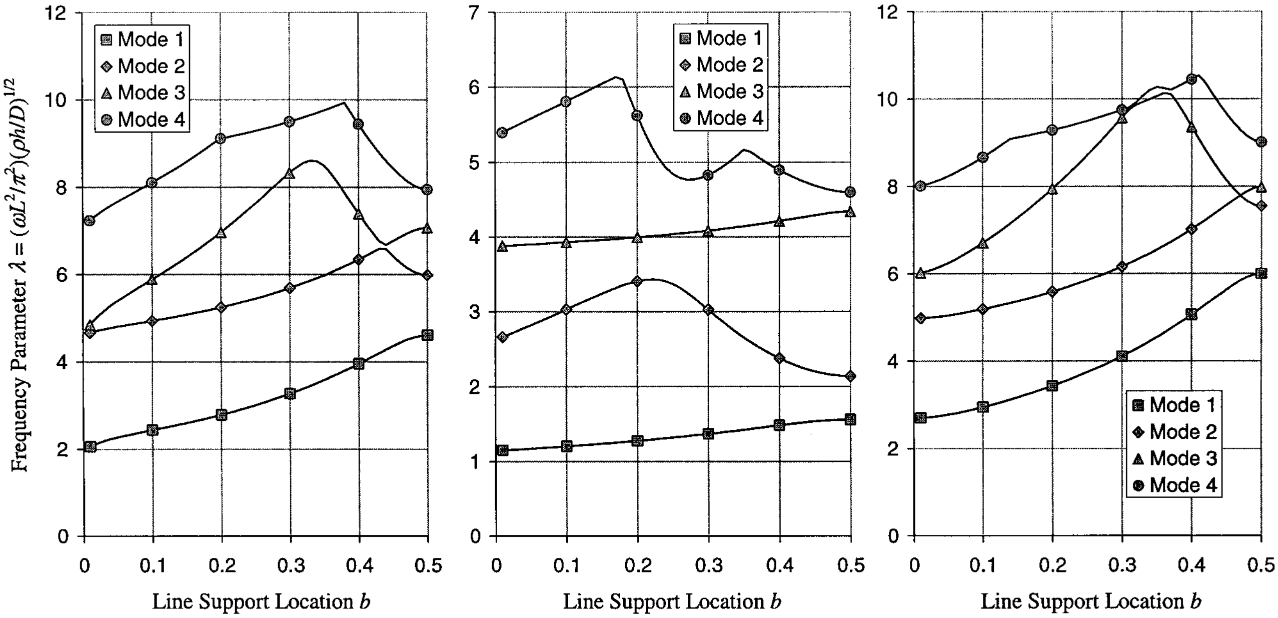
| h/L | b | Mode 1 | Mode 2 | Mode 3 | Mode 4 | Mode 5 | Mode 6 | Mode 7 | Mode 8 | Mode 9 | Mode 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.01 | 0.3 | 1.5211 | 4.4498 | 5.0600 | 8.2985 | 9.3796 | 11.676 | 13.371 | 14.985 | 15.748 | 16.292 |
| | 0.5 | 1.9456 | 4.9069 | 5.7162 | 8.6236 | 9.1039 | 9.8289 | 12.374 | 13.482 | 16.723 | 17.547 |
| | 0.7 | 2.7881 | 4.2836 | 5.9685 | 7.0169 | 10.155 | 11.005 | 11.790 | 13.011 | 17.836 | 17.975 |
| 0.05 | 0.3 | 1.4993 | 4.3475 | 4.9037 | 7.9353 | 8.9944 | 11.112 | 12.568 | 14.054 | 14.482 | 15.227 |
| | 0.5 | 1.9096 | 4.7706 | 5.5548 | 8.2757 | 8.6289 | 9.3885 | 11.557 | 12.709 | 15.582 | 16.113 |
| | 0.7 | 2.7372 | 4.1510 | 5.7939 | 6.7020 | 9.6786 | 10.490 | 11.083 | 12.258 | 16.519 | 16.704 |
| 0.10 | 0.3 | 1.4513 | 4.0970 | 4.5435 | 7.1458 | 8.1058 | 9.8480 | 10.911 | 11.961 | 12.138 | 13.041 |
| | 0.5 | 1.8341 | 4.4564 | 5.1450 | 7.4497 | 7.5883 | 8.4083 | 9.9432 | 11.034 | 13.288 | 13.472 |
| | 0.7 | 2.6251 | 3.8364 | 5.3805 | 6.0288 | 8.5772 | 9.3278 | 9.6585 | 10.629 | 13.903 | 14.139 |

**Table 6 Vibration frequency parameters $\lambda=(\omega L^2/\pi^2)\sqrt{\rho h/D}$ for a *CF* square plate with an internal line support**

| h/L | b | Mode 1 | Mode 2 | Mode 3 | Mode 4 | Mode 5 | Mode 6 | Mode 7 | Mode 8 | Mode 9 | Mode 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.01 | 0.3 | 1.5363 | 4.4554 | 5.1510 | 8.3541 | 9.3817 | 12.344 | 13.402 | 15.627 | 16.293 | 20.355 |
| | 0.5 | 1.9811 | 4.9168 | 7.2938 | 9.8321 | 9.8642 | 10.134 | 12.754 | 14.741 | 16.725 | 17.728 |
| | 0.7 | 3.1117 | 5.1104 | 6.2351 | 7.5397 | 11.227 | 12.108 | 12.151 | 14.784 | 18.151 | 18.799 |
| 0.05 | 0.3 | 1.5116 | 4.3514 | 4.9743 | 7.9734 | 8.9956 | 11.641 | 12.585 | 14.528 | 15.228 | 18.314 |
| | 0.5 | 1.9411 | 4.7782 | 6.9994 | 9.2334 | 9.3904 | 9.5773 | 11.818 | 13.697 | 15.582 | 16.212 |
| | 0.7 | 3.0339 | 4.9150 | 6.0097 | 7.1704 | 10.642 | 11.359 | 11.370 | 13.683 | 16.797 | 17.280 |
| 0.10 | 0.3 | 1.4578 | 4.0985 | 4.5769 | 7.1595 | 8.1061 | 10.141 | 10.915 | 12.372 | 13.042 | 14.317 |
| | 0.5 | 1.8566 | 4.4603 | 6.3020 | 7.9088 | 8.3530 | 8.4089 | 10.037 | 11.602 | 13.288 | 13.495 |
| | 0.7 | 2.8680 | 4.4463 | 5.5194 | 6.3685 | 9.3958 | 9.6938 | 9.8422 | 11.480 | 14.160 | 14.443 |

**Table 7 Vibration frequency parameters $\lambda=(\omega L^2/\pi^2)\sqrt{\rho h/D}$ for a *CS* square plate with an internal line support**

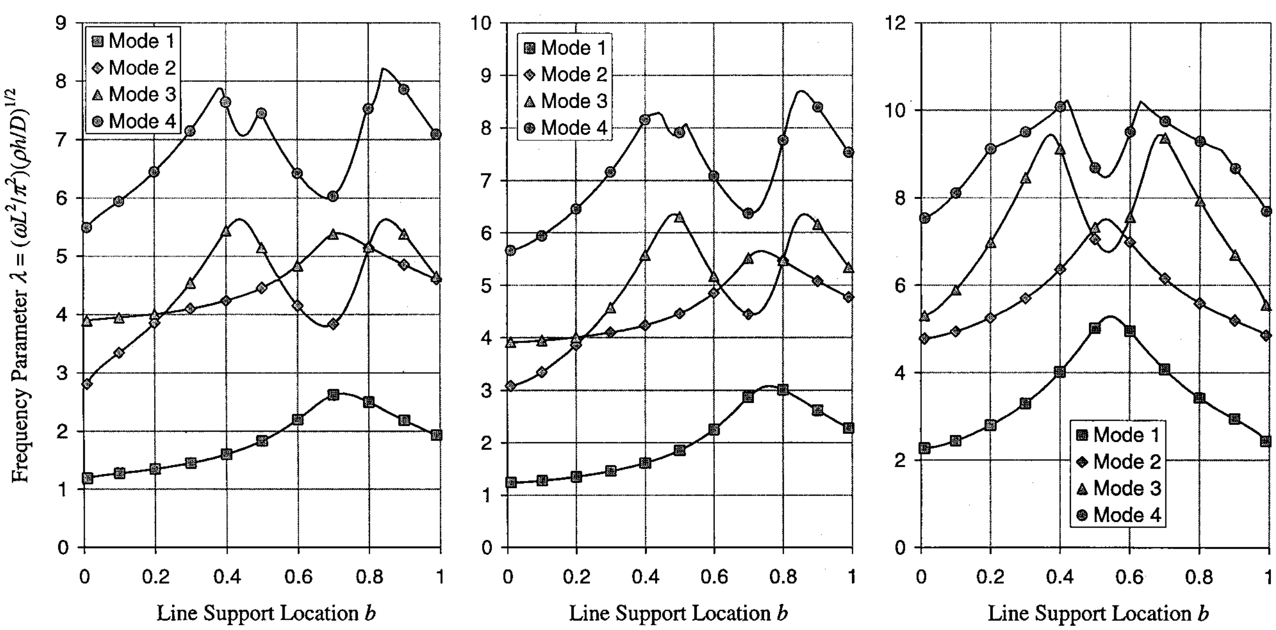
| h/L | b | Mode 1 | Mode 2 | Mode 3 | Mode 4 | Mode 5 | Mode 6 | Mode 7 | Mode 8 | Mode 9 | Mode 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.01 | 0.3 | 3.5944 | 6.3860 | 10.014 | 11.256 | 12.860 | 17.697 | 18.157 | 18.869 | 21.573 | 23.614 |
| | 0.5 | 5.5739 | 8.4207 | 8.7081 | 10.956 | 13.292 | 15.230 | 18.234 | 20.179 | 21.068 | 21.670 |
| | 0.7 | 4.5939 | 7.1024 | 11.408 | 11.761 | 14.156 | 15.642 | 18.163 | 18.533 | 18.856 | 22.603 |
| 0.05 | 0.3 | 3.5119 | 6.1887 | 9.5403 | 10.720 | 12.133 | 16.423 | 16.872 | 17.252 | 19.453 | 20.693 |
| | 0.5 | 5.4171 | 8.0994 | 8.1958 | 10.219 | 12.560 | 14.037 | 16.828 | 18.611 | 19.261 | 19.622 |
| | 0.7 | 4.4487 | 6.8211 | 10.780 | 11.128 | 13.233 | 14.343 | 16.513 | 17.143 | 17.336 | 20.286 |
| 0.10 | 0.3 | 3.2973 | 5.6959 | 8.4551 | 9.4984 | 10.551 | 13.862 | 13.954 | 14.271 | 15.354 | 16.095 |
| | 0.5 | 5.0135 | 7.0519 | 7.3241 | 8.6832 | 10.954 | 11.741 | 14.061 | 15.544 | 15.861 | 15.986 |
| | 0.7 | 4.0783 | 6.1534 | 9.3656 | 9.7448 | 11.283 | 11.809 | 13.452 | 14.388 | 14.408 | 16.270 |




(*a*) *SS* Plate (*b*) *FF* Plate (*c*) *CC* Plate

**Fig. 3 Frequency parameters $\lambda=(\omega L^2/\pi^2)(\rho h/D)^{1/2}$ versus line support location $b$ for *SS*, *FF* and *CC* square Mindlin plates with one internal line support ($h/L=0.1$)**

To further observe the influence of the line support location on the vibration of plates, the variation of frequency parameters λ of the first four modes against the line support location $b$ is plotted in Figs. 3 and 4 for square Mindlin plates with thickness ratio $h/L=0.1$. The line support location $b$ varies from 0.01 to 0.5 for the *SS*, *FF* and *CC* plates with 50 sample points on each curve (see Fig. 3) and from 0.01 to 0.99 for the *SF*, *CF* and *CS* plates with 99 sample points on each curve (see Fig. 4), respectively. It is seen that the internal line support strengthens the plates against vibration. However, the optimal location of the internal line support in increasing the frequency parameter varies from plate to plate and from mode to mode. For the three symmetric Levy square plates




(*a*) *SF* Plate (*b*) *CF* Plate (*c*) *CS* Plate

**Fig. 4 Frequency parameters $\lambda=(\omega L^2/\pi^2)(\rho h/D)^{1/2}$ versus line support location $b$ for *SF*, *CF* and *CS* square Mindlin plates with one internal line support ($h/L=0.1$)**

**Table 8 Vibration frequency parameters $\lambda=(\omega L^2/\pi^2)\sqrt{\rho h/D}$ for rectangular Mindlin plates with two equal spans ($a=2$, $h/L=0.10$)**

| Cases | Mode 1 | Mode 2 | Mode 3 | Mode 4 | Mode 5 | Mode 6 | Mode 7 | Mode 8 | Mode 9 | Mode 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| *SS* Plate | 1.9317 | 2.2663 | 4.6084 | 4.6084 | 4.7671 | 5.2781 | 7.0716 | 7.4914 | 8.6162 | 8.6162 |
| *FF* Plate | 1.1523 | 1.2406 | 2.6500 | 3.0780 | 3.8792 | 3.9134 | 5.3950 | 5.6358 | 5.6448 | 6.3488 |
| *CC* Plate | 2.2684 | 2.6992 | 4.7726 | 4.9693 | 5.2839 | 5.9928 | 7.5084 | 7.9664 | 8.7010 | 8.7906 |
| *SF* Plate | 1.1927 | 2.0689 | 2.8892 | 3.8954 | 4.6804 | 4.8667 | 5.5230 | 6.0679 | 7.2563 | 7.9474 |
| *CF* Plate | 1.1946 | 2.3943 | 2.9544 | 3.8956 | 4.8564 | 5.4232 | 5.5330 | 6.2076 | 7.6775 | 7.9474 |
| *CS* Plate | 2.0248 | 2.5548 | 4.6528 | 4.8153 | 4.9054 | 5.7650 | 7.1971 | 7.8210 | 8.6389 | 8.7618 |

**Table 9 Vibration frequency parameters $\lambda=(\omega L^2/\pi^2)\sqrt{\rho h/D}$ for rectangular Mindlin plates with three equal spans ($a=3$, $h/L=0.10$)**

| Cases | Mode 1 | Mode 2 | Mode 3 | Mode 4 | Mode 5 | Mode 6 | Mode 7 | Mode 8 | Mode 9 | Mode 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| *SS* Plate | 1.9317 | 2.0924 | 2.4619 | 4.6084 | 4.6084 | 4.6843 | 4.8588 | 4.9527 | 5.6123 | 7.0716 |
| *FF* Plate | 1.1873 | 1.2001 | 2.1960 | 2.8303 | 2.9891 | 3.8944 | 4.7561 | 5.0668 | 5.5031 | 5.9557 |
| *CC* Plate | 2.0931 | 2.4642 | 2.6983 | 4.6860 | 4.8648 | 4.9548 | 4.9670 | 5.6181 | 5.9906 | 7.2867 |
| *SF* Plate | 1.1936 | 1.9867 | 2.3440 | 2.9195 | 3.8955 | 4.6358 | 4.7212 | 4.8174 | 5.3669 | 5.5276 |
| *CF* Plate | 1.1936 | 2.1362 | 2.5680 | 2.9313 | 3.8955 | 4.7131 | 4.9230 | 5.0052 | 5.5280 | 5.7410 |
| *CS* Plate | 1.9739 | 2.2670 | 2.6296 | 4.6283 | 4.7049 | 4.7689 | 4.9374 | 5.2800 | 5.8834 | 7.1289 |

**Table 10 Vibration frequency parameters $\lambda=(\omega L^2/\pi^2)\sqrt{\rho h/D}$ for rectangular Mindlin plates with four equal spans ($a=4$, $h/L=0.10$)**

| Cases | Mode 1 | Mode 2 | Mode 3 | Mode 4 | Mode 5 | Mode 6 | Mode 7 | Mode 8 | Mode 9 | Mode 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| *SS* Plate | 1.9317 | 2.0246 | 2.2663 | 2.5527 | 4.6084 | 4.6084 | 4.7671 | 4.8144 | 4.9000 | 5.2781 |
| *FF* Plate | 1.1927 | 1.1945 | 2.0689 | 2.3925 | 2.8892 | 2.9538 | 3.8954 | 3.8956 | 4.6804 | 4.8505 |
| *CC* Plate | 2.0248 | 2.2674 | 2.5548 | 2.6978 | 4.6528 | 4.7698 | 4.8153 | 4.9054 | 4.9659 | 5.2810 |
| *SF* Plate | 1.1936 | 1.9608 | 2.1692 | 2.4853 | 2.9244 | 3.8955 | 4.6226 | 4.6706 | 4.7267 | 4.8799 |
| *CF* Plate | 1.1936 | 2.0440 | 2.3232 | 2.6309 | 2.9266 | 3.8955 | 4.6642 | 4.8050 | 4.8390 | 4.9432 |
| *CS* Plate | 1.9556 | 2.1321 | 2.4142 | 2.6582 | 4.6196 | 4.6637 | 4.7039 | 4.8383 | 4.9490 | 5.0305 |

(*i.e.*, *SS*, *FF* and *CC* plates), the best location of the internal line support in increasing the fundamental frequency is at the plate center ($b=0.5$), whereas for the three asymmetric Levy square plates (*i.e.*, *SF*, *CF* and *CS* plates), the best location of the internal line support for fundamental frequency shifts from the plate center to the side with a weaker edge constraint.

We observe that there are kink points on the curves for the second, third and fourth modes in the *SS*, *CC SF*, *CF* and *CS* plates and for the fourth mode in the *FF* plate (see Figs. 3 and 4). These kind points represent the locations where mode shape switching occurs when the line support location parameter $b$ varies across these kink points.

**3.3 Rectangular Mindlin Plates of Two-, Three- and Four-Equal Spans.** Tables 8–10 present the exact frequency parameters of the first ten modes for rectangular Mindlin plates with two-, three- and four-equal spans. The plate aspect ratio $a$ is set to be 2, 3 and 4 for the two-, three- and four-equal-span plates, respectively. Again we observe that the frequency parameters decrease as the plate thickness ratio $h/L$ increases. The frequency parameters also decrease as the number of spans increases for all considered cases, except for the first mode of the *SS* plates.

## 4 Conclusions

This paper presents the first-known exact solutions for the vibration of multi-span rectangular Mindlin plates. A rectangular plate is considered to have two parallel edges simply supported while the other two edges may have any combination of free, simply supported or clamped conditions. The internal line supports that divided the plate into multiple spans are arranged to be perpendicular to the two simply supported parallel edges. An analytical model based on the Levy solution method and the state-space technique is developed for the vibration analysis of multi-span rectangular Mindlin plates.

Tabulated in this paper are the first 10 exact frequency parameters for two-unequal-span square Mindlin plates and two-, three- and four-equal-span rectangular Mindlin plates. The influence of the internal line support on the frequency parameters of a square Mindlin plate is examined. It is observed that the optimal location of the internal line support in strengthening the plate against vibration varies from plate to plate and from mode to mode. However, for fundamental frequency, the best location of the internal line support is at the plate center for symmetric Levy square plates. For equal-span rectangular plates, the frequency parameters decrease with increasing number of spans. The frequency parameters also decrease as the plate thickness ratio increases due to the influence of transverse shear deformation and rotary inertia. We believe that the present exact vibration solutions may serve as benchmark values for Mindlin plates with internal line supports and are also useful to engineers who are designing multi-span thick plates.

## Acknowledgment

This work was supported by the University of Western Sydney and by the National University of Singapore.

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