Exercise 5.12

(a) The longer you live, the later you die, and vice versa. Hence, the longer you live, the more expensive life annuity would be; the later you die, the cheaper life insurance would be. And of course, vice versa. We therefore expect a negative covariance.

(b) Rewrite the product of $Y$ and $Z$ as

$$YZ = v^T \cdot \bar{a}_T = v^T \frac{1 - v^T \delta}{\delta} = \frac{1}{\delta} (v^T - v^T 2^T)$$

so that the covariance can be expressed as

$$\text{Cov}[Y, Z] = \text{E}[YZ] - \text{E}[Y]\text{E}[Z] = \frac{1}{\delta} (\text{E}[v^T] - \text{E}[v^T 2^T]) - \bar{A}_x \cdot \bar{a}_x$$

(c) We can re-express the covariance in (b) as

$$\text{Cov}[Y, Z] = \frac{1}{\delta} (\bar{A}_x - 2\bar{A}_x) - \bar{A}_x \left( \frac{1 - \bar{A}_x}{\delta} \right)$$

$$= \frac{1}{\delta} (\bar{A}_x - 2\bar{A}_x) - \frac{1}{\delta} [\bar{A}_x - (\bar{A}_x)^2]$$

$$= -\frac{1}{\delta} [2\bar{A}_x - (\bar{A}_x)^2]$$

$$= -\delta \text{Var}[\bar{a}_T].$$

This covariance is clearly negative because both $\delta$ and $\text{Var}[\bar{a}_T]$ are positive.