Exercise 8.5

(a) The Kolmogorov forward equations are

$$\frac{d}{dt}{}_tp_x^{00} = {}_tp_x^{01}\mu_{x+t}^{10} - {}_tp_x^{00}\left(\mu_{x+t}^{01} + \mu_{x+t}^{02}\right)$$

$$\frac{d}{dt}{}_tp_x^{01} = {}_tp_x^{00}\mu_{x+t}^{01} - {}_tp_x^{01}\left(\mu_{x+t}^{10} + \mu_{x+t}^{12}\right)$$

$$\frac{d}{dt}{}_tp_x^{02} = {}_tp_x^{00}\mu_{x+t}^{02} + {}_tp_x^{01}\mu_{x+t}^{12}$$

(b) $\bar{a}_{x:\overline{n}|}^{01}$ is the actuarial present value of a benefit of 1 per year, payable continuously to a healthy life now age x, while in state 1 (sick) within n years. It can be expressed as

$$\bar{a}_{x:\overline{n}|}^{01} = \int_0^n v^t\, {}_tp_x^{01} dt.$$

$\bar{A}_{x:\overline{n}|}^{02}$ is the actuarial present value of a death benefit of 1 payable to a healthy life now age x at the moment of his/her death, provided this occurs before n years. It can be expressed as

$$\bar{A}_{x:\overline{n}|}^{02} = \int_0^n v^t\left({}_tp_x^{01}\mu_{x+t}^{02} + {}_tp_x^{01}\mu_{x+t}^{12}\right) dt.$$

(c) The reserve at time t for a policy in state 1 is

$$\begin{aligned} {}_tV^{(1)} &= \text{APV(FDB)}_t + \text{APV(FSB)}_t - \text{APV(FP)}_t \\ &= B\bar{a}_{40+t:\overline{20-t}|}^{11} + S\bar{A}_{40+t:\overline{20-t}|}^{12} - P\bar{a}_{40+t:\overline{20-t}|}^{10} \end{aligned}$$

where APV = actuarial present value, FDB = future death benefit, FSB = future sickness benefit, FP = future premiums.

(d) The numerical approximations to use are:

$${}_{t+h}p_{40}^{00} \approx {}_tp_{40}^{00} + h\left[{}_tp_{40}^{01}\mu_{40+t}^{10} - {}_tp_{40}^{00}\left(\mu_{40+t}^{01} + \mu_{40+t}^{02}\right)\right]$$

$${}_{t+h}p_{40}^{01} \approx {}_tp_{40}^{01} + h\left[{}_tp_{40}^{00}\mu_{40+t}^{01} - {}_tp_{40}^{01}\left(\mu_{40+t}^{10} + \mu_{40+t}^{12}\right)\right]$$

with initial conditions ${}_0p_{40}^{00} = 1$, ${}_0p_{40}^{01} = 0$.

(i) With $h = 0.1$ and starting at $t = 0$, we get

$${}_{0.1}p_{40}^{00} \approx 1 + 0.1\left[0 - (0.01074 + 0.00328)\right] = 0.998598$$

$${}_{0.1}p_{40}^{01} \approx 0 + 0.1(1 \times 0.01071) = 0.001074$$

Doing another round of iteration, next starting with $t = 0.1$, we have

$${}_{0.2}p_{40}^{00} \approx {}_{0.1}p_{40}^{00} + 0.1\left[{}_{0.1}p_{40}^{01}(0.09003) - {}_{0.1}p_{40}^{00}(0.01094 + 0.00330)\right] = 0.9971857$$

$${}_{0.2}p_{40}^{01} \approx {}_{0.1}p_{40}^{01} + 0.1\left[{}_{0.1}p_{40}^{00}(0.01094) - {}_{0.1}p_{40}^{01}(0.09003 + 0.00719)\right] = 0.002156025$$

(ii) The Thiele's differential equations for the policy reserves satisfy

$$\frac{d}{dt}{}_tV^{(0)} = \delta\, {}_tV^{(0)} + P - \mu^{02}_{x+t}(S - {}_tV^{(0)}) - \mu^{01}_{x+t}({}_tV^{(1)} - {}_tV^{(0)})$$

and

$$\frac{d}{dt}{}_tV^{(1)} = \delta\, {}_tV^{(0)} - B - \mu^{12}_{x+t}(S - {}_tV^{(1)}) - \mu^{10}_{x+t}({}_tV^{(0)} - {}_tV^{(1)})$$

(iii) Using the backward Euler's method: $\frac{d}{dt}{}_tV^{(j)} \approx \frac{1}{h}({}_tV^{(j)} - {}_{t-h}V^{(j)})$ and the conditions that ${}_{20}V^{(0)} = {}_{20}V^{(1)} = 0$, with $h = 0.1$ we get

$${}_{19.9}V^{(0)} = -0.1(P - S\mu^{02}_{60}) = -0.1(6000 - 100000 * 0.01730) = -427$$

and

$${}_{19.9}V^{(1)} = -0.1(-B - S\mu^{12}_{60}) = -0.1(-20000 - 100000 * 0.05941) = 2,594.10$$