**Exercise 7.18**

First, we calculate the single premium, and let this be $P$. Then we have

$$
P = \text{APV}(FB_0) = \int_0^{20} P(1 + i)^t \nu^t d\mu_{x+t} + 50000_{20} E_x \bar{a}_{x+20}
$$

$$
= P_{20} q_x + 50000_{20} E_x \bar{a}_{x+20}
$$

Solving for $P$, we get

$$
P = 50000 \frac{\nu^{20} p_x \bar{a}_{x+20}}{1 - 20 q_x} = 50000 \nu^{20} \bar{a}_{x+20}
$$

(a) During the deferred period, we have $t \leq 20$. Prospectively, the reserve is the actuarial present value of future benefits minus the actuarial present value of future premiums; because it is a single premium, there are no premiums more to be collected. Thus, we have the prospective reserve formula:

$$
t V = \text{APV}(FB_t)
$$

$$
= \int_0^{20-t} P(1 + i)^s v^s \nu^s d\mu_{x+t+s} + 50000_{20-t} E_{x+t} \bar{a}_{x+20}
$$

$$
= P(1 + i)^t 20-t q_{x+t} + 50000_{20-t} E_{x+t} \bar{a}_{x+20}
$$

After the deferred period, $t > 20$, the prospective reserve formula is clearly

$$
t V = 50000 \bar{a}_{x+t}
$$

since only the annuity payments are being disbursed during this period.

(b) During the deferred period, we have $t \leq 20$. Retrospectively, the reserve is the actuarial accumulated value of past premiums minus the actuarial accumulated value of past benefits. Thus, we have the retrospective reserve formula:

$$
t V = \frac{P}{t E_x} - \int_0^t P(1 + i)^s v^s \nu^s d\mu_{x+s} ds
$$

$$
= \frac{P(1 - t q_x)}{v^t d\mu_x} = P(1 + i)^t
$$

After the deferred period, $t > 20$, the retrospective reserve formula is

$$
t V = \frac{P}{t E_x} - \frac{P_{20} q_x + 50000_{20} E_x \bar{a}_{x+20} - 20 \nu^{20} \bar{a}_{x+20}}{t E_x}
$$

(c) For $t \leq 20$, starting with the prospective formula, we have

**Prospective**

$$
= P(1 + i)^t 20-t q_{x+t} + 50000_{20-t} E_{x+t} \bar{a}_{x+20}
$$

$$
= P(1 + i)^t 20-t q_{x+t} + P(1 + i)^{20} \nu^{20-t} 20-p_{x+t}
$$

$$
= P(1 + i)^t \left(20-t q_{x+t} + 20-p_{x+t}\right) = P(1 + i)^t
$$

= **Retrospective**
The second line follows because

\[ P = 50000v^{20}a_{x+20} \quad \text{or equivalently} \quad P(1 + i)^{20} = 50000a_{x+20} \]

For \( t > 20 \), starting with the retrospective formula, by substituting

\[ \bar{a}_{x+20; t-20} = \bar{a}_{x+20} - 20-t\bar{E}_{x+t}\bar{a}_{x+t} \]

we have

\[
\text{Retrospective} = \frac{P}{t\bar{E}_x} - \frac{P_{20}q_x + 50000_20E_x\bar{a}_{x+20} - 50000_20E_x_{20-t}\bar{E}_{x+t}\bar{a}_{x+t}}{t\bar{E}_x} \\
= \frac{50000_tE_x\bar{a}_{x+t}}{t\bar{E}_x} = 50000\bar{a}_{x+t}
\]

The last line follows because

\[ P = P_{20}q_x + 50000_20E_x\bar{a}_{x+20} \]

and that

\[ _{20}E_x_{20-t}\bar{E}_{x+t} = t\bar{E}_x \]