Exercise 5.15

Recall from basic interest theory the following relationships:

\[ i = e^\delta - 1, \quad i^{(m)} = m \left( e^{\delta/m} - 1 \right), \]
\[ d = 1 - e^{-\delta}, \quad \text{and} \quad d^{(m)} = m \left( 1 - e^{-\delta/m} \right). \]

(a) Write \( \alpha(m) \) as a function of \( \delta \):

\[ \alpha(m) = \frac{i}{i^{(m)}}, \quad \frac{d}{d^{(m)}} = \frac{e^\delta - 1}{m(e^{\delta/m} - 1)}, \quad \frac{1 - e^{-\delta}}{m(1 - e^{-\delta/m})} = \frac{1}{m^2} e^{\delta[(1/m) - 1]} \left( \frac{e^\delta - 1}{e^{\delta/m} - 1} \right)^2. \]

Let this be \( g_1(\delta) \) and use Taylor’s series expansion to express \( g_1 \) in terms of powers of \( \delta \). It is a very tedious exercise to even show that \( g_1(0) = 1, \quad g_1'(0) = 0 \) and \( g_1''(0) = (m^2 - 1)/(6m^2) \) so that we can write

\[ \alpha(m) = g_1(0) + g_1'(0) \delta + \cdots = 1 + \frac{m^2 - 1}{12m^2} \delta^2 + \cdots \]

Thus, we see that removing powers of 2 and higher, we get the approximation \( \alpha(m) \approx 1 \). One can also verify, using Mathematica for example, that

\[ \alpha(m) = 1 + \frac{m^2 - 1}{12m^2} \delta^2 + \frac{2m^4 - 5m^2 + 3}{720m^4} \delta^4 + \cdots \]

(b) Similarly, write \( \beta(m) \) as a function of \( \delta \):

\[ \beta(m) = \frac{i - i^{(m)}}{i^{(m)}d^{(m)}} = \frac{(e^\delta - 1) - [m(e^{\delta/m} - 1)]}{m(e^{\delta/m} - 1) \cdot m(1 - e^{-\delta/m})} = \frac{1}{m^2} e^{\delta/m} \cdot \frac{(e^\delta - 1) - [m(e^{\delta/m} - 1)]}{(e^{\delta/m} - 1)^2}. \]

Let this be \( g_2(\delta) \) and again use Taylor’s series expansion. It is equally very tedious to show that \( g_2(0) = (m - 1)/2m, \quad g_2'(0) = (m^2 - 1)/(6m^2) \) so that we can write

\[ \beta(m) = g_2(0) + g_2'(0) \delta + \cdots = \frac{m - 1}{2m} + \frac{m^2 - 1}{6m^2} \delta + \cdots \]

Thus, we see that removing powers of 1 and higher, we get the approximation \( \beta(m) \approx (m - 1)/2m \). For additional terms in the series expansion, one can verify, again with Mathematica for example, that we have

\[ \beta(m) = \frac{m - 1}{2m} + \frac{m^2 - 1}{6m^2} \delta + \frac{24m^2 - 1}{24m^2} \delta^2 + \frac{3m^4 - 5m^2 + 2}{360m^4} \delta^3 + \cdots \]

The results in this problem indeed lead us to the common approximation

\[ \ddot{a}_x^{(m)} \approx \ddot{a}_x - \frac{m - 1}{2m}. \]