Name: ______________________________

Section: ____________ **Recitation Instructor**: ______________________

INSTRUCTIONS

- Fill in your name, etc. on this first page.
- Without fully opening the exam, check that you have pages 1 through 10.
- **Show all your work** on the standard response questions. Write your answers clearly! Include enough steps for the grader to be able to follow your work. Don't skip limits or equal signs, etc. Include words to clarify your reasoning.
- Do first all of the problems you know how to do immediately. Do not spend too much time on any particular problem. Return to difficult problems later.
- If you have any questions please raise your hand and a proctor will come to you.
- You will be given exactly 90 minutes for this exam.

ACADEMIC HONESTY

- Do not open the exam booklet until you are instructed to do so.
- Do not leave the exam room during the first 20 minutes of the exam.
- Do not seek or obtain any kind of help from anyone to answer questions on this exam. If you have questions, consult only the proctor(s).
- Books, notes, calculators, phones, or any other electronic devices are not allowed on the exam. Students should store them in their backpacks.
- No scratch paper is permitted. If you need more room use the back of a page.
- Anyone who violates these instructions will have committed an act of academic dishonesty. Penalties for academic dishonesty can be very severe. All cases of academic dishonesty will be reported immediately to the Dean of Undergraduate Studies and added to the student's academic record.

I have read and understand the above instructions and statements regarding academic honesty: ______________________________

SIGNATURE

Multiple Choice. Circle the best answer. No work needed. No partial credit available.

1. (6 points) Find the limit of the sequence $a_n = \dfrac{\ln n}{\ln 2n}$

   A. 0

   B. 1/2

   C. 1

   D. $\ln 2$

   E. The sequence diverges.

2. (6 points) Which statement is true about the series $\sum_{n=1}^{\infty} \cos\left(\frac{1}{n}\right)$

   A. The **n$^{\text{th}}$ term test** concludes that the series converges.

   B. The n$^{\text{th}}$ term test concludes that the series diverges.

   C. The **n$^{\text{th}}$ term test** hypotheses are not met by this series, so it cannot be applied.

   D. The **n$^{\text{th}}$ term test** hypotheses are met by this series however the test is inconclusive.

   E. None of the above are true.

3. (6 points) Which statement is true about the series $\sum_{n=1}^{\infty} \sin(n)$

   A. The **integral test** concludes that the series converges.

   B. The **integral test** concludes that the series diverges.

   C. The integral test hypotheses are not met by this series, so it cannot be applied.

   D. The **integral test** hypotheses are met by this series however the test is inconclusive.

   E. None of the above are true.

4. (6 points) Which statement is true concerning the following series:

$$(1)\ \sum_{n=1}^{\infty} \frac{1}{n+2} \quad \text{and} \quad (2)\ \sum_{n=1}^{\infty} \frac{n^2}{2n^3-1}$$

A. Both converge.

B. (1) converges, (2) diverges.

C. (1) diverges, (2) converges.

D. Both diverge.

E. Cannot be determined.

5. (6 points) Which statement is true? The series: $\sum_{k=1}^{\infty} \frac{(-1)^k}{2^k-3}$ is

I. Convergent and II. Absolutely convergent

A. Neither I nor II is true.

B. Only I is true.

C. Only II is true.

D. Both are true.

E. Cannot be determined.

6. (6 points) Find the radius of convergence of $\sum_{k=0}^{\infty} \frac{(-1)^k x^k}{3^k(k+1)}$

A. $1/4$

B. $1/3$

C. 3

D. 4

E. $+\infty$

7. (7 points) The function $\dfrac{3x}{3+x^2}$ is represented by the power series:

   A. $1 - 3x^2 + 9x^4 - ...$

   B. $1 + 3x^2 + 9x^4 - ...$

   C. $x + 3x^3 + 9x^5 - ...$

   D. $1 + \dfrac{x^2}{3} + \dfrac{x^4}{9} + ...$

   E. $x - \dfrac{x^3}{3} + \dfrac{x^5}{9} - ...$

8. (7 points) Find the second-degree Taylor polynomial generated by the function $f(x) = \dfrac{1}{x}$ about the point $x = 3$

   A. $\dfrac{1}{3} - \dfrac{1}{9}(x-3) + \dfrac{1}{27}\dfrac{(x-3)^2}{2!}$

   B. $\dfrac{1}{3} + \dfrac{1}{9}(x-3) + \dfrac{1}{27}\dfrac{(x-3)^2}{2!}$

   C. $\dfrac{1}{3} - \dfrac{1}{9}(x-3) + \dfrac{2}{27}\dfrac{(x-3)^2}{2!}$

   D. $\dfrac{1}{3} + \dfrac{1}{9}(x-3) + \dfrac{2}{27}\dfrac{(x-3)^2}{2!}$

   E. None of the above.

Standard Response Questions. Show all work to receive credit. Please **BOX** your final answer.

9. (15 points) Determine whether the **sequence** converges or diverges. If it converges, find the limit.

$$a_n = \sqrt[n]{2^{3n+1}}$$

Solution:

$$\lim_{n\to\infty} \sqrt[n]{2^{3n+1}}$$
$$= \lim_{n\to\infty} (2^{3n+1})^{\frac{1}{n}}$$
$$= \lim_{n\to\infty} 2^{\frac{3n+1}{n}}$$
$$= \lim_{n\to\infty} 2^{(3+1/n)}$$
$$= 2^3$$
$$= \boxed{8}$$

10. (15 points) Find the sum of the series $\sum_{n=0}^{\infty} \frac{2-3^{n+1}}{6^n}$

Solution:

$$\sum_{n=0}^{\infty} \frac{2-3^{n+1}}{6^n}$$
$$= \sum_{n=0}^{\infty} \left[\frac{2}{6^n} - \frac{3^{n+1}}{6^n}\right]$$
$$= \sum_{n=0}^{\infty} \left[2\left(\frac{1}{6}\right)^n - \frac{3\cdot 3^n}{6^n}\right]$$
$$= \sum_{n=0}^{\infty} \left[2\left(\frac{1}{6}\right)^n - 3\left(\frac{1}{2}\right)^n\right]$$
$$= 2\sum_{n=0}^{\infty} \left(\frac{1}{6}\right)^n - 3\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$$
$$= 2\left(\frac{1}{1-1/6}\right) - 3\left(\frac{1}{1-1/2}\right)$$
$$= 2\left(\frac{6}{5}\right) - 3\cdot 2$$
$$= \boxed{-\frac{18}{5}}$$

11. Determine whether the following series converge or diverge. **You must justify your answer with work and explicitly state which test(s) you use and whether the conditions for the test are met!**

(a) (10 points) $\sum_{n=2}^{\infty} \frac{5}{n \ln n}$

Solution:

Consider the **integral test**:

Since $\frac{5}{n \ln n}$ is continuous, positive, and decreasing for $n \geq 2$, the test may be applied

$$\int_2^{\infty} \frac{5}{x \ln x} dx$$

$$= \lim_{b \to \infty} 5 \int_2^b \frac{1}{x \ln x} dx$$

$$= \lim_{b \to \infty} 5 \ln(\ln x) \Big|_1^b = \infty$$

So the series $\boxed{\text{diverges}}$ as well.

(b) (10 points) $\sum_{n=1}^{\infty} \frac{3n^2 + n}{n^4 + \sqrt{n}}$

Solution:

Consider the **limit comparison test**:

All terms are positive for $n \geq 1$, so we may apply the test.

Let $a_n = \frac{3n^2 + n}{n^4 + \sqrt{n}}$ and $b_n = \frac{3n^2}{n^4} = \frac{3}{n^2}$

$$= \lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{3n^2 + n}{n^4 + n^{1/2}} \cdot \frac{n^2}{3}$$

$$= \lim_{n \to \infty} \frac{3n^4 + n^3}{3n^4 + 3n^{1/2}}$$

$$= \lim_{n \to \infty} \frac{3 + \frac{1}{n}}{3 + \frac{3}{n^{7/2}}}$$

$$= 1$$

Therefore, by the limit comparison test, either both a_n and b_n converge or they both diverge. But b_n is a p-series with $p = 2$ which converges. Therefore the series $\boxed{\text{converges.}}$

12. Consider the function $g(x) = \dfrac{5}{1-4x^2}$

(a) (10 points) Express g as a power series in sigma-notation.

Solution:

$\dfrac{5}{1-4x^2} = \dfrac{a}{1-r}$

So $g(x)$ is the sum of a geometric series with $a = 5$ and $r = 4x^2$

Therefore, $g(x) = \sum_{n=0}^{\infty} ar^n$

$= \boxed{\sum_{n=0}^{\infty} 5(4x^2)^n}$

(b) (5 points) What is the radius of convergence for this power series?

Solution:

A geometric series converges when $|r| < 1$, so

$|r| = |4x^2| < 1$

$x^2 < \dfrac{1}{4}$

$|x| < \dfrac{1}{2}$

Therefore, the radius of convergence $= \boxed{\dfrac{1}{2}}$

13. (20 points) Find the first three non-zero terms of the Maclaurin series for $f(x) = e^{2x}\cos(x)$

Solution:

$f(x) = e^{2x}\cos x$ $\quad f(0) = 1$

$f'(x) = 2e^{2x}\cos x - e^{2x}\sin x$ $\quad f'(0) = 2$

$f'(x) = 4e^{2x}\cos x - 2e^{2x}\sin x - 2e^{2x}\sin x - e^{2x}\cos x$ $\quad f'(0) = 3$

So the Maclaurin series is:

$$\boxed{1 + 2x + \frac{3x^2}{2!} + ...}$$

14. For the curve given by $y = \dfrac{x^3}{6} + \dfrac{1}{2x}$ for $1 \le x \le 3$

(a) (10 points) Write an integral that expresses the length of the arc.

Solution:

$y = \dfrac{x^3}{6} + \dfrac{1}{2x}$

$\dfrac{dy}{dx} = \dfrac{1}{2}x^2 - \dfrac{1}{2x^2}$

The arc length is given by:

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

So, L= $\boxed{\int_1^3 \sqrt{1 + \left(\frac{1}{2}x^2 - \frac{1}{2x^2}\right)^2}\,dx}$

(b) (5 points) Evaluate the integral to find the length of the arc.

Solution:

$$\int_1^3 \sqrt{1 + \left(\frac{1}{2}x^2 - \frac{1}{2x^2}\right)^2}\,dx$$
$$= \int_1^3 \sqrt{1 + \frac{1}{4}\left(x^2 - \frac{1}{x^2}\right)^2}\,dx$$
$$= \int_1^3 \sqrt{1 + \frac{1}{4}\left(\frac{x^4 - 1}{x^2}\right)^2}\,dx$$
$$= \int_1^3 \sqrt{1 + \frac{x^8 - 2x^4 + 1}{4x^4}}\,dx$$
$$= \int_1^3 \sqrt{\frac{4x^4 + x^8 - 2x^4 + 1}{4x^4}}\,dx$$
$$= \int_1^3 \sqrt{\frac{x^8 + 2x^4 + 1}{4x^4}}\,dx$$
$$= \int_1^3 \frac{x^4 + 1}{2x^2}\,dx$$
$$= \int_1^3 \left(\frac{x^2}{2} + \frac{1}{2}x^{-2}\right)dx$$
$$= \left[\frac{x^3}{6} - \frac{1}{2}x^{-1}\right]\Bigg|_1^3$$
$$= \left[\frac{27}{6} - \frac{1}{6}\right] - \left[\frac{1}{6} - \frac{1}{2}\right]$$
$$= \boxed{\frac{14}{3}}$$

Congratulations you are now done with the exam!

Go back and check your solutions for accuracy and clarity. Make sure your final answers are BOXED.

When you are completely happy with your work please bring your exam to the front to be handed in. **Please have your MSU student ID ready** so that is can be checked.

DO NOT WRITE BELOW THIS LINE.

Page	Points	Score
2	18	
3	18	
4	14	
5	30	
6	20	
7	15	
8	20	
9	15	
Total:	150	